Math 8802 Spring 2015

Homework Assignment 2

(due April 15)

Do two (or more) of the following six problems.

1. Let $M(n, \mathbf{C})$ be the algebra of $n \times n$ complex matrices. For $T \in M(n, \mathbf{C})$ let $A = A_T$ be the subalgebra of $M(n, \mathbf{C})$ generated by I and T (where I is the identity matrix).
(a) Determine the dimension of A.
(b) Identify all the multiplicative functionals on A.
Hint: Use the Jordan form of T.

2. A commutative algebra A over $\mathbf{C}$ (with a unit element, as usual) is called local if it has exactly one maximal ideal, or, equivalently, one multiplicative functional.
(a) Show that the algebra $A = A_T$ from Problem 1 is a finite direct sum of local algebras, i. e. there exist local algebras $A_1, \ldots, A_r$, such that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$. (The multiplication in $A_1 \oplus A_2 \oplus \cdots \oplus A_r$ is given by $(a_1, \ldots, a_r)(b_1, \ldots, b_r) = (a_1 b_1, \ldots, a_r b_r)$.)
(b) Show that T is diagonalizable if and only if all the algebras A_j in the decomposition above are isomorphic to $\mathbf{C}$.

3. In what following we will use the word "generic" in the following sense: if a statement is true for a "generic" matrix in $M(n, \mathbf{C})$, it means that there there exists a dense open set $\mathcal{O} \subset M(n, \mathbf{C})$ such that the statement is true for each matrix from $\mathcal{O}$. In a similar way, a statement is true for two "generic matrices" $S, T \in M(n, \mathbf{C})$, if it is true for (S, T) in an open dense subset $\mathcal{O}$ of $M(n, \mathbf{C}) \times M(n, \mathbf{C})$. (We could also impose additional requirements, e. g. that the complement of $\mathcal{O}$ has measure zero, or that it is an algebraic set, etc. However, in this problem you can only work with the weaker notion above.)
Which of the following statements are correct? (Please give a justification for your answers.)

- (i) For a generic matrix $T \in M(n, \mathbf{C})$, the algebra generated by I and T is isomorphic to the direct sum of the n copies of $\mathbf{C}$. (In terms of Problem 2, $r = n$ and $A_j \simeq \mathbf{C}$.)
- (ii) For every two generic matrices $S, T \in M(n, \mathbf{C})$, the only linear subspaces $Y \subset \mathbf{C}^n$ invariant under both S and T (in the sense that $S(Y) \subset Y$ and $T(Y) \subset Y$) are trivial: either $Y = \{0\}$ or $Y = \mathbf{C}^n$.
- (iii) For every two generic matrices $S, T \in M(n, \mathbf{C})$ the the algebra generated by I, S and T coincides with $M(n, \mathbf{C})$.

4. Let (Ω, Σ, μ) be a measure space. (As usual, Σ is a sigma-algebra of subsets of Ω and μ is a measure on Σ.) Let X be the Hilbert space $L^2(\Omega, \mu)$ and let $m \in L^\infty(\Omega, \mu)$. Consider the operator $T_m \colon X \to X$ defined by $f \to mf$. Assume that for each measurable set $E \in \Sigma$ with $\mu(E) > 0$ there exists $F \subset E$ and $F \in \Sigma$ such that $0 < \mu(F) < \mu(E)$ (strict inequalities). Show that T is compact if and only if $m \equiv 0$.

5. Let $D \subset \mathbf{C}$ be the unit disc centered at 0 and let $\Omega \subset D$ be the annulus $\{z, 1/2 < |z| < 1\}$. Let B be the algebra of bounded holomorphic functions in Ω which can be continuously extended to $\bar{\Omega}$ and let A be the algebra of all holomorphic functions in D which can be continuously extended to $\bar{D}$. For $f \in B$ we let $||f||_B = \sup_{z \in \Omega} |f(z)|$ and for $f \in A$ we let $||f||_A = \sup_{z \in D} |f(z)|$.

- (i) Show that for each $f \in A$ we have $||f|_\Omega||_B = ||f||_A$. This means that the map $f \to f|_\Omega$ is an isometric injection, and A can be considered as a subalgebra of B. (Note that this would not be the case if we worked with the class of continuous functions.)
- (ii) Show that the sets $A^\times$ and $B^\times \cap A$ are different.

6. (a) Let $X_m = C^m[0,1]$, the (complex) Banach space of complex-valued functions on $[0,1]$ with continuous derivatives up to order m and the norm $||f||_m = \sum_{k=0}^m \sup_{x\in[0,1]} |f^{(k)}(x)|$. Consider a linear operator $L\colon X_m \to X_0$ given by

$$Lf = f^{(m)} + a_1 f^{(m-1)} + \cdots + a_m f \, , \tag{1}$$

where $a_1, \dots a_m \in \mathbf{C}$. Show that L is Fredholm and determine its index.

(b) Let $\Omega \subset \mathbf{C}$ be a smooth bounded domain and consider the Banach space X_m of the holomorphic functions in Ω which can be continuously extended to $\bar{\Omega}$ together with their (complex) derivatives up to order m. The norm in X_m is given by $||f||_m = \sum_{k=0}^m \sup_{z\in\bar{\Omega}} |f^{(k)}(z)|$. Consider the operator $L\colon X_1 \to X_0$ given by

$$Lf = f' \tag{2}$$

Show that L is Fredholm and determine its index.

(c)* As an optional part, you can consider part (b) with X_1 replaced by X_m and L replaced by the more general operator of order m similar to (1), with $f^{(k)}$ denoting the usual k-th order complex derivative of a holomorphic function f.