Math 8802

Spring 2015

Homework Assignment 3

(due May 14, 1pm)

Do two (or more) of the following six problems.

1. Let X be a separable Hilbert space and let be $\mathcal{L}(X)$ the space of continuous linear operators on X. For $R > 0$ we denote by $\mathcal{B}_R(X)$ the set $\{T \in \mathcal{L}(X)\,,\, ||T|| \le R\}$.
Recall that the *strong operator topology* is defined on $\mathcal{L}(X)$ by the system of neighborhoods (of the zero operator) given by

$$\mathcal{O}_{x_1,\dots,x_n;\epsilon_1,\dots\epsilon_n} = \{T \in \mathcal{L}(X)\,,\, ||Tx_1|| < \epsilon_1, \dots, ||Tx_n|| < \epsilon_n\}\ , \tag{1}$$

where n is any natural number, $x_1,\dots,x_n \in X$ and $\epsilon_1,\dots,\epsilon_n$ are strictly positive real numbers. We will denote this topology by σ. Note that the topology can also be considered on $\mathcal{B}_R(X)$.
The *weak operator topology* is defined on $\mathcal{L}(X)$ by the system of neighborhoods

$$\mathcal{O}_{x_1,y_1,\dots,x_n,y_n;\epsilon_1,\dots\epsilon_n} = \{T \in \mathcal{L}(X)\,,\, |(Tx_1,y_1)| < \epsilon_1, \dots, |(Tx_n,y_n)| < \epsilon_n\}\ , \tag{2}$$

where n is any natural number, $x_1, y_1 \dots, x_n, y_n \in X$ and $\epsilon_1,\dots,\epsilon_n$ are strictly positive real numbers. We will denote this topology by τ. The topology can also be considered on $\mathcal{B}_R(X)$.

- (i) Show that neither $(\mathcal{L}(X),\sigma)$ nor $(\mathcal{L}(X),\tau)$ are metrizable.
- (ii) Show that both $(\mathcal{B}_R(X),\sigma)$ and $(\mathcal{B}_R(X),\tau)$ are metrizable and, moreover, $(\mathcal{B}_R(X),\tau)$ is compact, while $(\mathcal{B}_R(X),\sigma)$ is not compact.
- (iii) Show that the standard operator product $(S,T) \to ST$ is not continuous as a map from $(\mathcal{L}(X),\sigma) \times (\mathcal{L}(X),\sigma)$ to $(\mathcal{L}(X),\sigma)$.
- (iv) Show that the standard operator product $(S,T) \to ST$ is continuous as a map from $(\mathcal{B}_{R_1}(X),\sigma) \times (\mathcal{B}_{R_2}(X),\sigma)$ to $(\mathcal{B}_{R_1R_2}(X),\sigma)$.
- (v) Decide whether the standard operator product $(S,T) \to ST$ is continuous as a map from $(\mathcal{B}_{R_1}(X),\tau) \times (\mathcal{B}_{R_2}(X),\sigma)$ to $(\mathcal{B}_{R_1R_2}(X),\tau)$.
- (vi) Decide whether the standard operator product $(S,T) \to ST$ is continuous as a map from $(\mathcal{B}_{R_1}(X),\sigma) \times (\mathcal{B}_{R_2}(X),\tau)$ to $(\mathcal{B}_{R_1R_2}(X),\tau)$.
- (vii) Decide whether the map $T \to T^*$ of taking adjoint is continuous from $(\mathcal{B}_R(X),\sigma)$ to $(\mathcal{B}_R(X),\sigma)$.
- (viii) Decide whether the map $T \to T^*$ of taking adjoint is continuous from $(\mathcal{B}_R(X),\tau)$ to $(\mathcal{B}_R(X),\tau)$.
- (ix) Decide whether the map $T \to T^*$ of taking adjoint is continuous from $(\mathcal{L}(X),\tau)$ to $(\mathcal{L}(X),\tau)$.

2. Let $S^1 \subset \mathbf{C}$ be the unit circle centered at 0. Let λ be the standard one-dimensional Lebesgue measure on S^1 (normalized so that $\lambda(S^1) = 2\pi$). Let us write complex numbers $z = x + iy$. let $X = L^2(S^1,\lambda)$ and let $T\colon X \to X$ be defined by $Tf(z) = xf(z)$, or, equivalently, $Tf(z) = (\operatorname{Re} z)f(z)$. Show that for a suitable measure μ on $(-1,1)$ and $Y = L^2((-1,1),\mu)$ there exists a bijective isomorphism $U\colon X \to Y \oplus Y$ such that we have

$$Tf = U^{-1}MUf \tag{3}$$

where the operator $M\colon Y\oplus Y \to Y \oplus Y$ is given by $(f_1,f_2) \to (xf_1, xf_2)$ (with the slight abuse of notation that xf_1 means the function $x \to xf_1(x)$).

3. Let X be a separable Hilbert space and let T_1, T_2 be two bounded self-adjoint operators on X which commute with each other, i. e. $T_1T_2 = T_2T_1$. Assume moreover that there is a vector $x_0 \in X$ such that the vectors of the form $\{\sum_{k,l} c_{k,l} T_1^k T_2^l x_0\,, \quad c_{k,l} \in \mathbf{C}$ with only finitely many non-zero $\}$ are dense in X. (The last condition means that the natural representation of the algebra generated by T_1, T_2 in X is cyclic.) Show that there is a compact set $K \subset \mathbf{R}^2$ and a Borel measure μ on K such that X can be identified with $L^2(K, \mu)$ and the operator T_j can be identified with the operators $f \to x_j f$ (multiplication by x_j).
As an optional part of the problem, you can formulate a suitable generalization of the statement to the situation when

(a) we have n mutually commuting self-adjoint operators $T_1, \dots, T_n$, and
(b) the representation of the algebra is not cylic.

4. Let X be a Hilbert space and let $T \in \mathcal{L}(X)$ be self-adjoint. Prove that $\lambda \in \mathbf{C}$ is in the spectrum of T if and only if there exists a sequence $x_n \in X$ with $||x_n|| = 1$, such that $||Tx_n - \lambda x_n|| \to 0$ as $n \to \infty$. (This statement is due to H. Weyl. You can also try to decide whether it remains true without the assumption that T be self-adjoint.)

5. True or false? (Please give a reason for your answer.)
Assume $u = u(k,t)$ is a real-valued function on $\mathbf{Z} \times (-\infty, \infty)$ which is smooth in t for each k, satisfies

$$\sum_k \left(|u(k,t)|^2 + |\partial_t u(k,t)|^2\right) \le c$$

for all t for some $c \ge 0$, and solves the equation

$$\frac{\partial^2}{\partial t^2} u(k,t) = u(k+1,t) - 2u(k,t) + u(k-1,t)$$

(which is a discrete version of the wave equation). If $u(k,0)$ and $\frac{\partial}{\partial t} u(k,0)$ vanish for $|k| > k_0$, then, for each $T > 0$, there exists $m = m(T, k_0) > 0$ such that $u(k,t)$ vanishes for $|t| \le T$ and $|k| > m(T)$. (In other words, disturbances propagate with finite speed.)
Hint: use the spectral representation of the operator on the right-hand side of the equation.

6. True or false? (Please give a reason for your answer.)
Let X be a separable Hilbert space and let $T \in \mathcal{L}(X)$ be a Fredholm operator. Then the following two conditions are equivalent:

(i) There exist a self-adjoint operator $P \in \mathcal{L}(X)$ and a unitary operator[1] $U \in \mathcal{L}(X)$ such that $T = UP$.
(ii) $\operatorname{Ind}(T) = 0$.

Hint: Note that when $T = UP$ as above then $T^*T = P^2$.

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[1]Recall that an operator is called unitary if $U^*U = UU^* = I$.