

Non-blowup at large times and stability for global solutions to the Navier-Stokes equations[†]

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Abstract

Suppose there exists a global solution u to the incompressible Navier-Stokes equations such that $u \in C_t(\dot{H}^{1/2})$. We prove that its $\dot{H}^{1/2}$ norm goes to 0 at infinity. We next use this fact to control the $L_t^2(\dot{H}^{1/2})$ norm of u , and finally we prove that such a solution is stable.

Introduction. – We consider the incompressible Navier-Stokes equations in the whole space,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \nabla \cdot u = 0. \\ u(x, 0) = u_0, \quad x \in \mathbb{R}^3, t \geq 0. \end{cases} \quad (1)$$

It is well-known that there are two different theories for the Cauchy problem: the Leray weak solutions [8], for initial data $u_0 \in L^2$, which are global but for which the uniqueness (or the propagation of regularity) is an open problem; and the Fujita-Kato “strong” solutions [4] for initial data $u_0 \in \dot{H}^{1/2}$, which are unique and local in time, i.e. $u \in C([0, T^*), \dot{H}^{1/2})$. The goal of this paper is to study a solution for which one assumes a priori that $T^* = \infty$. Note that if one supposes the initial data to be small then the solution is actually global. We show that a “large” global solution necessarily becomes small after a certain time, which implies in particular that it is stable. More precisely, we obtain the following result.

Theorem 1. *Let $u \in C([0, \infty), \dot{H}^{1/2})$ be a solution of (1). Then,*
– *there cannot be a blowup at $t = \infty$, and more precisely $\lim_{t \rightarrow \infty} \|u(t)\|_{\dot{H}^{1/2}} = 0$.*
– *the solution u belongs to $L^2((0, \infty), \dot{H}^{3/2})$.*

Remark. – An L^3 version of the first part of Theorem 1 is implied in [7], where it is showed that a solution $u \in C_t(L^3)$ satisfies $\lim_{t \rightarrow \infty} \sqrt{t} \|u(t)\|_\infty = 0$.

Note that no assumption is made on the rate of increase of the $\dot{H}^{1/2}$ norm of the solution. The second part of the theorem can be seen as a consequence

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of the first part and the following theorem. That is a result of persistency, which does not seem to exist in any references although it should be of folklore for strong solutions. It is the reverse of the result that is well-known [3] as follows: if a solution $u \in C([0, T^*), \dot{H}^{1/2})$ belongs to $L^2((0, T^*), \dot{H}^{1/2})$ then it can extend in $\dot{H}^{1/2}$ beyond T^* .

Theorem 2. *Let $T^* < \infty$ and let $u \in C([0, T^*], \dot{H}^{1/2})$ be a solution of (1). Then $\int_0^{T^*} \|u\|_{\dot{H}^{3/2}}^2 ds < \infty$.*

We therefore obtain the second part in Theorem 1 in the following way. There exists T such that $u(T)$ is small in norm $\dot{H}^{1/2}$; by the theory of solutions of small initial data $u \in L^2((T, \infty), \dot{H}^{3/2})$ and by Theorem 2 (the case T^* is finite), $u \in L^2((0, T), \dot{H}^{3/2})$ from which one concludes that $u \in L^2((0, \infty), \dot{H}^{3/2})$.

An important consequence of Theorem 1 is that it helps prove a stability theorem under the general assumption $u \in C_t(\dot{H}^{1/2})$. Note that if one assumes that the solution is slightly more regular, $u \in L_{t,\text{loc}}^\infty(H^1) \cap L_{t,\text{loc}}^2(H^2)$, then a stability theorem was proved in [10] under the assumption on the integrability at infinity $\nabla u \in L_t^4(L_x^2)$, which can be excluded thanks to Theorem 1.

Theorem 3. *Let $u \in C_t(\dot{H}^{1/2})$ be a global solution of (1). Then this solution is stable. That is, there exists $\varepsilon(u)$ such that if $\|u_0 - v_0\|_{\dot{H}^{1/2}} < \varepsilon(u)$ then v , the solution corresponding to initial data v_0 , is global and*

$$\begin{aligned} & \| (u - v)(t) \|_{\dot{H}^{1/2}}^2 + \int_0^t \| \nabla (u - v)(s) \|_{\dot{H}^{1/2}}^2 ds \\ & \leq C \| u_0 - v_0 \|_{\dot{H}^{1/2}} e^{C(\int_0^t \| u \|_{\dot{H}^{3/2}}^2 ds + \int_0^t \| u \|_{\dot{H}^1}^4 ds)}. \end{aligned}$$

We refer to [6] for the complete and extensive proofs in the setting of Besov spaces, which includes in particular the case $u \in C_t(L^3)$. Note that the stability of L^3 solutions was recently obtained by Tchamitchian [11] by completely different techniques, with an additional assumption on the smallness in large time, which can be excluded by taking into account the results in Lemarié [7]. We now give an idea of the proofs, which adapts to the case of Besov spaces modeled on L^p using a combination of techniques introduced in [2] and [5].

Proof of Theorem 1. – The important point is the following: if $u_0 \in \dot{H}^{1/2}$, the inhomogeneous Sobolev space, then by the weak-strong uniqueness [12] the solution u remains in L^2 and satisfies the energy inequality $\forall t \geq 0$, $E(u) \stackrel{\text{def}}{=} \|u(t)\|_{L^2}^2 + 2 \int_0^t \| \nabla u(s) \|_{\dot{H}^{1/2}}^2 ds \leq \|u_0\|_{L^2}^2$. As a result, $u \in L_t^\infty(L^2) \cap L_t^2(\dot{H}^1)$, and by interpolation u belongs to $L_t^4(\dot{H}^{1/2})$. For every $\varepsilon_0 > 0$, there exists, therefore, a time t_0 such that $\|u(t_0)\|_{\dot{H}^{1/2}} \leq \varepsilon_0$, and starting from this time one can apply the theory of small solutions. Thus, the solution remains small and tends to zero at infinity in norm $\dot{H}^{1/2}$

(see for example [9]). To reduce the general case to this case, one uses the method of high/low frequency decomposition which is introduced in [1] in the context of Navier-Stokes and reused with success in [5] to obtain the global solutions of infinite energy in dimension 2. More precisely, one decomposes $u_0 = v_0 + w_0$ where $w_0 \in \dot{H}^{1/2}$ with small norm and $v_0 \in H^{1/2}$ with large norm. One solves the equation $\partial w / \partial t = \Delta w - \nabla \cdot (w \otimes w) - \nabla p$, with $\nabla \cdot w = 0$, $w(x, 0) = w_0(x)$, by the theory of small initial data, to obtain a solution $w \in C_t(\dot{H}^{1/2}) \cap L_t^2(\dot{H}^{3/2})$ with small norm (see for example [3]) and which tends to zero at infinity. We have in particular

$$\|w(t)\|_{\dot{H}^{1/2}}^2 + \int_0^t \|w(s)\|_{\dot{H}^{3/2}}^2 ds \leq \|w_0\|_{\dot{H}^{1/2}}^2. \quad (2)$$

Then $v \stackrel{\text{def}}{=} u - w$ satisfies the equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \cdot (v \otimes v) - \nabla \cdot (v \otimes w) - \nabla \cdot (w \otimes v) - \nabla p, \quad \text{with } \nabla \cdot v = 0, v(x, 0) = v_0(x) \quad (3)$$

and v belongs to the space $C_t(\dot{H}^{1/2}) \cap L_{t,\text{loc}}^2(\dot{H}^{3/2})$ since w belongs to this space, as well as u , thanks to Theorem 2. Thus, from the equation of v one obtains $\partial_t v \in L_{t,\text{loc}}^2(\dot{H}^{-1/2})$. Recalling that $v_0 \in L^2$ and $v \in L_{t,\text{loc}}^\infty(\dot{H}^{1/2}) \subset L_{t,\text{loc}}^2(\dot{H}^{1/2})$, we deduces that $v \in C_t(L^2)$. We can thus write an energy inequality for v : multiply the equation of v by v and integrate in time and space to get

$$\|v(t)\|_{L^2}^2 + 2 \int_0^t \|v\|_{\dot{H}^1}^2 ds \leq \|v_0\|_{L^2}^2 + 2 \left| \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla w) \cdot v dx ds \right|. \quad (4)$$

One now estimates by the product rule [of derivatives]

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^3} (v \cdot \nabla w) \cdot v dx ds \right| &= \left| \int_0^t \int_{\mathbb{R}^3} (v \otimes w) \cdot \nabla v dx ds \right| \leq C \int_0^t \|w\|_{\dot{H}^{1/2}} \|v\|_{\dot{H}^1}^2 ds \\ &\leq C \|w_0\|_{\dot{H}^{1/2}} \int_0^t \|v\|_{\dot{H}^1}^2 ds, \end{aligned}$$

where the last inequality is a consequence of (2). Supposing that $C\|w_0\|_{\dot{H}^{1/2}} \leq \frac{1}{2}$ we can reduce the above term by the left hand side of (4), which allows us to conclude that the energy of v , $E(v)$, remains bounded. We then obtain by the previous argument that there exists T such that $\|v(T)\|_{\dot{H}^{1/2}} \leq \|w_0\|_{\dot{H}^{1/2}}$. We deduce that $\|u(T)\|_{\dot{H}^{1/2}} \leq 2\|w_0\|_{\dot{H}^{1/2}}$ and by the theory of small solutions that $\lim_{t \rightarrow \infty} \|u(t)\|_{\dot{H}^{1/2}} = 0$. Note that it is in fact not necessary to use the theory of small solutions: a self-contained argument is simply by noticing that from $\|u(T)\|_{\dot{H}^{1/2}} \leq 2\|w_0\|_{\dot{H}^{1/2}}$ and from the relation (2) applied for $u|_{[T,\infty)}$ one can deduce that $\limsup_{t \rightarrow \infty} \|u(t)\|_{\dot{H}^{1/2}} \leq \|u(T)\|_{\dot{H}^{1/2}} \leq 2\|w_0\|_{\dot{H}^{1/2}}$, which completes the proof of the first part of Theorem 1 because $\|w_0\|_{\dot{H}^{1/2}}$ can be chosen arbitrarily small. We already showed the second part from the first part and Theorem 2. Theorem 1 is thus proved.

Proof of Theorem 2. – We start by noticing that the norm $L_t^2(\dot{H}^{3/2})$ is finite for a small period of time after the initial time (by the uniqueness). We show that the solution cannot blow up until T^* inclusively. We write the energy inequality for the scalar product in $\dot{H}^{1/2}$, denoted by $(\cdot|\cdot)_{\dot{H}^{1/2}}$. It comes from

$$\|u(t)\|_{\dot{H}^{1/2}}^2 + 2 \int_0^t \|u(s)\|_{\dot{H}^{3/2}}^2 ds \leq \|u_0\|_{\dot{H}^{1/2}}^2 + 2 \left| \int_0^t (\nabla(u \otimes u)|u)_{\dot{H}^{1/2}} ds \right|. \quad (5)$$

By the density of regular functions in $C_t(\dot{H}^{1/2})$, one can split u into two terms : a small term $w \in C_t(\dot{H}^{1/2})$ and a regular and large term v . By the product rule, one can estimate

$$\begin{aligned} |(\nabla(u \otimes u)|u)_{\dot{H}^{1/2}}| &\leq |(w \cdot \nabla u|u)_{\dot{H}^{1/2}}| + |(v \cdot \nabla u|u)_{\dot{H}^{1/2}}| \\ &\leq C \|w\|_{\dot{H}^{1/2}} \|u\|_{\dot{H}^{3/2}}^2 + C \|v\|_{\dot{H}^{3/2} \cap L^\infty} \|u\|_{\dot{H}^{3/2}} \|u\|_{\dot{H}^{1/2}} \\ &\leq \left(C \|w\|_{\dot{H}^{1/2}} + \frac{1}{2} \right) \|u\|_{\dot{H}^{3/2}}^2 + C \|v\|_{\dot{H}^{3/2} \cap L^\infty}^2 \|u\|_{\dot{H}^{1/2}}^2. \end{aligned}$$

By choosing $\|w\|_{\dot{H}^{1/2}}$ to be sufficiently small, one can reduce the first term on the right hand side of the inequality by the left hand side of (5). The result follows by applying Gronwall's lemma since $v \in L^2((0, T^*), \dot{H}^{3/2} \cap L^\infty)$.

Remark 1. – The proof also applies directly to the case $T^* = \infty$, given the first part of Theorem 1 which assures $\lim_{t \rightarrow \infty} \|u(t)\|_{\dot{H}^{1/2}} = 0$.

Remark 2. – The previous proof is based on the separation argument which plays as the core of each of the results mentioned here. One can also obtain Theorem 2 in the following way, which was pointed out to us by J.-Y. Chemin: by examining the proof of the local existence of $\dot{H}^{1/2}$ -solutions, it is easy to see that the [maximal] time of existence is uniformly bounded from the below by the initial data in a compact subset of $\dot{H}^{1/2}$. Then Theorem 2 is a simple consequence of the uniqueness of the $\dot{H}^{1/2}$ -solutions since the set of initial data $v_t(x) = u(x, t)$ for $t \in [0, T^*]$ is compact (as the image of a compact set).

Proof of Theorem 3. – As remarked, the global solution $u \in C_t(\dot{H}^{1/2})$ is automatically in $L_t^2(\dot{H}^{3/2})$. We then consider the equation satisfied by the difference $w = u - v$, that is $\frac{\partial w}{\partial t} = \Delta w - \nabla \cdot (w \otimes w) - \nabla \cdot (w \otimes u) - \nabla \cdot (u \otimes w) - \nabla p$, with $\nabla \cdot w = 0$ and $w(x, 0) = w_0(x) \stackrel{\text{def}}{=} u_0(x) - v_0(x)$, and search for an a priori estimate for this equation. We write the new energy inequality in $\dot{H}^{1/2}$ to obtain

$$\begin{aligned} &\|w(t)\|_{\dot{H}^{1/2}}^2 + 2 \int_0^t \|w\|_{\dot{H}^{3/2}}^2 ds \\ &\leq \|w_0\|_{\dot{H}^{1/2}}^2 + C \left| \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla w|w)_{\dot{H}^{1/2}} + (w \cdot \nabla u|w)_{\dot{H}^{1/2}} + (u \cdot \nabla w|w)_{\dot{H}^{1/2}} ds \right|. \end{aligned}$$

The first two terms are estimated easily: the first term which is nonlinear in w can be absorbed into the left hand side because we suppose a priori that the member on the left hand side is small and $\left| \int_0^t (w \cdot \nabla w|w)_{\dot{H}^{1/2}} ds \right| \leq$

$\int_0^t \|w(s)\|_{\dot{H}^{1/2}} \|w(s)\|_{\dot{H}^{3/2}}^2 ds$. The second term is bounded easily by using the product rule and the fact that $\nabla u \in L_t^2(\dot{H}^{1/2})$. The last term is little more delicate because a priori it is not possible to estimate (the low frequencies of) u by the norm $L_t^2(\dot{H}^{3/2})$. We call on the norm $L^4(\dot{H}^1)$ by estimating via the product rule

$$\begin{aligned} |(u \cdot \nabla w)_{\dot{H}^{1/2}}| &\leq C \|u\|_{\dot{H}^1} \|w\|_{\dot{H}^1} \|w\|_{\dot{H}^{3/2}} \leq C \|u\|_{\dot{H}^1} \|w\|_{\dot{H}^{1/2}}^{1/2} \|w\|_{\dot{H}^{3/2}}^{3/2} \\ &\leq \frac{1}{4} \|w\|_{\dot{H}^{3/2}}^2 + C \|u\|_{\dot{H}^1}^4 \|w\|_{\dot{H}^{1/2}}^2. \end{aligned}$$

This allows us to treat this term like the second term. Note that the second term becomes similar to the above term, the only difference being that the norm $\|u\|_{\dot{H}^1}^4$ is replaced by the norm $\|u\|_{\dot{H}^{3/2}}^2$. Note also that an application of Littlewood-Paley theory and the commutators allows us to get rid of the presence of the norm $\|u\|_{\dot{H}^1}^4$ in the upper bound (and thus in the final estimate), with the cost of a little more techniques. We then complete [the proof] a usual way by applying Gronwall's lemma.

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