Hilbert-Schmidt operators and Tensor Product of two vector spaces

[I] Hilbert-Schmidt operators

(1) Definition

Let V and W be two separable Hilbert spaces, T a bounded operator from V to W; i.e. $T \in B(V,W)$. Let $(v_n)_{n\in\mathbb{N}}$ be an orthonormal basis of V. We shall prove that the series

$\sum_{n=1}^{\infty} |Tv_n|^2$ is independent of the choice of basis (v_n).

Indeed, let (w_n) be an orthonormal basis of W. Then

$$Tv_n = \sum_{m=1}^{\infty} \langle Tv_n, w_m\rangle w_m = \sum_{m=1}^{\infty} \langle v_n, T^* w_m\rangle w_m$$

Thus

$$|Tv_n|^2 = \sum_{m=1}^{\infty} |\langle v_n, T^* w_m\rangle|^2$$

and

$$\sum_{n=1}^{\infty} |Tv_n|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle v_n, T^* w_m\rangle|^2$$

$$\underset{\text{Fubini}}{=} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\langle v_n, T^* w_m\rangle|^2$$

$$= \sum_{m=1}^{\infty} |T^* w_m|^2$$

Thus, the series $\sum_{n=1}^{\infty} |Tv_n|^2$ is independent of the choice of orthonormal basis (v_n).

Definition: $T \in B(V, W)$ is called Hilbert-Schmidt operator if the sum $\sum_{n=1}^{\infty} |Tv_n|^2$ is finite.

By this definition and the above calculation, we easily see that T is Hilbert-Schmidt if and only if T^* is Hilbert-Schmidt.

(2) If T is Hilbert-Schmidt (HS) then T is compact.

Proof To show that T is compact, we only need to show that T is a normlimit of a sequence of finite-rank operators. For each $v \in V$ with unit norm,

$$Tv = \sum_{n=1}^{\infty} \langle Tv, w_n \rangle w_n = \sum_{n=1}^{\infty} \langle v, T^* w_n \rangle w_n$$

For each $m \in \mathbb{N}$, we put

$$T_m v = \sum_{n=1}^{m} \langle v, T^* w_n \rangle w_n \qquad \forall v \in V$$

Then $$Tv - T_m v = \sum_{n=m+1}^{\infty} \langle v, T^* w_n \rangle w_n$$

and $$|Tv - T_m v|^2 = \sum_{n=m+1}^{\infty} |\langle v, T^* w_n \rangle|^2 \leq \sum_{n=m+1}^{\infty} |T^* w_n|^2$$

Thus,

$$\|T-T_m\|^2 \le \sum_{n \ge m+1}^{\infty} |T^* w_n|^2$$

We know that $\sum_{n=1}^{\infty} |T^* w_n|^2 = \sum_{n=1}^{\infty} |T v_n|^2 < \infty$. Hence,

$$\lim_{m \to \infty} \|T - T_m\|^2 = 0.$$

(3) The space of HS operators contains the space of trace class operators

Proof Let T be a trace class operator, then

$$\sum_{n=1}^{\infty} |T v_n| < \infty$$

Thus, $\lim_{n \to \infty} |T v_n| = 0$. There exists $N \in \mathbb{N}$ such that

$$|T v_n|^2 \le |T v_n| \quad \forall n \ge N$$

Hence, the series $\sum_{n=1}^{\infty} |T v_n|^2 < \infty$.

(4) Norm on the space of HS operator

Hereafter, the space of HS operator from Hilbert space V to Hilbert space W is denoted $B_2(V, W)$. We will show that

$$\|T\|_{HS} = \left(\sum_{n=1}^{\infty} |T v_n|^2 \right)^{1/2}$$

is actually a norm on $B_2(V, W)$. We have to check 3 ~~pro~~ criteria

* Positive definite: $\|T\|_{HS} \ge 0$.

If $\|T\|_{HS} = 0$ then $Tv_n = 0$ $\forall n \in \mathbb{N}$, then $T \equiv 0$.

* Homogeneous: let $\lambda \in \mathbb{C}$ then

$$\|\lambda T\|_{HS} = \left(\sum_{n=1}^{\infty} |\lambda T v_n|^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} |\lambda|^2 |Tv_n|^2\right)^{1/2}$$

$$= |\lambda| \left(\sum_{n=1}^{\infty} |Tv_n|^2\right)^{1/2} = |\lambda| \, \|T\|_{HS}.$$

* Triangle inequality:

Let S and T be two HS operator

$$\|S+T\|_{HS} = \left(\sum_{n=1}^{\infty} |(S+T)(v_n)|^2\right)^{1/2} = \left(\sum_{n=1}^{\infty} |Sv_n + Tv_n|^2\right)^{1/2}$$

$$\overset{\text{Minkowski}}{\leq} \left(\sum_{n=1}^{\infty} |Sv_n|^2\right)^{1/2} + \left(\sum_{n=1}^{\infty} |Tv_n|^2\right)^{1/2} = \|S\|_{HS} + \|T\|_{HS}$$

5 $\|T\|_{HS} = \|T^*\|_{HS}$ and $\|\cdot\|_{\infty} \leq \|\cdot\|_{HS}$

Proof By the calculation in Point 1, we have

$$\|T\|_{HS}^2 = \sum_{n=1}^{\infty} |Tv_n|^2 = \sum_{n=1}^{\infty} |T^* w_n|^2 = \|T^*\|_{HS}^2$$

Thus $\|T\|_{HS} = \|T^*\|_{HS}$. For each $v \in V$ with unit norm,

$$Tv = \sum_{n=1}^{\infty} \langle Tv, w_n \rangle w_n = \sum_{n=1}^{\infty} \langle v, T^* w_n \rangle w_n$$

Thus, $$|Tv|^2 = \sum_{n=1}^{\infty} |\langle v, T^* w_n \rangle|^2 \leq \sum_{n=1}^{\infty} |T^* w_n|^2 = \|T^*\|_{HS}^2 = \|T\|_{HS}^2$$

or $|Tv| \leq \|T\|_{HS}$. Hence $\|T\|_{\infty} = \sup_{|v|=1} |Tv| \leq \|T\|_{HS}$.

⑥ The space of finite-rank operators is dense on $B_2(V,W)$

Proof Hereafter, the space of finite-rank operators is denoted $B_{fin}(V,W)$. First, we'll show that $B_{fin}(V,W) \subset B_2(V,W)$.

Let $S \in B_{fin}(V,W)$. Then $\mathrm{Im}\, S$ is finite dimensional, with a finite orthonormal basis $\{w_1', w_2', \dots, w_k'\}$. ~~Let $\{w_1', \dots, w_k', w_{k+1}', \dots\}$ be an orthonormal~~ Thus $\mathrm{Im}\, S$ is generated by finitely many elements $S(v_1'), \dots, S(v_k')$. Let $\{v_1'', \dots, v_j''\}$ be an orthonormal set generating $v_1', \dots, v_k'$. ~~Let $\{v_1'', \dots, v_j'', v_{j+1}'', \dots\}$ be an orthonormal basis of V. Then $S(v_{j+1}'') = S(v_{j+2}'') = \dots = 0$~~

Let $\{v_{j+1}'', \dots\}$ be an orthonormal basis of $\ker S$. Then $\{v_1'', \dots, v_j'', v_{j+1}'', \dots\}$ is an orthonormal basis of V. We have

$$\sum_{n=1}^{\infty} |S(v_n'')|^2 = \sum_{n=1}^{j} |S(v_n'')|^2 < \infty$$

thus $S \in B_2(V,W)$.

Next, we'll show that $B_{fin}(V,W)$ is dense in $B_2(V,W)$. For each $T \in B_2(V,W)$, we have

$$Tv = \sum_{n=1}^{\infty} \langle Tv, w_n \rangle w_n$$

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For each $m \in \mathbb{N}$, we define the finite-rank operator

$$T_m v = \sum_{k=1}^{m} \langle Tv, w_k \rangle w_k \qquad \forall v \in V$$

Then
$$(T - T_m)(v) = \sum_{k=m+1}^{\infty} \langle Tv, w_k \rangle w_k$$

and
$$\left|(T - T_m)(v)\right|^2 = \sum_{k=m+1}^{\infty} |\langle Tv, w_k \rangle|^2 = \sum_{k=m+1}^{\infty} |\langle v, T^* w_k \rangle|^2$$

Then
$$\left|(T - T_m) v_n\right|^2 = \sum_{k=m+1}^{\infty} |\langle v_n, T^* w_k \rangle|^2$$

and
$$\|T - T_m\|_{HS}^2 = \sum_{n=1}^{\infty} \left|(T - T_m) v_n\right|^2 = \sum_{n=1}^{\infty} \sum_{k=m+1}^{\infty} |\langle v_n, T^* w_k \rangle|^2$$

$$\underset{\text{Fubini}}{=} \sum_{k=m+1}^{\infty} \sum_{n=1}^{\infty} |\langle v_n, T^* w_k \rangle|^2$$

$$= \sum_{k=m+1}^{\infty} |T^* w_k|^2$$

Since $\sum_{k=1}^{\infty} |T^* w_k|^2 = \sum_{k=1}^{\infty} |T v_k|^2 < \infty$, we have $\sum_{k=m+1}^{\infty} |T^* w_k|^2 \to 0$

as $m \to \infty$. Thus, $\|T - T_m\|_{HS} \to 0$ as $m \to \infty$.

7 ~~There exists a linear isomorphic isometrism between~~ $\cancel{B_2(V, W)}$

~~and~~ $\cancel{\ell^2(\mathbb{N})}$. ~~Consequently,~~ $\cancel{B_2(V, W)}$ ~~is a Hilbert space isomorphic~~

~~to~~ $\cancel{\ell^2(\mathbb{N})}$.

7. $B_2(V,W)$ is a Hilbert space with inner product

$$\langle T, S\rangle = \sum_{i=1}^{\infty} \langle S^*Tv_i, v_i\rangle = \sum_{i=1}^{\infty} \langle Tv_i, Sv_i\rangle$$

Proof First we show that the sum $\sum_{i=1}^{\infty} \langle Tv_i, Sv_i\rangle$ does not depend on the choice of orthonormal basis (v_i).

$$Tv_i = \sum_{n=1}^{\infty} \langle Tv_i, w_n\rangle w_n$$

$$Sv_i = \sum_{m=1}^{\infty} \langle Sv_i, w_m\rangle w_m$$

Then
$$\langle Tv_i, Sv_i\rangle = \sum_{n=1}^{\infty} \langle Tv_i, w_n\rangle\langle w_n, Sv_i\rangle$$

$$= \sum_{n=1}^{\infty} \underbrace{\langle v_i, T^*w_n\rangle\langle S^*w_n, v_i\rangle}_{a_{in}}$$

We have
$$|a_{in}| \le \frac{1}{2}\left(|\langle v_i, T^*w_n\rangle|^2 + |\langle S^*w_n, v_i\rangle|^2\right)$$

Thus
$$\sum_{i,n} |a_{in}| \le \frac{1}{2}\left\{\sum_n \sum_i |\langle v_i, T^*w_n\rangle|^2 + \sum_n \sum_i |\langle S^*w_n, v_i\rangle|^2\right\}$$

$$= \frac{1}{2}\left\{\sum_n |T^*w_n|^2 + \sum_n |S^*w_n|^2\right\} < \infty$$

By Fubini's Theorem,

$$\sum_{i=1}^{\infty} \langle Tv_i, Sv_i\rangle = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} a_{in} = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} a_{in}$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \left\langle \langle S^* w_n, v_i \rangle v_i, T^* w_n \right\rangle$$

$$= \sum_{n=1}^{\infty} \left\langle \sum_{i=1}^{\infty} \langle S^* w_n, v_i \rangle v_i, T^* w_n \right\rangle$$

$$= \sum_{n=1}^{\infty} \langle S^* w_n, T^* w_n \rangle$$, which is independent of $(v_i)_{i \in \mathbb{N}}$.

Next, we have to check the following 3 properties

* Linear with respect to the first argement
* Conjugate symmetric.
* Positive definite

Obviously, $\langle \cdot, \cdot \rangle$ is linear on the first argument.

$$\langle S, T \rangle = \sum_{n=1}^{\infty} \langle S v_n, T v_n \rangle = \sum_{n=1}^{\infty} \overline{\langle T v_n, S v_n \rangle} = \overline{\sum_{n=1}^{\infty} \langle T v_n, S v_n \rangle}$$

$$= \overline{\langle T, S \rangle}.$$

Thus, $\langle \cdot, \cdot \rangle$ is conjugate symmetric.

$$\langle T, T \rangle = \sum_{n=1}^{\infty} \langle T v_n, T v_n \rangle = \sum_{n=1}^{\infty} |T v_n|^2 = \|T\|_{HS}^2$$

Thus, $\langle \cdot, \cdot \rangle$ is positively definite. Up to now, we verified that $\langle \cdot, \cdot \rangle$ is an inner product on $B_2(V, W)$ which generates the norm $\|\cdot\|_{HS}$.

Next, we have to show that this norm is complete.

Let (T_m) be a Cauchy sequence in $B_2(V,W)$. Then

$$\sum_{n=1}^{\infty} |T_m v_n|^2 < \infty$$

Thus, $\{T_m v_n\}_n \in \ell^2(\mathbb{N})$. Put $u_m = \{T_m v_n\}_n$. Then

$$\|u_m - u_k\|_{\ell^2}^2 = \sum_{n=1}^{\infty} |T_m v_n - T_k v_n|^2 = \|T_m - T_k\|_{HS}^2,$$

or $\|u_m - u_k\|_{\ell^2} = \|T_m - T_k\|_{HS}$. That means $\{u_m\}$ is a Cauchy sequence on $\ell^2(\mathbb{N})$. Since $\ell^2(\mathbb{N})$ is complete, there exists $u \in \ell^2(\mathbb{N})$ such that $u_m \to u$. ~~Deno~~ We write $u = \{a_n\}_{n\in\mathbb{N}}$. Define a mapping $T \in B(V,W)$ such that $Tv_n = a_n$. We have

$$\sum_{n=1}^{\infty} |T_m v_n - Tv_n|^2 = \sum_{n=1}^{\infty} |T_m v_n - a_n|^2 = \|u_m - u\|_{\ell^2}^2 \to 0$$

Thus $T_m \to T$ in $B_2(V,W)$.

$\boxed{\text{I}}$ Tensor product of two vector spaces

① Definition

Let V and W be two modules on ring K. Then a tensor product of V and W is a pair (L, ϕ) consisting of a ~~vector space L~~ free module L and a bilinear mapping ϕ from $V \times W$ to L such that for each bilinear map h from $V\times W$ to a vector space

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Y, there exists uniquely a linear map $\tilde{h}$ from L to Y such that $h = \tilde{h} \circ \phi$, i.e. the following diagram is commutative

$$\begin{tikzcd} V \times W \arrow[r, "\phi"] \arrow[d, "h"'] & L \arrow[dl, "\tilde{h}"] \\ Y & \end{tikzcd}$$

(2) ~~Suppose that~~ (L_1, ϕ_1) and (L_2, ϕ_2) are two tensor products of vector space V and W if and only if there exists a linear isomorphism between L_1 and L_2.

Prop (2) Let (L_1, ϕ_1) be a tensor product of V and W. Then a pair (L_2, ϕ_2) is also a tensor product of V and W if and only if there exists a linear isomorphism between L_1 and L_2.

Proof The backward part

Let $\psi : L_1 \to L_2$ be an isomorphism. For each vector space Y and bilinear map $h : V \times W \to Y$, we'll show that there exists a unique linear map $\tilde{h} : L_2 \to Y$ such that $h = \tilde{h} \psi \phi_1$.

$$\begin{tikzcd} V \times W \arrow[r, "\phi_1"] \arrow[d, "h"'] & L_1 \arrow[r, "\psi"] \arrow[dl, dashed, "h_1"'] & L_2 \arrow[dll, dashed, "h_1 \psi^{-1} = \tilde{h}"] \\ Y & & \end{tikzcd}$$

Since (L_1, ϕ_1) is a tensor product, there exists a linear map $h_1 : L_1 \to Y$ such that $h = h_1\phi_1$. Put $\tilde{h} = h_1\psi^{-1}$. Then $\tilde{h} : L_2 \to Y$ and $h = \tilde{h}\psi\phi_1$. If there is another $\tilde{h}_1 : L_2 \to Y$ such that $h = \tilde{h}_1\psi\phi_1$ then $\tilde{\tilde{h}}_1 = \tilde{h}_1\psi$ is a linear map from L_1 to Y such that $h = \tilde{\tilde{h}}_1\phi_1$. Since (L_1, ϕ_1) is a tensor product, $\tilde{\tilde{h}}_1 \equiv h_1$. Then $\tilde{h}_1 = \tilde{\tilde{h}}_1\psi^{-1} \equiv h_1\psi^{-1} = \tilde{h}$. Thus, $\tilde{h}$ is unique. That means $(L_2, \psi\phi_1)$ is also a tensor product.

The forward part;

$$\begin{tikzcd}
V \times W \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & L_1 \arrow[dl, "h_1"] \\
L_2 \arrow[ur, dashed, bend right, "h_2"'] &
\end{tikzcd}$$

Since (L_1, ϕ_1) is a tensor product, there exists uniquely a linear map $h_1 : L_1 \to L_2$ such that
$$\phi_2 = h_1\phi_1.$$

Since (L_2, ϕ_1) is a tensor product there exists uniquely a linear map $h_2 : L_2 \to L_1$ such that $\phi_1 = h_2\phi_2$.

Thus, $\phi_1 = h_2\phi_2 = h_2h_1\phi_1$, i.e. $h_2h_1 \equiv id$ on $\mathrm{Im}\,\phi_1$.

To show that $h_2h_1 \equiv id$ on L_1, we have to show that ~~Im~~ L_1 can be linearly spanned from ~~by~~ $\mathrm{Im}\,\phi_1$. Put

$S = \langle \mathrm{Im}\,\phi_1 \rangle$. Suppose by contradiction that $S \subsetneq L_1$. Since L_1 is a semisimple module S is its direct summand. there exists a vector space T such that $S \oplus T = L_1$. Let $\{v_i\}_{i \in I}$ be a basis of T.

$$\begin{tikzcd}
V \times W \arrow[r, "\phi_1"] \arrow[d, "\phi_1"'] & L_1 \arrow[dl, "k_1"] \arrow[dl, dashed, bend left, "k_2"] \\
S &
\end{tikzcd}$$

Let $k_1 : L_1 \to S$ be such that

$k_1(x) = x \quad \forall x \in S$

and $k_1(v_i) = 0 \quad \forall i \in I$.

Then $\phi_1 = k_1 \phi_1$.

Let $k_2 : L_1 \to S$ be such that

$k_2(x) = x \quad \forall x \in S$

$k_2(v_i) = 0 \quad \forall i \in I \setminus \{i_0\}$

$k_2(v_{i_0}) = u_0 \neq 0$

Then $\phi_1 = k_2 \phi_1$. Since (L_1, ϕ_1) is a tensor product, k_1 and k_2 must be the same. This is a contradiction. ∎

In short, $h_2 h_1 = \mathrm{id}_{L_1}$. Thus, $h_2 = h_1^{-1}$ is the linear isomorphism between L_1 and L_2.

Point ② guarantees that the property mentioned in Point ① is a universal property, i.e. it contains all attributes of tensor product.

Accordingly, tensor product is unique up to a linear isomorphism.

③ Construction of tensor product

Let $F = K^{V\times W}$ be a direct sum, i.e. each element of F is a map from $V\times W$ to K that is zero for all but finitely many elements in $V\times W$. Then F is also a module with addition

$$(f+g)(x) := f(x) + g(x) \quad \forall x \in V\times W,$$

and scalar multiplication

$$(\lambda f)(x) := \lambda f(x) \quad \forall \lambda \in K, x \in V\times W.$$

For each $(v,w) \in V\times W$, we denote $\mathbb{1}_{(v,w)}$ the mapping from $V\times W$ to K such that

$$\mathbb{1}_{(v,w)}(u) = \begin{cases} 1 & \text{if } u = (v,w) \\ 0 & \text{otherwise} \end{cases}$$

Then F is a free module with basis $\{\mathbb{1}_{(v,w)} : v \in V, w \in W\}$.

Let R be a submodule of F spanned by the set

$$\{\mathbb{1}_{(av_1+bv_2,w)} - a\mathbb{1}_{(v_1,w)} - b\mathbb{1}_{(v_2,w)} \mid a,b \in K, v_1,v_2 \in V, w \in W\}$$

$$\cup \{\mathbb{1}_{(v,aw_1+bw_2)} - a\mathbb{1}_{(v,w_1)} - b\mathbb{1}_{(v,w_2)} \mid a,b \in K, v \in V, w_1,w_2 \in W\}$$

On F, we define an equivalence relation

$$x \sim y \Leftrightarrow x-y \in R$$

and denote F/R the set of all equivalence class.

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We define the map
$$\phi : V \times W \longrightarrow F/R$$
$$(v,w) \longmapsto \mathbb{1}_{(v,w)} + R$$

Then ~~we claim~~ $(F/R, \phi)$ is a tensor product between V and W.

Proof. First, we show that ϕ is bilinear.

$$\phi(v_1+v_2, w) = \mathbb{1}_{(v_1+v_2,w)} + R = \mathbb{1}_{(v_1,w)} + \mathbb{1}_{(v_2,w)} + \underbrace{\mathbb{1}_{(v_1+v_2,w)} - \mathbb{1}_{(v_1,w)} - \mathbb{1}_{(v_2,w)}}_{\in R} + R$$

$$= \mathbb{1}_{(v_1,w)} + \mathbb{1}_{(v_2,w)} + R$$

$$= \phi(v_1,w) + \phi(v_2,w)$$

$$\phi(av, w) = \mathbb{1}_{(av,w)} + R = a\mathbb{1}_{(v,w)} + \underbrace{\mathbb{1}_{(av,w)} - a\mathbb{1}_{(v,w)}}_{\in R} + R$$

$$= a\mathbb{1}_{(v,w)} + R = a\phi(v,w)$$

Similarly, ϕ is linear with repect to W.

Let Y be a vector space. Suppose $h : V\times W \to Y$ is a bilinear map. We'll show that there exists uniquely a linear map $\tilde{h} : F/R \to Y$ such that $h = \tilde{h}\phi$.

$$\begin{tikzcd}
V\times W \arrow[r, "\phi"] \arrow[d, "h"'] & F/R \arrow[dl, dashed, "\tilde{h}"] \\
Y &
\end{tikzcd}$$

The uniqueness:

If $\tilde{h} : F/R \to Y$ is a linear map such that $h = \tilde{h}\phi$ then

$$h(v,w) = \tilde{h}\,\phi(v,w) \qquad \forall (v,w) \in V \times W$$

Then $$\tilde{h}\left(\mathbb{1}_{(v,w)} + R\right) = h(v,w)$$

Since the set $\{\mathbb{1}_{(v,w)} + R : (v,w) \in V\times W\}$ generates F/R, $\tilde{h}$ is determined uniquely over F/R.

The existence:

Since $\{\mathbb{1}_{(v,w)} : v \in V, w \in W\}$ is a basis of F, there exists a linear map $h_1 : F \to Y$ such that $h_1(\mathbb{1}_{(v,w)}) = h(v,w)$ $\forall (v,w) \in V\times W$

~~Let~~ Suppose $u_1, u_2 \in F$ ~~such that~~ satisfy $u_1 - u_2 \in R$. Then

$$u_1 - u_2 = \sum_{i=1}^{m} \alpha_i \left[\mathbb{1}_{(a_i f_i^1 + b_i f_i^2,\, g_i)} - a_i \mathbb{1}_{(f_i^1, g_i)} - b_i \mathbb{1}_{(f_i^2, g_i)} \right]$$

$$+ \sum_{j=1}^{n} \beta_j \left[\mathbb{1}_{(f_j,\, c_j g_j^1 + d_j g_j^2)} - c_j \mathbb{1}_{(f_j, g_j^1)} - d_j \mathbb{1}_{(f_j, g_j^2)} \right]$$

By definition, h_1 is linear. Thus,

$$h_1(u_1 - u_2) = \sum_{i=1}^{m} \alpha_i \left[h_1\left(\mathbb{1}_{(a_i f_i^1 + b_i f_i^2,\, g_i)}\right) - a_i h_1\left(\mathbb{1}_{(f_i^1, g_i)}\right) - b_i h_1\left(\mathbb{1}_{(f_i^2, g_i)}\right) \right]$$

$$+ \sum_{j=1}^{n} \beta_j \left[h_1\left(\mathbb{1}_{(f_j,\, c_j g_j^1 + d_j g_j^2)}\right) - c_j h_1\left(\mathbb{1}_{(f_j, g_j^1)}\right) - d_j h_1\left(\mathbb{1}_{(f_j, g_j^2)}\right) \right]$$

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$$= \sum_{i=1}^{m} \alpha_i \left[h(a_i f_i^1 + b_i f_i^2, g_i) - a_i h(f_i^1, g_i) - b_i h(f_i^2, g_i) \right]$$

$$+ \sum_{j=1}^{n} \beta_j \left[h(f_j, c_j g_j^1 + d_j g_j^2) - c_j h(f_j, g_j^1) - d_j h(f_j, g_j^2) \right]$$

$= 0$ because h is bilinear.

Thus, $h_1(u_1) = h_2(u_2)$. By this reason, we can define a map

$$\tilde{h} : F/R \longrightarrow Y$$
$$\tilde{h}(f + R) = h_1(f)$$

For each $(v, w) \in V \times W$,

$$\tilde{h}\phi(v, w) = \tilde{h}(\mathbb{1}_{(v,w)} + R) = h_1(\mathbb{1}_{(v,w)}) = h(v, w)$$

Thus, $\tilde{h}\phi = h$.

(4) Let $\{v_i\}_{i \in I}$ be a basis of V, $\{w_j\}_{j \in J}$ a basis of W. Then $S = \{\mathbb{1}_{(v_i, w_j)} + R : i \in I, j \in J\}$ is a basis of F/R.

Proof: First we show that S can linearly generate F/R. We know the set $\{\mathbb{1}_{(v,w)} + R : v \in V, w \in W\}$ can generate F/R. Thus, it is sufficient to show that for each $v \in V$, $w \in W$, $\mathbb{1}_{(v,w)} + R$ is a linear combination of elements of S. We can write

$$v = \sum_{i=1}^{n} \alpha_i v_i \quad , \quad w = \sum_{j=1}^{m} \beta_j w_j$$

Then $\mathbb{1}_{(v,w)} + R = \mathbb{1}_{(\sum \alpha_i v_i, \sum \beta_j w_j)} + R = \sum_{ij} \alpha_i \beta_j \mathbb{1}_{(v_i, w_j)} + R$

Next, we'll show that S is linearly independent. Suppose that $c_{ij} \in K$, $\forall i = 1, \dots, n;\ j = 1, \dots, m$ be such that

$$\sum_{ij} c_{ij} \left(\mathbb{1}_{(v_i, w_j)} + R \right) = 0 \quad \text{or} \quad \sum_{ij} c_{ij} \mathbb{1}_{(v_i, w_j)} \in R.$$

For each bilinear map $h : V \times W \to Y$, we define as in Point 3 the ~~map~~ linear map $h_1 : F \to Y$ such that $h_1(\mathbb{1}_{(v,w)}) = h(v,w)$. By Point 3, if $u_1 - u_2 \in R$ then $h_1(u_1) = h_1(u_2)$. Note that

$$\sum_{ij} c_{ij} \mathbb{1}_{(v_i, w_j)} - 0 \in R$$

Thus, $$0 = h_1\left(\sum_{ij} c_{ij} \mathbb{1}_{(v_i, w_j)} \right) = \sum_{ij} c_{ij} h_1\left(\mathbb{1}_{(v_i, w_j)}\right) = \sum_{ij} c_{ij} h(v_i, w_j)$$

Therefore, $$\sum_{ij} c_{ij} h(v_i, w_j) = 0 \quad (*)$$

$\forall$ bilinear h from $V \times W$ to Y.

For each pair of indices (i_0, j_0), we define the bilinear map

$$h_{i_0 j_0} : V \times W \to K$$
$$(v, w) \mapsto \alpha_{i_0} \beta_{j_0}$$

where $v = \sum \alpha_i v_i$, $w = \sum \beta_j w_j$

$$h_{i_0 j_0}(v_i, w_j) = \begin{cases} 1 & \text{if } i=i_0, j=j_0 \\ 0 & \text{otherwise} \end{cases}$$

Applying (*) for $h = h_{i_0 j_0}$, we get $a_{i_0 j_0} = 0$. ⊠

Hereafter, we denote $V \otimes W$ the space F/R together with the bilinear map ϕ. That means, $V \otimes W$ is the tensor product of V and W. Also, we define $v \otimes w := \mathbb{1}_{(v,w)} + R$.

$\boxed{\text{III}}$ Hilbert-space tensor product

Let V and W be two separable Hilbert spaces. Let $(v_i)_{i\in\mathbb{N}}$ and $(w_j)_{j\in\mathbb{N}}$ be repectively an orthonormal basis of V and W.

$$V_0 = \langle \{v_1, v_2, \dots\} \rangle,$$

$$W_0 = \langle \{w_1, w_2, \dots\} \rangle,$$

i.e. each element of V_0 is a finite linear combination of $\{v_1, v_2, \dots\}$.
Note that $V_0 \subsetneq V$ and $W_0 \subsetneq W$ if V and W are infinite dimensional. In part $\boxed{\text{II}}$, we defined the <u>algebraic tensor product</u> of V and W. In this case, V and W have noncountable bases. However, orthogonal bases (v_i) and (w_j) are very important in those spaces. We should introduce another tensor product between V and W that involves these orthogonal bases. Such a kind of tensor product

is Hilbert-space tensor product.

① $V_0 \otimes W_0$ is independent of the choice of orthonormal bases of V and W.

Proof Let (v_i') and (w_j') be respectively orthonormal bases of V and W. We denote

$$V_0' = \langle \{ v_1', v_2', \dots \} \rangle$$
$$W_0' = \langle \{ w_1', w_2', \dots \} \rangle$$

By Point ④, Part $\boxed{\text{II}}$, $\{ v_i \otimes w_j \,/\, i,j \in \mathbb{N} \}$ is a basis of $V_0 \otimes W_0$, and $\{ v_i' \otimes w_j' \,/\, i,j \in \mathbb{N} \}$ is a basis of $V_0' \otimes W_0'$. We can introduce a linear isomorphism between $V_0 \otimes W_0$ and $V_0' \otimes W_0'$

$$\psi : V_0 \otimes W_0 \longrightarrow V_0' \otimes W_0'$$
$$\sum \alpha_{ij} v_i \otimes w_j \longmapsto \sum \alpha_{ij} v_i' \otimes w_j'$$

Thus, $V_0' \otimes W_0'$ is simply a tensor product of V_0 and W_0.

② $V_0 \otimes W_0$ can be equiped with the following inner product

$$\left\langle \sum_{ij} \alpha_{ij} v_i \otimes w_j , \sum_{kl} \beta_{kl} v_k \otimes w_l \right\rangle = \sum_{ij} \alpha_{ij} \overline{\beta_{ij}}$$

Proof Because (α_{ij}) and (β_{kl}) vanish at all but finitely many entries, the map $\langle \cdot , \cdot \rangle$ is well-defined. Moreover, by its definition,

$\langle \cdot, \cdot \rangle$ is linear in the first argument. We have

$$\left\langle \sum_{kl} \beta_{kl} v_k \otimes w_l, \sum_{ij} \alpha_{ij} v_i \otimes w_j \right\rangle = \sum \beta_{kl} \overline{\alpha_{ij}} = \overline{\sum \alpha_{ij} \overline{\beta_{kl}}}$$

$$= \overline{\left\langle \sum_{ij} \alpha_{ij} v_i \otimes w_j, \sum_{kl} \beta_{kl} v_k \otimes w_l \right\rangle}$$

Thus, $\langle \cdot, \cdot \rangle$ is conjugate symmetric. We have

$$\left\langle \sum \alpha_{ij} v_i \otimes w_j, \sum \alpha_{ij} v_i \otimes w_j \right\rangle = \sum \alpha_{ij} \overline{\alpha_{ij}} = \sum |\alpha_{ij}|^2 \geq 0$$

The equality holds if and only if $\alpha_{ij} = 0 \ \forall ij$, i.e. $\sum \alpha_{ij} v_i \otimes w_j = 0$

Thus, $\langle \cdot, \cdot \rangle$ is an inner product and induces a norm of $V_0 \otimes W_0$

$$\left\| \sum \alpha_{ij} v_i \otimes w_j \right\| = \left(\sum |\alpha_{ij}|^2 \right)^{1/2} = \| (\alpha_{ij}) \|_{\ell^2(\mathbb{N} \times \mathbb{N})}$$

③ By the previous point, $(V_0 \otimes W_0, \| \cdot \|)$ is a norm space.

Definition: ~~The~~ A completion of $(V_0 \otimes W_0, \| \cdot \|)$ is called Hilbert-space tensor product of V and W, and denoted $V \hat{\otimes} W$. ∎

By this definition, Hilbert-space tensor product of V and W is unique up to a linear isometric isomorphism.

④ In this point, we'll construct a specific Hilbert-space tensor product of two separable Hilbert spaces V and W.

Put $G = \mathbb{C}^{V\times W}$ - the set a all maps from $V\times W$ to $\mathbb{C}$.

We define the following subsets of G:

$$R_0 = \left\langle \left\{ \mathbb{1}_{(af_1+bf_2,\, cg_1+dg_2)} - ac\mathbb{1}_{(f_1,g_1)} - ad\mathbb{1}_{(f_1,g_2)} - bc\mathbb{1}_{(f_2,g_1)} - bd\mathbb{1}_{(f_2,g_2)} \,/\, a,b,c,d \in \mathbb{C};\ f_1,f_2 \in V_0;\ g_1,g_2 \in W_0 \right\} \right\rangle$$

$$R = \left\langle \left\{ \mathbb{1}_{(\sum \alpha_i v_i, \sum \beta_j w_j)} - \sum_{ij} \alpha_i \beta_j \mathbb{1}_{(v_i, w_j)} \,/\, (\alpha_i), (\beta_j) \in \ell^2(\mathbb{N}) \right\} \right\rangle$$

Remember that an element $v \in V$ corresponds one to one to a sequence $(\alpha_i) \in \ell^2(\mathbb{N})$ by the relation

$$v = \sum_{i=1}^{\infty} \alpha_i v_i$$

there is one thing worth noticing in the definition of R. The map

$$f = \sum_{ij} \alpha_i \beta_j \mathbb{1}_{(v_i, w_j)}$$

is simply a map from $V\times W$ to $\mathbb{C}$ such that

$$f(x) = \begin{cases} \alpha_i \beta_j & \text{if } x = (v_i, w_j) \\ 0 & \text{otherwise} \end{cases}$$

By definition, R_0 and R are vector spaces and $R_0 \subset R$. Then we have two equivalence relations on G

$$u \overset{R_0}{\sim} u' \text{ if and only if } u - u' \in R_0.$$

$$u \overset{R}{\sim} u' \text{ if and only if } u - u' \in R.$$

As we know, a special ~~subset~~ of the R_0-equivalence classes is the algebraic tensor product of V_0 and W_0

$$V_0 \otimes W_0 = \tilde{F}_0 = \left\{ \sum_{ij} \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R_0 \;/\; (\gamma_{ij}) \text{ vanishes a all but finitely many entries} \right\}$$

We define a special set of the R-equivalence classes

$$\tilde{F} = \left\{ \sum_{ij} \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R \;/\; (\gamma_{ij}) \in \ell^2(\mathbb{N} \times \mathbb{N}) \right\}$$

Then $\tilde{F} = V \hat{\otimes} W$.

Proof Put $\varphi : \tilde{F}_0 \longrightarrow \tilde{F}$

$$\sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R_0 \longmapsto \sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R$$

Then φ is well-defined and linear. To show that φ is injective, we only need to show that $\ker \varphi = 0$. Suppose that

$$\sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R = 0,$$

i.e.

$$\sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} \in R$$

thus,

$$\sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} = \sum_{k=1}^{N} c_k \left[\mathbb{1}_{(\sum \alpha_i^k v_i, \sum \beta_j^k w_j)} - \sum_{ij} \alpha_i^k \beta_j^k \mathbb{1}_{(v_i, w_j)} \right] \qquad (**)$$

$$= \sum_{k=1}^{N} c_k \mathbb{1}_{(\sum \alpha_i^k v_i, \sum \beta_j^k w_j)} - \sum_{ij} \tilde{\alpha}_{ij} \mathbb{1}_{(v_i, w_j)}$$

where $\tilde{\alpha}_{ij} = \sum_{k=1}^{N} \alpha_i^k \beta_j^k$

We can assume that $c_k \neq 0$ and $\mathbb{1}_{(\sum \alpha_i^k v_i, \sum \beta_j^k w_j)}$'s are distinct.

Then $\left(\sum \alpha_i^k v_i, \sum \beta_j^k w_j\right) = (v_{i_k}, w_{j_k}) \quad \forall k = 1, \dots, N$ and

therefore the sum ~~with re~~ over i and j on the right hand side of $(**)$ must be finite. Thus,

$$\sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} \in R_0,$$

i.e.

$$\sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R_0 = 0.$$

Hence, ψ is linear and injective. Then $\psi(\tilde{F}_0)$ is ~~a~~ linear isomorphic to $\tilde{F}_0$ and thus a tensor product of V_0 and W_0.

$$\psi(\tilde{F}_0) = V_0 \otimes W_0$$

(Here the corresponding bilinear map is implicitly understood). Since $\tilde{F}_0$ is endowed with the inner product mentioned in Point ②, we can define an inner product on $\psi(\tilde{F}_0)$ as follow

$$\langle \psi(x), \psi(y) \rangle_{\psi(\tilde{F}_0)} := \langle x, y \rangle_{\tilde{F}_0}$$

Then this inner product induces a norm on $\Psi(\tilde{F}_0)$, and $(\Psi(\tilde{F}_0), \|.\|)$ is linearly isometrically isomorphic to $(\tilde{F}_0, \|.\|)$. Thus, the completion space of $(\Psi(\tilde{F}_0), \|.\|)$ is a Hilbert-space tensor product of V and W. The problem now becomes to show that this completion is actually $\tilde{F}$.

We know that each element in $\tilde{F}$ has the form

$$f = \sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R \text{ , where } (\gamma_{ij}) \in \ell^2(\mathbb{N}\times\mathbb{N})$$

First, we'll show that this representation is unique. Suppose that

$$f = \sum \gamma'_{ij} \mathbb{1}_{(v_i, w_j)} + R \quad \text{where } (\gamma'_{ij}) \in \ell^2(\mathbb{N}\times\mathbb{N}).$$

Put $\alpha_{ij} = \gamma_{ij} - \gamma'_{ij}$. Then $(\alpha_{ij}) \in \ell^2(\mathbb{N}\times\mathbb{N})$ and

$$\sum \alpha_{ij} \mathbb{1}_{(v_i, w_j)} \in R$$

Thus,

$$\sum \alpha_{ij} \mathbb{1}_{(v_i, w_j)} = \sum_{k=1}^{N} c_k \left[\mathbb{1}_{(\sum \alpha_i^k v_i, \sum \beta_j^k w_j)} - \sum_{ij} \alpha_i^k \beta_j^k \mathbb{1}_{(v_i, w_j)}\right] \quad (***)$$

$$= \sum_{k=1}^{N} c_k \mathbb{1}_{(\sum \alpha_i^k v_i, \sum \beta_j^k w_j)} - \sum_{ij} \bar{\alpha}_{ij} \mathbb{1}_{(v_i, w_j)}$$

where $\bar{\alpha}_{ij} = \sum_{k=1}^{N} \alpha_i^k \beta_j^k$

Then $\left(\sum \alpha_i^k v_i, \sum \beta_j^k w_j\right) = (v_{i_k}, w_{j_k})$, and the sum over ij at $(***)$ is actually a finite sum.

Thus, $\sum \alpha_{ij} \mathbb{1}_{(v_i, w_j)} \in R_0$. Hence $\alpha_{ij} = 0$, $\forall i,j \in \mathbb{N}$ ∎

We introduce an inner product on $\tilde{F}$:

$$\left\langle \sum \alpha_{ij} \mathbb{1}_{(v_i, w_j)} + R , \sum \beta_{k\ell} \mathbb{1}_{(v_k, w_\ell)} + R \right\rangle_{\tilde{F}} = \sum \alpha_{ij} \overline{\beta_{ij}}$$

$$\forall (\alpha_{ij}), (\beta_{k\ell}) \in \ell^2(\mathbb{N} \times \mathbb{N})$$

Notice that the sum at the right hand side always converges. It is easy to see that $\langle \cdot, \cdot \rangle_{\tilde{F}}$ is linear in the first argument, conjugate symmetric and positively definite. Thus, $\langle \cdot, \cdot \rangle_{\tilde{F}}$ is indeed an inner product on $\tilde{F}$. In fact, $\langle \cdot, \cdot \rangle_{\tilde{F}}$ is an extension of $\langle \cdot, \cdot \rangle_{\varphi(\tilde{F}_0)}$ onto $\tilde{F}$. It induces a norm on $\tilde{F}$. Define

$$\lambda : \ell^2(\mathbb{N} \times \mathbb{N}) \longrightarrow \tilde{F}$$
$$(\gamma_{ij}) \longmapsto \sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R$$

Then λ is well-defined, surjective, injective, linear and norm-preserving

$$\| \lambda((\gamma_{ij})) \| = \left\| \sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R \right\| = \| (\gamma_{ij}) \|_{\ell^2}$$

Since $\ell^2(\mathbb{N} \times \mathbb{N})$ is complete, $(\tilde{F}, \|.\|)$ is also complete. The only task left is to show that $\varphi(\tilde{F}_0)$ is dense in $\tilde{F}$. Let

$$f = \sum \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R \quad \in \tilde{F}$$

For each $M \in \mathbb{N}$, we define

$$f_m = \sum_{i,j=1}^{m} \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R \quad \in \varphi(\tilde{F}_0)$$

Then

$$f - f_m = \sum_{i,j > m} \gamma_{ij} \mathbb{1}_{(v_i, w_j)} + R$$

$$\|f - f_m\|^2 = \sum_{i,j > m} |\gamma_{ij}|^2$$

Because $\sum_{i,j=1}^{\infty} |\gamma_{ij}|^2$ converges, $\lim_{m \to \infty} \sum_{i,j>m} |\gamma_{ij}|^2 = 0$. Thus $\|f - f_m\| \to 0$, or $f_m \to f$. Therefore, $\varphi(\tilde{F}_0)$ is dense in $(\tilde{F}, \|\cdot\|)$.

In conclusion, $(\tilde{F}, \langle \cdot, \cdot \rangle_{\tilde{F}})$ is the Hilbert-space tensor product of V and W.

(5) With the notation $v \hat{\otimes} w := \mathbb{1}_{(v,w)} + R \quad \forall v \in V, w \in W$, we have $\langle v \hat{\otimes} w, v' \hat{\otimes} w' \rangle_{\tilde{F}} = \langle v, v' \rangle \langle w, w' \rangle$.

<u>Proof</u> We can write

$$v = \sum_{i=1}^{\infty} \alpha_i v_i, \quad w = \sum_{j=1}^{\infty} \beta_j w_j$$

$$v' = \sum_{k=1}^{\infty} \alpha'_k v_k, \quad w' = \sum_{l=1}^{\infty} \beta'_l w_l$$

where $(\alpha_i), (\beta_j), (\alpha'_k), (\beta'_l) \in \ell^2(\mathbb{N} \times \mathbb{N})$. Then

$$v \hat{\otimes} w = \mathbb{1}_{(v,w)} + R = \mathbb{1}_{(\sum \alpha_i v_i, \sum \beta_j w_j)} + R$$

$$= \sum \alpha_i \beta_j \mathbb{1}_{(v_i, w_j)} + R$$

Similarly, $v' \hat{\otimes} w' = \sum \alpha'_k \beta'_\ell \mathbb{1}_{(v_k, w_\ell)} + R$

By definition,

$$\langle v \hat{\otimes} w, v' \hat{\otimes} w' \rangle = \left\langle \sum \alpha_i \beta_j \mathbb{1}_{(v_i, w_j)} + R, \sum \alpha'_k \beta'_\ell \mathbb{1}_{(v_k, w_\ell)} + R \right\rangle$$

$$= \sum_{ij \in \mathbb{N}} \alpha_i \beta_j \overline{\alpha'_i} \overline{\beta'_j} = \sum_{ij} (\alpha_i \overline{\alpha'_i})(\beta_j \overline{\beta'_j})$$

We have

$$\sum_{ij} |(\alpha_i \overline{\alpha'_i})(\beta_j \overline{\beta'_j})| \overset{\text{Fubini}}{=} \underbrace{\left(\sum_i |\alpha_i| |\alpha'_i|\right)}_{<\infty} \underbrace{\left(\sum_j |\beta_j| |\beta'_j|\right)}_{<\infty} < \infty$$

Thus, $\sum_{ij} (\alpha_i \overline{\alpha'_i})(\beta_j \overline{\beta'_j}) = \left(\sum_i \alpha_i \overline{\alpha'_i}\right)\left(\sum_j \beta_j \overline{\beta'_j}\right)$

$$= \left\langle \sum \alpha_i v_i, \sum \alpha'_j v_j \right\rangle_V \left\langle \sum \beta_k w_k, \sum \beta'_\ell w_\ell \right\rangle_W$$

$$= \langle v, v' \rangle \langle w, w' \rangle$$

Therefore, $\langle v \hat{\otimes} w, v' \hat{\otimes} w' \rangle = \langle v, v' \rangle \langle w, w' \rangle$.

$\boxed{\text{IV}}$ Some examples of Hilbert-space tensor product

① $V \hat{\otimes} W = B_2(V, W)$

Proof Let $(v_i)_{i \in \mathbb{N}}$ be an orthonormal basis of V

$(w_j)_{j \in \mathbb{N}}$ be an orthonormal basis of W

As in previous points, we put

$$V_0 = \langle \{v_1, v_2, \dots\} \rangle$$

$$W_0 = \langle \{w_1, w_2, \dots\} \rangle$$

First, we'll show that $V_0 \otimes W_0 = B_{fin}(V_0, W_0)$. For each $i \in \mathbb{N}$, we denote v_i^* the linear map from V to $\mathbb{C}$ such that

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

More explicitly, $\quad v_i^*(v) = \alpha_i$, where $v = \sum_{j=1}^{\infty} \alpha_j v_j$.

Then $v_i^* \in V^*$. We define the map

$$\phi : V_0 \otimes W_0 \longrightarrow B_{fin}(V_0, W_0)$$

$$\phi\left(\sum \alpha_{ij} v_i \otimes w_j\right) = \sum \alpha_{ij} v_i^*(\cdot) w_j$$

Then ϕ is well-define and linear. If $\phi(\sum \alpha_{ij} v_i \otimes w_j) = 0$ then

$\sum \alpha_{ij} v_i^*(\cdot) w_j = 0$, i.e. $\quad \sum \alpha_{ij} v_i^*(v) w_j = 0 \quad \forall v \in V_0$.

For each $k \in \mathbb{N}$, we substitute v by v_k and obtain

$$0 = \sum_{ij} \alpha_{ij} v_i^*(v_k) w_j = \sum_j \alpha_{kj} w_j$$

Then $\alpha_{kj} = 0 \ \forall j$. Thus. $\alpha_{ij} = 0 \ \forall ij$. Hence, ϕ is injective.

For each $f \in B_{fin}(V_0, W_0)$, we have

$$f(v) = \sum_j f_j(v) w_j \qquad \text{(finite sum)}$$

It is easy to see that f_j is linear $\forall j$. Since $v \in V$, it can be expressed as the finite sum $v = \sum \alpha_i v_i = \sum_i v_i^*(v) v_i$.

Then
$$f(v) = \sum_j f_j(v) w_j = \sum_j f_j\left(\sum_i v_i^*(v) v_i\right) w_j$$
$$= \sum_{ij} v_i^*(v) f_j(v_i) w_j$$

Put $\alpha_{ij} = f_j(v_i)$. Then $f(v) = \sum_{ij} \alpha_{ij} v_i^*(v) w_j$, or
$$f = \sum_{ij} \alpha_{ij} v_i^*(\cdot) w_j = \phi\left(\sum \alpha_{ij} v_i \otimes w_j\right)$$

Thus, ϕ is surjective. That means ϕ is a linear isomorphism. Hence, $B_{fin}(V_0, W_0)$ is a tensor product of V_0 and W_0. The inner product on $B_{fin}(V_0, W_0)$ induced by ϕ is
$$\left\langle \sum \alpha_{ij} v_i^*(\cdot) w_j, \sum \beta_{kl} v_k^*(\cdot) w_l \right\rangle = \sum \alpha_{ij} \overline{\beta_{ij}}$$

Let $f = \sum \alpha_{ij} v_i^*(\cdot) w_j$. Then
$$\langle f, f \rangle = \sum \alpha_{ij} \overline{\alpha_{ij}} = \sum |\alpha_{ij}|^2$$

We have
$$f(v_k) = \sum_{ij} \alpha_{ij} v_i^*(v_k) w_j = \sum_{ij} \alpha_{ij} \delta_{ik} w_j = \sum_j \alpha_{kj} w_j$$

Thus $\|f(v_k)\|^2 = \sum_j |\alpha_{kj}|^2$, and $\sum_k \|f(v_k)\|^2 = \sum_{kj} |\alpha_{kj}|^2$.

Hence $\langle f, f \rangle = \sum_k \|f(v_k)\|^2$.

The norm on $B_{fin}(V_0, W_0)$ induced by this inner product is therefore

$$\|f\| = \left(\sum_k |f(v_k)|^2\right)^{1/2},$$

i.e. the Hilbert-Schmidt norm. For each $f \in B_{fin}(V_0, W_0)$ and $x \in V$, there exists a sequence (x_n) in V_0 such that $x_n \to x$. ~~We define $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$.~~ Then (x_n) is a Cauchy sequence in V.

$$|f(x_n) - f(x_m)| = |f(x_n - x_m)| \leq \|f\|\,|x_n - x_m|$$

Thus $\{f(x_n)\}$ is a Cauchy sequence in W. Since W is complete, the sequence converges. Moreover, the limit is independent of the choice of sequence $\{x_n\}$. Thus, we can define

$$\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$$

Then f is a finite-rank operator from V to W_0. Define

$$\psi : B_{fin}(V_0, W_0) \longrightarrow B_2(V, W)$$
$$f \longmapsto \tilde{f}$$

Then ψ is well-defined and linear. If $\tilde{f} = 0$ then $\tilde{f}(v_i) = 0$ $\forall i$ i.e. $f(v_i) = 0$ $\forall i$, i.e. $f = 0$. Thus ψ is injective. Hence,

$G = \psi(B_{fin}(V_0, W_0)) = V_0 \otimes W_0$. We can define an inner product on G

$$\langle \psi(f), \psi(g) \rangle_G = \langle f, g \rangle_{B_{fin}(V_0, W_0)}.$$

This product is ~~my~~ in fact the restriction of Hilbert-Schmidt inner product on G

$$\langle \tilde{f}, \tilde{g} \rangle_{B_2(V,W)} = \sum_k \langle \tilde{f}(v_k), \tilde{g}(v_k) \rangle_W$$

To show that $V \hat{\otimes} W = B_2(V,W)$, we have to show that $B_2(V,W)$ is the completion of $(G, \|.\|_{HS})$. By Point ⑦, Part $\boxed{I}$, $B_2(V,W)$ is a Banach space. The task left is to show that $(G, \|.\|_{HS})$ is dense in $B_2(V,W)$. For each $T \in B_2(V,W)$,

$$Tv = \sum_{n=1}^{\infty} \langle Tv, w_n \rangle w_n$$

For each $m \in \mathbb{N}$, we define

$$T_m v = \sum_{n=1}^{m} \langle Tv, w_n \rangle w_n$$

Then $T_m \in \varphi(B_{fin}(V_0, W_0)) = G$ and

$$\|T - T_m\|_{HS}^2 = \sum_{k=1}^{\infty} |Tv_k - T_m v_k|^2 = \sum_{k=1}^{\infty} \Big| \sum_{n=m+1}^{\infty} \langle Tv_k, w_n \rangle w_n \Big|^2$$

$$= \sum_{k=1}^{\infty} \sum_{n=m+1}^{\infty} |\langle Tv_k, w_n \rangle|^2$$

$$= \sum_{n=m+1}^{\infty} \sum_{k=1}^{\infty} |\langle v_k, T^* w_n \rangle|^2 = \sum_{n=m+1}^{\infty} |T^* w_n|^2$$

Since $\sum_{n=1}^{\infty} |T^* w_n|^2 = \sum_{n=1}^{\infty} |T v_n|^2 = \|T\|_{HS}^2 < \infty$,

$\lim_{m\to\infty} \sum_{n=m+1}^{\infty} |T^* w_n|^2 = 0$. Thus $\|T - T_m\|_{HS} \to 0$, and

$T_m \to T$. Therefore, $(G, \|.\|_{HS})$ is dense in $B_2(V,W)$ and

$$B_2(V,W) = V \hat{\otimes} W.$$

② $L^2(X,\mu) \hat{\otimes} L^2(Y,\nu) = L^2(X\times Y, \mu\times\nu)$

Proof Let (f_i) be an orthonormal basis of $L^2(X,\mu)$,

(g_j) ——— $L^2(Y,\nu)$. $(i\in\mathbb{N}, j\in\mathbb{N})$

Put $\varphi : L^2(X,\mu) \otimes L^2(Y) \to L^2(X\times Y, \mu\times\nu)$ be a linear map such that $\varphi(f_i \otimes g_j) = h_{ij}$ where $h_{ij}(x,y) = f_i(x) g_j(y)$. To make sure that φ is well-defined, we show that $h_{ij} \in L^2(X\times Y)$. Let $\mathcal{S}\times\mathcal{T}$ be the product σ-algebra on $X\times Y$. Then f_i and g_j are also $\mathcal{S}\times\mathcal{T}$-measurable. Thus h_{ij} is $\mathcal{S}\times\mathcal{T}$-measurable. Moreover,

$$\int_{X\times Y} |h_{ij}|^2 dx dy \overset{\text{Fubini}}{=} \int_X \int_Y |h_{ij}|^2 dy dx = \int_X |f_i(x)|^2 dx \int_Y |g_j(y)|^2 dy$$

$$= \|f_i\|_{L^2(X)}^2 \|g_j\|_{L^2(Y)}^2 = 1 < \infty$$

Thus, $h_{ij} \in L^2(X\times Y)$, and φ is well-defined.

Next, we show that Ψ is injective. Suppose that $\alpha_{ij} \in \mathbb{C}$ and

$$\sum \alpha_{ij} h_{ij} = 0$$

then $$\sum \alpha_{ij} f_i(x) g_j(y) = 0 \quad \text{for a.e. } (x,y) \in X \times Y$$

Then

$$0 = \sum_{ij} \alpha_{ij} f_i(x) g_j(y) \overline{\sum_{kl} \alpha_{kl} f_k(x) g_l(y)}$$

$$= \sum_{ijkl} \alpha_{ij} \overline{\alpha_{kl}} f_i(x) \overline{f_k(x)} g_j(y) \overline{g_l(y)}$$

Then

$$0 = \int_{X \times Y} \sum_{ijkl} \alpha_{ij} \overline{\alpha_{kl}} f_i(x) \overline{f_k(x)} g_j(y) \overline{g_l(y)} \, dx\, dy$$

$$= \sum_{ijkl} \alpha_{ij} \overline{\alpha_{kl}} \int_{X \times Y} f_i(x) \overline{f_k(x)} g_j(y) \overline{g_l(y)} \, dx\, dy$$

$$= \sum_{ijkl} \alpha_{ij} \overline{\alpha_{kl}} \int_X f_i(x) \overline{f_k(x)} \, dx \int_Y g_j(y) \overline{g_l(y)} \, dy$$

$$= \sum_{ijkl} \alpha_{ij} \overline{\alpha_{kl}} \langle f_i, f_k \rangle_{L^2(X)} \langle g_j, g_l \rangle_{L^2(Y)}$$

$$= \sum_{ij} \alpha_{ij} \overline{\alpha_{ij}} = \sum_{ij} |\alpha_{ij}|^2$$

Thus, $\alpha_{ij} = 0 \ \forall i,j$, and Ψ is injective. Thus, $\Psi(L^2(X) \times L^2(Y))$ is linearly isomorphic to $L^2(X) \otimes L^2(Y)$. Hence $\operatorname{Im} \Psi = L^2(X) \otimes L^2(Y)$.

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The inner product on $L^2(X) \otimes L^2(Y)$ is

$$\left\langle \sum \alpha_{ij} f_i \otimes g_j , \sum \beta_{kl} f_k \otimes g_l \right\rangle = \sum \alpha_{ij} \overline{\beta_{ij}}$$

The inner product on $G = \Psi(L^2(X) \otimes L^2(Y))$ induced by Ψ is

$$\left\langle \sum \alpha_{ij} h_{ij} , \sum \beta_{kl} h_{kl} \right\rangle_G = \sum \alpha_{ij} \overline{\beta_{ij}}$$

This is simply the restriction of the inner product on $L^2(X \times Y)$ onto G.

Indeed,

$$\left\langle \sum \alpha_{ij} h_{ij} , \sum \beta_{kl} h_{kl} \right\rangle_{L^2(X\times Y)} = \sum \alpha_{ij} \overline{\beta_{kl}} \int_{X\times Y} h_{ij} \overline{h_{kl}} \, dx dy$$

$$= \sum \alpha_{ij} \overline{\beta_{kl}} \int_{X\times Y} f_i(x) g_j(y) \overline{f_k(x)} \, \overline{g_l(y)} \, dx dy$$

$$= \sum \alpha_{ij} \overline{\beta_{kl}} \int_X f_i(x) \overline{f_k(x)} \, dx \int_Y g_j(y) \overline{g_l(y)} \, dy$$

$$= \sum \alpha_{ij} \overline{\beta_{kl}} \langle f_i, f_k \rangle_{L^2(X)} \langle g_j, g_l \rangle_{L^2(Y)}$$

$$= \sum \alpha_{ij} \overline{\beta_{ij}}$$

$$= \left\langle \sum \alpha_{ij} h_{ij} , \sum \beta_{kl} h_{kl} \right\rangle_G$$

We know that $\left(L^2(X\times Y), \|\cdot\|_{L^2(X\times Y)}\right)$ is a complete space. Thus, to show that $L^2(X\times Y) = L^2(X) \hat{\otimes} L^2(Y)$, we only need to show that $(G, \|\cdot\|)$ is dense in $\left(L^2(X\times Y), \|\cdot\|\right)$.

Each function $h \in L^2(X\times Y)$ can be written as

$$h = h^+ - h^-$$

where $h^+ = \max\{0, h\}$, $h^- = \max\{0, -h\}$ and $h^+, h^- \in L^2(X\times Y)$.

There exist sequences of simple functions (s_n), (t_n) in $L^2(X\times Y)$ such that $s_n \uparrow h^+$ and $t_n \uparrow h^-$. Thus $u_n = s_n - t_n \in L^2(X\times Y)$ is a simple function and

$$\|h - u_n\|_{L^2} = \|(h^+ - h^-) - (s_n - t_n)\|_{L^2} \le \underbrace{\|h^+ - s_n\|_{L^2}}_{\to 0} + \underbrace{\|h^- - t_n\|_{L^2}}_{\to 0}$$

Hence $u_n \to h$ in $L^2(X\times Y)$.

That means the set of simple functions in $L^2(X\times Y)$ is dense in $L^2(X\times Y)$. Thus we only need to show that G is dense in this set. Moreover, each simple function in $L^2(X\times Y)$ is a linear combination of characteristic function in $L^2(X\times Y)$. Hence ~~we~~ the task is left as follow.

[Let A be $\mathcal{S}\times\mathcal{T}$-measurable and $(\mu\times\nu)(A) < \infty$. Find a sequence in G that converges to χ_A.]

We have 3 following lemmas that will be proved in the end.

$\boxed{\text{Lemma 1: Let } D = A_1\times B_1 \text{ be a measurable rectangle in } X\times Y. \text{ Then } \chi_D \text{ is the limit of a sequence in } G \text{ if } \mu(A), \nu(B) < \infty.}$

We called $Q \subset X \times Y$ an elementary set if $Q = R_1 \cup \dots \cup R_n$ where each R_i is a measurable rectangle and $R_i \cap R_j = \emptyset$ for $i \neq j$. The class of all elementary sets is denoted by $\mathcal{E}$.

Lemma 2: If $P, Q \in \mathcal{E}$ then $P \cap Q, P \cup Q, P \setminus Q \in \mathcal{E}$.

Lemma 3: Let $Q \in \mathcal{E}$ (with $(\mu \times \nu)(Q) < \infty$). Then χ_Q is a limit of a sequence in G.

By Lemma 3, we only have to show that the set $\{\chi_Q : Q \in \mathcal{E}\}$ is dense in $\{\chi_A : A \in \mathcal{S} \times \mathcal{T}\}$. We have

$$\| \chi_Q - \chi_A \|_{L^2}^2 = \int_{X \times Y} |\chi_Q - \chi_A|^2 = (\mu \times \nu)(Q \Delta A)$$

where $Q \Delta A = (Q \setminus A) \cup (A \setminus Q)$ is the symmetric difference of Q and A. For simplicity, we put $\lambda = \mu \times \nu$. Put

$$\mathcal{N} = \{ A \in \mathcal{S} \times \mathcal{T} : \ \forall \varepsilon > 0, \exists Q_\varepsilon \in \mathcal{E} \text{ such that } \lambda(Q_\varepsilon \Delta A) < \varepsilon \}$$

The task left is to show that $\mathcal{N} = \mathcal{S} \times \mathcal{T}$. Because $\mathcal{E} \subset \mathcal{N} \subset \mathcal{S} \times \mathcal{T}$, and $\mathcal{S} \times \mathcal{T}$ is the smallest σ-algebra on $X \times Y$ containing all elementary sets, we only need to show that $\mathcal{N}$ is a σ-algebra.

We have $\emptyset, X \times Y \in \mathcal{E} \subset \mathcal{N}$.

Let $A \in \mathcal{N}$. We show that $X \times Y \setminus A \in \mathcal{N}$. For each $\varepsilon > 0$, there exists $Q_\varepsilon \in \mathcal{E}$ such that $\lambda(Q_\varepsilon \triangle A) < \varepsilon$. By Lemma 2,

$Q'_\varepsilon = (X \times Y) \setminus Q_\varepsilon \in \mathcal{E}$. Put $A' = X \times Y \setminus A$.

$$Q'_\varepsilon \triangle A' = (Q'_\varepsilon \setminus A') \cup (A' \setminus Q'_\varepsilon) = (A \setminus Q_\varepsilon) \cup (Q_\varepsilon \setminus A)$$
$$= Q_\varepsilon \triangle A$$

Thus $\lambda(Q'_\varepsilon \triangle A') = \lambda(Q_\varepsilon \triangle A) < \varepsilon$. Hence $A' \in \mathcal{N}$.

Let (A_n) be a sequence in $\mathcal{N}$ and $A = \bigcup_{n=1}^{\infty} A_n$. We'll show that $A \in \mathcal{N}$. For each $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $Q_n^\varepsilon \in \mathcal{E}$ such that $\lambda(Q_n^\varepsilon \triangle A_n) < \frac{\varepsilon}{4} \frac{1}{2^n}$.

For each $m \in \mathbb{N}$, we put $P_m^\varepsilon = \bigcup_{n=1}^{m} Q_n^\varepsilon$. By Lemma 2, $P_m^\varepsilon \in \mathcal{E}$.

We have

$$\lambda\left(A \setminus \bigcup_{m=1}^{\infty} P_m^\varepsilon\right) = \lambda\left[\bigcup_{n=1}^{\infty}\left(A_n \setminus \bigcup_{m=1}^{\infty} P_m^\varepsilon\right)\right]$$
$$\leq \sum_{n=1}^{\infty} \lambda\left(A_n \setminus \bigcup_{m=1}^{\infty} P_m^\varepsilon\right) \leq \sum_{n=1}^{\infty} \lambda(A_n \setminus Q_n^\varepsilon)$$
$$\leq \sum_{n=1}^{\infty} \lambda(Q_n^\varepsilon \triangle A_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{4} \frac{1}{2^n} = \frac{\varepsilon}{4}$$

Since $\lambda\left(A \setminus \bigcup_{m=1}^{\infty} P_m^\varepsilon\right) = \lim_{m \to \infty} \lambda(A \setminus P_m^\varepsilon)$, we get

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$$\cancel{\lambda(A \setminus \bigcup_{m=1}^{\infty} P_m^\varepsilon} \quad \liminf_{m\to\infty} \lambda(A \setminus P_m^\varepsilon) \le \frac{\varepsilon}{4}. \qquad (1)$$

Moreover,

$$\lambda(P_m^\varepsilon \setminus A) = \lambda\Big(\bigcup_{n=1}^{m} Q_n^\varepsilon \setminus A\Big) \le \sum_{n=1}^{m} \lambda(Q_n^\varepsilon \setminus A) \le \sum_{n=1}^{m} \lambda(Q_n^\varepsilon \setminus A_n)$$

$$\le \sum_{n=1}^{\infty} \lambda(Q_n^\varepsilon \Delta A_n) \le \sum_{n=1}^{\infty} \frac{\varepsilon}{4}\frac{1}{2^n} = \frac{\varepsilon}{4}$$

Together with (1), we have

$$\liminf_{m\to\infty} \lambda(A \Delta P_m^\varepsilon) = \liminf \big[\lambda(A \setminus P_m^\varepsilon) + \lambda(P_m^\varepsilon \setminus A)\big] \le \frac{\varepsilon}{2}$$

Thus, there exists $m_0 \in \mathbb{N}$ such that $\lambda(A \Delta P_{m_0}^\varepsilon) < \varepsilon$. Therefore, $A \in \mathcal{N}$. That completes the proof. ∎

* Proof of Lemma 1

Let $A \in \mathcal{S}$ and $B \in \mathcal{T}$ such that $\mu(A), \nu(B) < \infty$. Then $\chi_A \in L^2(X)$ and $\chi_B \in L^2(Y)$. For each $\varepsilon > 0$, there exist a finite sum $\sum \alpha_i f_i \in L^2(X)$ such that

$$\Big\| \sum \alpha_i f_i - \chi_A \Big\|_{L^2(X)} < \varepsilon$$

and a finite sum $\sum \beta_j g_j \in L^2(Y)$ such that

$$\Big\| \sum \beta_j g_j - \chi_B \Big\|_{L^2(Y)} < \varepsilon$$

Put $h \in L^2(X\times Y)$ given by

$$h(x,y) = \Big(\sum \alpha_i f_i(x)\Big)\Big(\sum \beta_j g_j(y)\Big)$$

~~We~~

$$= \sum_{ij} \alpha_i \beta_j f_i(x) g_j(y)$$

then $h \in G$. We have

$$\|h - \chi_{A\times B}\|_{L^2(X\times Y)} = \Big\| \Big(\sum \alpha_i f_i\Big)\Big(\sum \beta_j g_j\Big) - \chi_A \chi_B \Big\|_{L^2(X\times Y)}$$

Put $f = \sum \alpha_i f_i$ and $g = \sum \beta_j g_j$. We have

$$\|h - \chi_{A\times B}\|_{L^2(X\times Y)} = \|f(x)g(y) - \chi_A(x)\chi_B(y)\|_{L^2(X\times Y)}$$

$$= \|(f(x) - \chi_A(x))g(y) + \chi_A(x)(g(y) - \chi_B(y))\|_{L^2(X\times Y)}$$

$$\leq \|(f(x) - \chi_A(x)) g(y)\|_{L^2(X\times Y)} + \|\chi_A(x)(g(y) - \chi_B(y))\|_{L^2(X\times Y)}$$

$$= \|f - \chi_A\|_{L^2(X)} \|g\|_{L^2(Y)} + \|\chi_A\|_{L^2(X)} \|g - \chi_B\|_{L^2(Y)}$$

$$\leq \varepsilon \|g\|_{L^2(Y)} + \mu(A)^{1/2}\varepsilon$$

$$\leq \varepsilon\big(\|\chi_B\|_{L^2(Y)} + \varepsilon\big) + \mu(A)^{1/2}\varepsilon$$

$$= \varepsilon\big(\nu(B) + \varepsilon\big) + \mu(A)^{1/2}\varepsilon$$

Thus $\chi_{A\times B}$ is a limit of a sequence in G.

* <u>Proof of Lemma 2</u>

First, we see that

$$(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D),$$

i.e. the intersection of two measurable rectangles is also a measurable rectangle. Let $P, Q \in \mathcal{E}$

$$P = \cup R_i \ , \quad Q = \cup R_j'$$

Then $P\cap Q = (\cup R_i)\cap(\cup R_j') = \cup \underbrace{(R_i \cap R_j')}_{\text{measurable rectangle}} \quad \in \mathcal{E}$

Consequently, every finite intersection of elements in $\mathcal{E}$ belongs to $\mathcal{E}$.

Let $A\times B \in \mathcal{E}$ be a measurable rectangle. Then

$$(X\times Y)\setminus(A\times B) = \underbrace{[A\times(Y\setminus B)]}_{R_1'} \cup \underbrace{[(X\setminus A)\times(Y\setminus B)]}_{R_2'} \cup \underbrace{[(X\setminus A)\times B]}_{R_3'}$$

R_1', R_2', R_3' are measurable rectangles and pairwise disjoint. Thus, $(X\times Y)\setminus(A\times B)\in\mathcal{E}$

For each $P = \cup R_i \in \mathcal{E}$, we have

$$(X\times Y)\setminus P = (X\times Y)\setminus \cup R_i = \cap \underbrace{[(X\times Y)\setminus R_i]}_{\in \mathcal{E}}$$

This is a finite intersection of elements in $\mathcal{E}$. Thus, $(X\times Y)\setminus P \in \mathcal{E}$.

For each $P, Q \in \mathcal{E}$, we have

$$P\setminus Q = P \cap \underbrace{[(X\times Y)\setminus Q]}_{\in\mathcal{E}} \quad \in \mathcal{E}$$

To show that $P \cup Q \in \mathcal{E}$, we only need to show that $(X \times Y) \setminus (P \cup Q) \in \mathcal{E}$. We have

$$(X \times Y) \setminus (P \cup Q) = [\underbrace{(X \times Y) \setminus P}_{\in \mathcal{E}}] \cap [\underbrace{(X \times Y) \setminus Q}_{\in \mathcal{E}}] \quad \in \mathcal{E}$$

<u>Proof of Lemma 3</u>

Let $Q \in \mathcal{E}$ and $\lambda(Q) < \infty$. We can write $Q = \bigcup R_i$ where R_i is a measurable rectangle and $R_i \cap R_j = \phi$ for $i \neq j$. Thus,

$$\lambda(Q) = \sum_{i=1}^{n} \lambda(R_i),$$

and hence $\lambda(R_i) < \infty$ $\forall i$. By Lemma 1, for each $\varepsilon > 0$, there exists $f_i^\varepsilon \in G$ such that $\|f_i^\varepsilon - \chi_{R_i}\|_{L^2(X \times Y)} < \frac{\varepsilon}{n}$. Put

$f^\varepsilon = \sum_i f_i^\varepsilon \in G$. Then

$$\|f^\varepsilon - \chi_Q\|_{L^2(X \times Y)} = \|\sum_{i=1}^{n} f_i^\varepsilon - \sum_{i=1}^{n} \chi_{R_i}\| \leq \sum_{i=1}^{n} \|f_i^\varepsilon - \chi_{R_i}\| < \sum_{i=1}^{n} \frac{\varepsilon}{n}$$

Thus

$$\|f^\varepsilon - \chi_Q\|_{L^2(X \times Y)} < \varepsilon$$

Therefore, χ_Q is a limit of a sequence in $(G, \|\cdot\|_{L^2})$.