Part 1: Lie Groups, Algebraic Groups and Lie Algebras

This is an exposition for Chapter 1 of the book Goodman-Wallach "Symmetry, Representations and Invariants". Our goal is to introduce some basic notions about Lie groups, algebraic groups and Lie algebras. Everything will be described in terms of matrices because we choose the matrix realization for Lie groups and Lie algebras. Although this is not the most general method to study the subject, it is perhaps sufficient for many applications. In this write-up, we will discuss:

- Bilinear forms on a finite-dimensional vector space (including their matrix representations, isometry groups and associated Lie algebras).
- Two types of bilinear forms (symmetric and skew symmetric); the classical groups.
- Lie groups (including topological groups, morphisms among them, the exponential and logarithm maps).
- Differential of a topological-group morphism; vector fields on a Lie group.
- Algebraic groups and regular functions.
- Representations, regular representations and their differentials.
- Examples of representations.
- Jordan decomposition.

1 Bilinear forms on a finite-dimensional vector space

Let V be an n-dimensional vector space over $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathrm{End}(V)$ the space of all linear maps from V to itself. Denote by $GL(V)$ the subset of $\mathrm{End}(V)$ consisting of all invertible maps. It is a group under map composition.

A bilinear map $B: V \times V \to \mathbb{F}$ is called a bilinear form. Let $(v_1, \dots, v_n)$ be a basis of V and let $\Gamma \in M_n(\mathbb{F})$ be a matrix with $\Gamma_{ij} = B(v_i, v_j)$. Then Γ is called the representation matrix of B with respect to basis $(v_1, \dots, v_n)$. With the chosen basis, we can identify $GL(V)$ with $GL(n, \mathbb{F})$, and $\mathrm{End}(V)$ with $M_n(\mathbb{F})$ in the usual sense.

The isometry group of B is defined as

$$\mathcal{O}(B) = \{ g \in GL(V) : B(gv, gw) = B(v, w) \ \forall v, w \in V \}. \tag{1}$$

It is a subgroup of $GL(V)$. In terms of matrix,

$$\mathcal{O}(B) = \{ A \in GL(n, \mathbb{F}) : A^T \Gamma A = \Gamma \}. \tag{2}$$

The bilinear form B is called nondegenerate if

$$B(v, w) = 0 \ \forall w \in V \Rightarrow v = 0,$$

$$B(v, w) = 0 \ \forall v \in V \Rightarrow w = 0.$$

In terms of matrix, B is nondegenerate if and only if $\det(\Gamma) \neq 0$.

The Lie algebra associated with B is defined as

$$so(B) = \{ f \in End(V) : B(fv, w) + B(v, fw) = 0 \quad \forall v, w \in V \}. \qquad (3)$$

In terms of matrix,

$$so(B) = \{ A \in M_n(\mathbb{F}) : A^T \Gamma + \Gamma A = 0 \}. \qquad (4)$$

Let us recall the general definition of Lie algebras:

A vector space $\mathfrak{g}$ over $\mathbb{F}$ together with a bilinear map $(X, Y) \in \mathfrak{g} \times \mathfrak{g} \mapsto [X, Y] \in \mathfrak{g}$ is said to be a Lie algebra if the following conditions are satisfied.

Skew symmetry: $[X, Y] = -[Y, X] \quad \forall X, Y \in \mathfrak{g}$.

Jacobi identity: $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \quad \forall X, Y, Z \in \mathfrak{g}$.

We can easily turn an algebra $\mathfrak{g}$ into a Lie algebra by defining the Lie bracket $[X, Y] := XY - YX \quad \forall X, Y \in \mathfrak{g}$.

Thus, $M_n(\mathbb{F})$ and $End(V)$ are Lie algebras. A vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a Lie subalgebra of $\mathfrak{g}$ if it is closed under the Lie bracket, i.e.

$$[X, Y] \in \mathfrak{h} \quad \forall X, Y \in \mathfrak{h}.$$

We can check by direct calculation that $so(B)$ is a Lie subalgebra of $M_n(\mathbb{F})$. This is why $so(B)$ is called the Lie algebra associated with B.

2 Two types of bilinear forms; the classical groups

Two special types of bilinear forms are

Symmetric: $B(v,w) = B(w,v) \quad \forall v,w \in V$.

Skew-symmetric: $B(v,w) = -B(w,v) \quad \forall v,w \in V$.

<u>Theorem 1</u> Let V be an n-dimensional vector space over $\mathbb{C}$ and $B: V \times V \to \mathbb{C}$ be a nondegenerate bilinear form.

(a) If B is symmetric then there exists a basis of V with respect to which the representation matrix of B is the identity matrix I_n.

(b) If B is skew-symmetric then $n = 2m$ and there exists a basis of V with respect to which the representation matrix of B is $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$.

The proof of this theorem is given at the end of this section.

If B is symmetric and nondegenerate, the isometry group $O(B)$ is called the <u>orthogonal group</u>. With respect to the suitable basis mentioned in Theorem 1,

$$O(B) = O(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) : A^T A = I_n\}. \qquad (5)$$

If B is skew-symmetric and nondegenerate, the isometry group $O(B)$ is called the symplectic group. With respect to the suitable basis mentioned in Theorem 1,

$$O(B) = Sp(m,\mathbb{C}) = \{A \in GL(2m,\mathbb{C}) : A^T J A = J\}. \qquad (6)$$

It is interesting to note that the <u>special linear group</u>

$$SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) : \det(A) = 1\}$$

is not the isometry group of any bilinear form. Indeed, suppose otherwise:

there exists $\Gamma \in M_n(\mathbb{C})$ such that $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) : A^T \Gamma A = \Gamma\}$. Take $A = I_n + N$, where N is a strictly upper triangle matrix. Then the equation $A^T\Gamma A = \Gamma$ becomes $N^T\Gamma + \Gamma N + N^T \Gamma N = 0$. Let e_{ij} be the matrix with the entry at position (i,j) equal to 1 while all other entries are 0. Take $N = \alpha e_{ij}$ where $\alpha \in \mathbb{C}$, $1 \le i < j \le n$. Then the above equation becomes

$$\alpha(e_{ji}\Gamma + \Gamma e_{ij}) + \alpha^2(e_{ji}\Gamma e_{ij}) = 0 \qquad \forall \alpha \in \mathbb{C}.$$

This implies $e_{ji}\Gamma + \Gamma e_{ij} = 0$ and $e_{ji}\Gamma e_{ij} = 0$ for all $1 \le i < j \le n$. Thus, Γ is diagonal. The matrix $A = \begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$ satisfies $A^T\Gamma A = \Gamma$ and $\det(A) = -1$. This is a contradiction.

If B is symmetric and nondegenerate, we denote

$$SO(B) := SO(n,\mathbb{C}) := O(n,\mathbb{C}) \cap SL(n,\mathbb{C}).$$

Likewise, if B is skew-symmetric and nondegenerate, we denote

$$Sp(B) := Sp(m,\mathbb{C}) \cap SL(2m,\mathbb{C}).$$

It is not clear at the moment whether $Sp(m,\mathbb{C}) \subset SL(2m,\mathbb{C})$. This is actually true and will be proved in Part 2 by showing that $Sp(n,\mathbb{C})$ is a connected topological group.

The groups $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$ are called classical groups. The term "classical groups", first coined by Hermann Weyl in his book "The Classical Groups, Their Invariants and Representations", refers to a class of subgroups of $GL(n,\mathbb{C})$ which preserve a volume form, a bilinear form, or a sesquilinear form. Those four classical groups, however, are our main consideration in the sequel.

With the suitable basis of V mentioned in Theorem 1, we now can rewrite the Lie algebra associated with B in two cases: symmetric and skew-symmetric.

* B is symmetric and nondegenerate.

$$so(n,\mathbb{C}) := so(B) = \{A \in M_n(\mathbb{C}) : A^T + A = 0\}$$

* B is skew-symmetric and nondegenerate.

$$sp(m,\mathbb{C}) := sp(B) := so(B) = \{A \in M_{2m}(\mathbb{C}) : A^T J + JA = 0\},$$

where $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$.

Proof of Theorem 1

(a) We will show by induction in $n \in \mathbb{N}$ that there exists a basis $(w_1, \dots, w_n)$ of V such that $B(w_i, w_j) = 0$ if $i \neq j$ and $B(w_i, w_i) \neq 0$.

Because B is symmetric,

$$B(v,w) = \frac{B(v,w)+B(w,v)}{2} = \frac{B(v+w,v+w) - B(v-w,v-w)}{4} \quad \forall v,w \in V.$$

Because B is also degenerate, there exists $w_1 \in V$ such that $B(w_1,w_1) \neq 0$. Thus, our claim is true if $n=1$. Consider the case $n>1$. Suppose our claim is true for $n-1$. The space $V' = \{v \in V ; B(w_1,v) = 0\}$ is a linear complement of $\mathbb{C}\{w_1\}$ in V because

$$v = \underbrace{\left(v - \frac{B(w_1,v)}{B(w_1,w_1)} w_1\right)}_{\in V'} + \underbrace{\frac{B(w_1,v)}{B(w_1,w_1)} w_1}_{\in \mathbb{C}\{w_1\}} .$$

$B|_{V' \times V'}$ is a symmetric nondegenerate bilinear form on V'. By the induction hypothesis, there exists a basis $(w_2, \dots, w_n)$ of V' such that $B(w_i,w_j) = 0$ if $i \neq j$, $i,j > 1$ and $B(w_i,w_i) \neq 0$ if $i > 1$. Then the basis $(w_1, w_2, \dots, w_n)$ proves our claim. Now put

$$v_i = \frac{w_i}{\sqrt{B(w_i,w_i)}} .$$

Then $(v_1, v_2, \dots, v_n)$ is a basis of V such that $B(v_i,v_j) = \delta_{ij}$.

(b) Let Γ be the representation matrix of B with respect to some basis of V. Because B is skew-symmetric, so is Γ, i.e. $\Gamma^T = -\Gamma$. Thus,

$$\det(\Gamma) = \det(\Gamma^T) = \det(-\Gamma) = (-1)^n \det(\Gamma).$$

Because B is nondegenerate, $\det(\Gamma) \neq 0$. Thus, n must be even. Write $n = 2m$. We will prove our claim by induction in $m \in \mathbb{N}$.

Since B is nondegenerate, there exist $v, w \in V\setminus\{0\}$ such that $B(v,w)=1$. Since B is skew-symmetric, $B(v,v)= B(w,w)=0$. This implies that v and w are linearly independent. Put $W_1 = \langle\{v,w\}\rangle$ and

$$W_2 = \{x \in V : B(v,x) = B(w,x) = 0\}.$$

We can check that W_2 is a subspace of V and $W_1 \cap W_2 = \{0\}$. Moreover, $V = W_1 \oplus W_2$ because

$$y = \underbrace{(y + B(w,y)v - B(v,y)w)}_{\in W_2} + \underbrace{(-B(w,y)v + B(v,y)w)}_{\in W_1} \quad \forall y \in V.$$

We get $\dim W_1 = 2$ and $\dim W_2 = 2m-2$. If $m=1$ then $V = W_1$; the representation matrix of B with respect to the basis (v,w) is

$$\begin{pmatrix} B(v,v) & B(v,w) \\ B(w,v) & B(w,w) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_1 \\ -I_1 & 0 \end{pmatrix}.$$

Thus, our claim is true for $m=1$. Suppose that it is true for $m-1$. Because $B|_{W_2 \times W_2}$ is a skew-symmetric nondegenerate bilinear form on W_2 and $\dim W_2 = 2(m-1)$, there exists a basis of W_2, namely $(v_2, \ldots, v_m, v_{m+2}, \ldots, v_{2m})$ such that

$$(B(v_i, v_j)) = \begin{pmatrix} 0 & I_{m-1} \\ -I_{m-1} & 0 \end{pmatrix}.$$

Put $v_1 = v$ and $v_{n+1} = w$. We get

$$(B(v_i, v_j))_{1 \le i,j \le 2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

3 Lie groups

The vector space $M_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n^2}$, so it can be endowed with a topology. The set $GL(n,\mathbb{R})$ and all of its subgroups thus have both group structure and topology structure. The intertwining of these two structures yields an interesting method of studying topological properties through algebraic properties. For example, we can show that a group generated by unipotent elements is connected (Part 2).

Whenever we talk about topology on a classical group in $M_n(\mathbb{C})$, we are referring to its image under the injective map

$$\pi : M_n(\mathbb{C}) \to M_{2n}(\mathbb{R}), \quad \pi(A+iB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad \forall A, B \in M_n(\mathbb{R}).$$

π is actually a Lie algebra morphism, i.e. a linear map (over $\mathbb{R}$) which preserves the Lie bracket. It also preserves the group structure when restricted to $GL(n,\mathbb{C})$ because $\pi(XY) = \pi(X)\pi(Y)$ for all $X, Y \in M_n(\mathbb{C})$.

By direct calculation, we have $\pi(M_n(\mathbb{C})) = \{X \in M_{2n}(\mathbb{R}) : XJ = JX\}$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Thus, $M_n(\mathbb{C})$ can be identified with the subspace of $M_{2n}(\mathbb{R})$ consisting of all matrices which commute with J. Recall the general definition of <u>topological group</u>:

A group G together with a Hausdorff topology is called a topological group if the multiplication map $G \times G \to G$, $(x,y) \mapsto xy$ and the inversion map

$G \to G, x \mapsto x^{-1}$ are continuous.

A topological-group morphism $\varphi: G \to H$ between two topological groups G and H is a continuous group morphism. If, in addition, φ has a continuous inverse, it is called a topological-group isomorphism. The inversion, left-translation and right-translation are topological-group isomorphisms. So is the conjugation $\tau(g): G \to G, x \mapsto gxg^{-1}$.

It is interesting to note that open subgroups are always closed. Indeed, if H is an open subgroup of a topological group G, then

$$H = G \setminus \underbrace{\left(\bigcup_{x \in G \setminus H} xH \right)}_{\text{open in } G}$$

which is closed in G. The converse is not true: the trivial subgroup is closed but not open.

If a subgroup of $GL(n, \mathbb{C})$ locally looks like a Euclidean space, we can endow it with a manifold or smooth manifold structure (that would indicate some connection between analytical and algebraic approaches in studying the group). The exponential map helps us do so. Many maps from a subset of $M_n(\mathbb{C})$ to $M_n(\mathbb{C})$ can be defined in the following way:

Starting with a holomorphic map $f: \Gamma \to \mathbb{C}$ where Γ is an open simply connected subset of $\mathbb{C}$, we get a continuous map $\tilde{f}: E_f \to M_n(\mathbb{C})$, where

$$E_f = \{A \in M_n(\mathbb{C}) : \text{all eigenvalues of } A \text{ lie in } \Gamma\}$$

and $\tilde{f}(P^{-1}\operatorname{diag}(\lambda_1,\dots,\lambda_n)P) = P^{-1}\operatorname{diag}(f(\lambda_1),\dots,f(\lambda_n))P$.

The fact that the set of diagonalizable matrices is dense in $M_n(\mathbb{C})$ then gives a unique continuous extension of $\tilde{f}$ on E_f. This is Theorem 6.2.27, p. 426, Horn – Johnson "Topics in Matrix Analysis", 1991.

For $\Gamma = \mathbb{C}$ and $f(z) = e^z$, we obtain the <u>exponential map</u> $\exp = \tilde{f}$: $M_n(\mathbb{C}) \to M_n(\mathbb{C})$,

$$\exp(X) = \sum_{m=0}^{\infty} \frac{X^m}{m!}. \qquad (7)$$

For $\Gamma = \mathbb{D}(1,1)$ and $f(z) = \log(1+z)$, we obtain the logarithm map $\log : B(I_n, 1) \to M_n(\mathbb{C})$,

$$\log(I_n + X) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{X^m}{m}. \qquad (8)$$

For $n \geq 2$, the identity $\exp(X+Y) = \exp(X)\exp(Y)$ is not true. It is true if $XY = YX$. In particular, $\exp(X)\exp(-X) = \exp(0) = I_n$. Thus, $\exp(X) \in GL(n,\mathbb{R})$.

Using the fact that the set of diagonalizable matrices in $M_n(\mathbb{C})$ is dense in $M_n(\mathbb{C})$, we can prove the following identities.

$$\log \circ \exp(A) = A \qquad \forall A \in B(0, \log 2),$$
$$\exp \circ \log(A) = A \qquad \forall A \in B(I_n, 1),$$
$$\det(\exp(A)) = e^{\operatorname{tr}(A)} \quad \forall A \in M_n(\mathbb{C}).$$

Thus, in the ball $B(0, \log 2)$ of $M_n(\mathbb{C})$, $\exp$ is a homeomorphism onto its image. The same is true for the ball $B(0,\log 2)$ of $M_n(\mathbb{R})$. The following theorem describes how a line in $GL(n,\mathbb{R})$ looks like.

<u>Theorem 2</u> Let $\varphi : (\mathbb{R}, +) \to GL(n, \mathbb{R})$ be a topological-group morphism. Then there exists a unique $X \in M_n(\mathbb{R})$ such that $\varphi(t) = \exp(tX)$ for all $t \in \mathbb{R}$.

The proof of this theorem is given at the end of this section.

Now given a closed subgroup G of $GL(n, \mathbb{R})$, we define

$$\mathrm{Lie}(G) = \{X \in M_n(\mathbb{R}) : \exp(tX) \in G \quad \forall t \in \mathbb{R}\}. \tag{9}$$

Using the following identities,

$$\exp(X+Y) = \lim_{k\to\infty} \left(\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\right)^k, \tag{10}$$

$$\exp([X,Y]) = \lim_{k\to\infty} \left(\exp\left(\frac{X}{k}\right)\exp\left(\frac{Y}{k}\right)\exp\left(-\frac{X}{k}\right)\exp\left(-\frac{Y}{k}\right)\right)^{k^2}, \tag{11}$$

we can show that $\mathrm{Lie}(G)$ is a Lie subalgebra of $M_n(\mathbb{R})$. It is called the <u>Lie algebra of G</u>. So far we have seen Lie algebras arise in 3 situations: (1) from the general definition, (2) from a bilinear form, (3) from a closed subgroup of $GL(n, \mathbb{R})$. The connections are:

- The Lie algebra associated with a bilinear form on V (or a closed subgroup) is a Lie subalgebra of $\mathrm{End}(V)$ (or $M_n(\mathbb{R})$).
- The Lie algebra associated with a bilinear form is the same as the Lie algebra of its isometry group if viewed as a closed subgroup of $GL(n, \mathbb{R})$.
- The Lie algebra of a closed subgroup can be obtained by differentiating the defining equation of that group along a one-parameter curve.

We explain the last bullet through the example of isotropy group of a bilinear form B:

$$G = O(B) = \{g \in GL(V) : B(gv, gw) = B(v,w) \ \forall v, w \in V\}.$$

Consider a one-parameter curve of this group $(\sigma(t))_{t\in\mathbb{R}}$, where $\sigma(0) = I_n$.
We have $B(\sigma(t)v, \sigma(t)w) = B(v,w) \ \forall t \in \mathbb{R}$.

Thus,
$$0 = \frac{d}{dt}\Big|_{t=0} B(\sigma(t)v, \sigma(t)w) = B(\sigma'(0)v, \sigma(0)w) + B(\sigma(0)v, \sigma'(0)w)$$
$$= B(\sigma'(0)v, w) + B(v, \sigma'(0)w).$$

Thus, the set of all $\sigma'(0)$ satisfying the above equation is exactly the Lie algebra associated with B. The Lie algebra of a closed subgroup can be ~~vted~~ viewed as the tangent space of that group at the identity element I_n (in fact <u>any</u> element, because the left translation is a topological-group isomorphism). This viewpoint will be reaffirmed in the rest of this section.

The exponential map yields a local chart on a closed subgroup G by which it locally looks like its Lie algebra $\mathrm{Lie}(G)$. The local chart turns G into a manifold whose dimension is the same as its Lie algebra's dimension. The following theorem summarizes what we have discussed.

<u>Theorem 3</u> Let H be a closed subgroup of $G = GL(n,\mathbb{R})$. Denote $\mathfrak{h} = \mathrm{Lie}(H)$. Then there exists an open neighborhood V of 0 in $\mathfrak{h}$ an an open neighborhood Ω of I_n in G such that $\exp : V \to H \cap \Omega$ is a homeomorphism.

The proof of this theorem is given at the end of this section.

In fact, the manifold structure on a closed subgroup G turns out to be a real-analytic structure.

Theorem 4 Let H be a closed subgroup of $G = GL(n, \mathbb{R})$. Then H can be endowed with an (real) ~~any~~ analytic manifold structure which fits with the topology on H. Moreover, the multiplication $H \times H \to H$ and the inversion $H \to H$ are real-analytic maps.

With such an analysis-friendly structure, the algebraic object G is now given a different name:

A topological group G is called a Lie group if

(i) The manifold topology on G is the same as the group topology,

(ii) The multiplication map $G \times G \to G$ and the inversion map $G \to G$ are real analytic.

Proof of Theorem 2

Put $\Gamma' = \exp(B(0, \log 2)) \subset M_n(\mathbb{R})$. By the Invariance of Domain theorem, Γ' is an open neighborhood of I_n in $GL(n, \mathbb{R})$. Moreover, the maps

$$B(0, \log 2) \xrightarrow{\exp} \Gamma', \quad \Gamma' \xrightarrow{\log} B(0, \log 2)$$

are inverses of each other. Because φ is continuous at 0 and $\varphi(0) = I_n$, there is $\varepsilon > 0$ such that $\varphi((-\varepsilon, \varepsilon)) \subset \Gamma'$. Put $\Gamma'' = \exp\left(B\left(0, \frac{\log 2}{2}\right)\right)$. Then Γ''

is also an open neighborhood of I_n in $GL(n,\mathbb{R})$. We could have chosen $\varepsilon>0$ such that $\varphi((-\varepsilon,\varepsilon))\subset\Gamma''$.

Pick any number $N\in\mathbb{N}$ with $1/N<\varepsilon$. Then $\varphi(\frac{1}{N})\in\Gamma''$. Put $X=N\log\varphi(\frac{1}{N})\in M_n(\mathbb{R})$. Then $\frac{1}{N}X=\log\varphi(\frac{1}{N})$. Taking the exponential both sides, we get $\exp(\frac{X}{N})=\varphi(\frac{1}{N})$. We will prove by induction in $k\in\{0,1,2,\ldots\}$ that $\exp\left(\frac{1}{2^kN}X\right)=\varphi\left(\frac{1}{2^kN}\right)$. $\quad(*)$

Suppose that $(*)$ is true for $k\geq 0$. Because $\frac{1}{2^{k+1}N}<\frac{1}{N}<\varepsilon$, $\varphi\left(\frac{1}{2^{k+1}N}\right)\in\Gamma''$. Thus, there exists $Y\in B(0,\frac{\log 2}{2})$ such that $\varphi\left(\frac{1}{2^{k+1}N}\right)=\exp Y$. Then

$$\varphi\left(\frac{1}{2^kN}\right)=\varphi\left(\frac{1}{2^{k+1}N}+\frac{1}{2^{k+1}N}\right)=\varphi\left(\frac{1}{2^{k+1}N}\right)^2=\exp(Y)^2=\exp(2Y).$$

Thus, by the induction hypothesis, $\exp\left(\frac{X}{2^kN}\right)=\varphi\left(\frac{1}{2^kN}\right)=\exp(2Y)$.

Because $\frac{X}{N}=\log\varphi(\frac{1}{N})\in B(0,\frac{\log 2}{2})$, we have $\frac{X}{2^kN}\in B\left(0,\frac{\log 2}{2}\right)$.

Because $Y\in B(0,\frac{\log 2}{2})$, $2Y\in B(0,\log 2)$. Thus both $\frac{X}{2^kN}$ and $2Y$ belong to $B(0,\log 2)$. Because $\exp$ is injective on $B(0,\log 2)$, we have $\frac{X}{2^kN}=2Y$.

Thus, $Y=\frac{1}{2^{k+1}N}X$ and $\varphi\left(\frac{1}{2^{k+1}N}X\right)=\exp\left(\frac{1}{2^{k+1}N}X\right)$. This means $(*)$ is true for $k+1$.

For each number $r\in[0,1]$, we can write r as a series

$$r = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \quad \text{where } a_k \in \{0,1\}.$$

Since φ is continuous, $\varphi\left(\frac{r}{N}\right) = \varphi\left(\sum_{k=1}^{\infty} \frac{a_k}{2^k N}\right) = \prod_{k=1}^{\infty} \varphi\left(\frac{1}{2^k N}\right)^{a_k}$

$$= \prod_{k=1}^{\infty} \left(\exp\left(\frac{1}{2^k N} X\right)\right)^{a_k} = \exp\left(\sum_{k=1}^{\infty} \frac{a_k}{2^k N} X\right)$$

Thus, $\varphi\left(\frac{r}{N}\right) = \exp\left(\frac{r}{N} X\right)$. Taking the N^{th} power of both sides, we get $\varphi(r) = \exp(rX)$ for all $r \in [0,1]$. For each $t \in \mathbb{R}$, $t > 0$, there is an integer $m \in \mathbb{N}$ such that $t/m \in [0,1]$. Then $\varphi\left(\frac{t}{m}\right) = \exp\left(\frac{t}{m} X\right)$. Taking the m'th power of both sides, we get $\varphi(t) = \exp(tX)$. Thus, $\varphi(t) = \exp(tX)$ for all $t \geq 0$. For $t < 0$, $\varphi(t) = \varphi(-t)^{-1} = (\exp(-tX))^{-1}$
$= \exp(tX)$.

Therefore, $\varphi(t) = \exp(tX)$ for all $t \in \mathbb{R}$.

Next, we prove that X is unique. Suppose there is $Y \in M_n(\mathbb{R})$ such that $\varphi(t) = \exp(tY)$ for all $t \in \mathbb{R}$. Choose $\delta > 0$ small enough such that $\delta X, \delta Y \in B(0, \log 2)$. Then

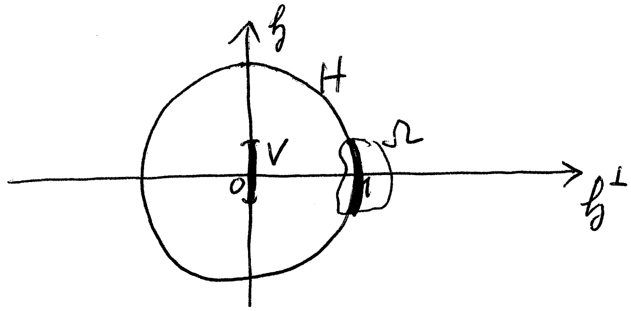
$$\exp(\delta Y) = \varphi(\delta) = \exp(\delta X).$$

Because exp is injective on $B(0, \log 2)$, $\delta Y = \delta X$. Thus, $X = Y$.

<u>Proof of Theorem 3</u>

In $M_n(\mathbb{R})$, we have a bilinear form $(a,b) := tr(a^T b)$. This form is also symmetric and possitively definite, which turns itself into an inner product on $M_n(\mathbb{R})$. This structure also induces a norm on $M_n(\mathbb{R})$ via

$$\|a\| := \sqrt{tr(a^T a)} \quad \forall a \in M_n(\mathbb{R}).$$



Picture for $G = GL(1, \mathbb{C})$,
$H = S^1$,
$\mathfrak{h} = \{iy \mid y \in \mathbb{R}\}$.

Since $\mathfrak{h}$ is a vector subspace of $M_n(\mathbb{R})$, it has an orthogonal complement

$$\mathfrak{h}^\perp = \{a \in G \mid (a, b) = 0 \quad \forall b \in \mathfrak{h}\}.$$

Then $M_n(\mathbb{R}) = \mathfrak{h} \oplus \mathfrak{h}^\perp$ as vector spaces. Denote $\pi : M_n(\mathbb{R}) \to \mathfrak{h}$ the projection map. Since π is a linear map between two finite dimensional vector spaces, it is smooth and analytic. Define a map $\varphi : M_n(\mathbb{R}) \to G$, $\varphi(X) = \exp(X_1)\exp(X_2)$, where $X_1 = \pi(X) \in \mathfrak{h}$, $X_2 = X - X_1 \in \mathfrak{h}^\perp$. In other words,

$$\varphi(X) = \exp(\pi(X))\exp(X - \pi(X)).$$

Since the exponential map and the map π are analytic, φ is also analytic. We'll show that it is nonsingular at 0. For any $X \in M_n(\mathbb{R})$, the directional direction of φ at 0 along X-direction is by definition

$$\frac{d}{dt}\Big|_{t=0} \varphi(tX) = \frac{d}{dt}\Big|_{t=0} [\exp(tX_1)\exp(tX_2)].$$

By definition,
$$\exp(tX_1) = I_n + \sum_{k=1}^{\infty} \frac{(tX_1)^k}{k!},$$
$$\exp(tX_2) = I_n + \sum_{j=1}^{\infty} \frac{(tX_2)^j}{j!}.$$

Thus, $\exp(tX_1)\exp(tX_2) = \left(I_n + tX_1 + \frac{t^2X_1^2}{2} + \cdots\right)\left(I_n + tX_2 + \frac{t^2X_2^2}{2} + \cdots\right)$

$$= I_n + t(X_1 + X_2) + O(t^2)$$
$$= I_n + tX + O(t^2).$$

Thus, $\left.\frac{d\varphi}{dt}\right|_{t=0} = X$. Thus, $d\varphi(0)(X) = X$. This means $d\varphi(0)$ is the identity matrix, which is invertible. By the Inverse Function theorem, there exists $\varepsilon_0 > 0$ such that φ is a diffeomorphism from $\cancel{B_{\varepsilon_0}(0)}\ B(0,\varepsilon_0)$ to $\varphi(B(0,\varepsilon_0))$, and that $\varphi(B(0,\varepsilon_0))$ is open in $GL(n,\mathbb{R})$.

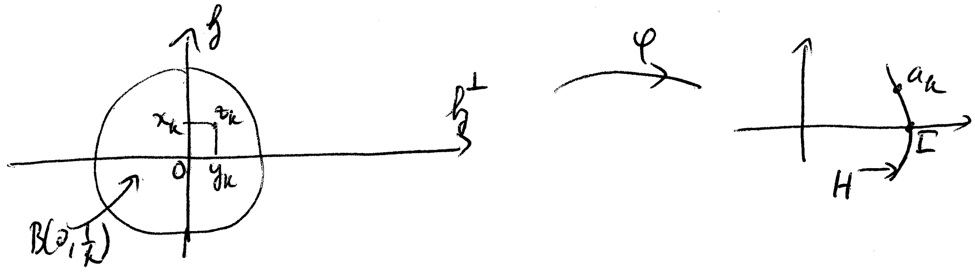
Consequently, $\varphi(\cancel{B_{\varepsilon}(0)}\ B(0,\varepsilon))$ is open in $\varphi(B(0,\varepsilon_0))$ and thus open in $GL(n,\mathbb{R})$, for all $\varepsilon \in (0,\varepsilon_0)$. Next, we'll show that there exists $\varepsilon \in (0,\varepsilon_0)$ such that $\varphi(B(0,\varepsilon)\cap \mathfrak{h}) = \varphi(B(0,\varepsilon)) \cap H$. Suppose otherwise. For every $\varepsilon \in (0,\varepsilon_0)$, $\varphi(B(0,\varepsilon)\cap\mathfrak{h}) \neq \varphi(B(0,\varepsilon))\cap H$. Note that it is always true that

$$\varphi(B(0,\varepsilon)\cap\mathfrak{h}) \subset \varphi(B(0,\varepsilon))\cap H.$$

Thus, for every $\varepsilon \in (0,\varepsilon_0)$, $\varphi(B(0,\varepsilon))\cap H \not\subset \varphi(B(0,\varepsilon)\cap\mathfrak{h})$. Take $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{N}$, $k > \varepsilon_0^{-1}$. Then there exists ~~$z_k \in B(0,\frac{1}{k})$ such that~~

$$a_k \in \left(\varphi(B(0,\tfrac{1}{k}))\cap H\right) \setminus \varphi(B(0,\tfrac{1}{k})\cap\mathfrak{h}).$$

Since $a_k \in \varphi(B(0,\frac{1}{k}))$, there is $z_k \in B(0,\frac{1}{k})$ such that $a_k = \varphi(z_k)$.



Write $z_k = x_k + y_k$ where $x_k = \pi(z_k) \in \mathfrak{h}$ and $y_k \in \mathfrak{h}^\perp$. Because $a_k \notin \varphi(B(0,\frac{1}{k}) \cap \mathfrak{h})$, $z_k \notin \mathfrak{h}$. Thus, $y_k \neq 0$. We have $a_k = \varphi(z_k) = \exp(x_k)\exp(y_k)$. Since $x_k \in \mathfrak{h}$, $\exp(x_k) \in H$. Thus, $\exp(y_k) = \exp(x_k)^{-1} a_k \in H$. By the parallelogram rule,

$$\|x_k\|^2 + \|y_k\|^2 = \|z_k\|^2 < \left(\frac{1}{k}\right)^2.$$

Thus, $\|y_k\| < \frac{1}{k} \leq 1$ for all $k \in \mathbb{N}$, $k > \varepsilon_0^{-1}$. Thus, there exists a number $m_k \in \mathbb{N}$ such that $1 \leq m_k \|y_k\| \leq 2$. Then the sequence $(m_k y_k)$ is contained in the compact set $\{z \in M_n(\mathbb{R}) : 1 \leq \|z\| \leq 2\}$. Thus, it has a convergent subsequente, namely $m_{k_j} y_{k_j} \to y$ as $j \to \infty$, for some $y \in M_n(\mathbb{R})$, $1 \leq \|y\| \leq 2$. On the other hand, $m_k y_k \in \mathfrak{h}^\perp$ since $y_k \in \mathfrak{h}^\perp$. Since $\mathfrak{h}^\perp$ is a closed subset of $M_n(\mathbb{R})$, $y \in \mathfrak{h}^\perp$. We will show that $y \in \mathfrak{h}$. If this can be done, then $y \in \mathfrak{h} \cap \mathfrak{h}^\perp = \{0\}$, which contradicts the fact that $1 \leq \|y\| \leq 2$.

Take any $t \in \mathbb{R}$, we show that $\exp(ty) \in H$. For each $j \in \mathbb{N}$, we put $\alpha_j = [t m_{k_j}]$, i.e. the greatest integer that is $\leq t m_{k_j}$. Then $ty \cancel{= \lim_{j\to\infty}}$

$$t m_{k_j} = \alpha_j + \beta_j \quad \text{for some } 0 \leq \beta_j < 1.$$

We have $ty = \lim_{j\to\infty} t m_{k_j} y_{k_j}$. Since the exponential map is continuous,

$$\exp(ty) = \lim_{j\to\infty} \exp(t m_{k_j} y_{k_j}) = \lim_{j\to\infty} \exp(\alpha_j y_{k_j} + \beta_j y_{k_j})$$

$$= \lim_{j\to\infty} \left(\exp(y_{k_j})\right)^{\alpha_j} \exp(\beta_j y_{k_j}). \quad (*)$$

Since $y_{k_j} \to 0$ as $j \to \infty$, we have $\|\exp(\beta_j y_{k_j}) - I_n\| \leq \exp(\beta_j \|y_{k_j}\|) - 1$

$$\leq \exp(\|y_{k_j}\|) - 1 \to 0.$$

Thus, $\exp(\beta_j y_{k_j}) \to I_n$ as $j \to \infty$. Then $(*)$ implies

$$\exp(ty) = \lim_{j\to\infty} \left(\exp(y_{k_j})\right)^{\alpha_j}.$$

We proved earlier that $\exp(y_{k_j}) \in H$. Thus, $\left(\exp(y_{k_j})\right)^{\alpha_j} \in H$. Since H is a closed subgroup of $GL(n,\mathbb{R})$, the limit is also in H. Thus, $\exp(ty) \in H$. Since this is true for all $t \in \mathbb{R}$, we have $y \in \mathfrak{h}$. This is a contradiction.

So far, we have showed that $\varphi(B(0,\varepsilon)\cap\mathfrak{h}) = \varphi(B(0,\varepsilon))\cap H$ for all $\varepsilon \in (0,\varepsilon_0)$. We have $\varphi(B(0,\varepsilon)\cap\mathfrak{h}) = \{\varphi(X) \mid X \in B(0,\varepsilon)\cap\mathfrak{h}\}$. Note that if $X \in \mathfrak{h}$ then $\varphi(X) = \exp(X)\exp(0) = \exp(X)$. Therefore,

$$\varphi(B(0,\varepsilon)\cap\mathfrak{h}) = \{\exp(X) \mid X \in B(0,\varepsilon)\cap\mathfrak{h}\} = \exp(B(0,\varepsilon)\cap\mathfrak{h}).$$

Therefore, $\exp(B(0,\varepsilon)\cap\mathfrak{h}) = \varphi(B_\varepsilon(0))\cap H$ for all $\varepsilon \in (0,\varepsilon_0)$. We pick any $\varepsilon \in (0,\varepsilon_0)$ and put $V = B(0,\varepsilon)\cap\mathfrak{h}$, $\Omega = \varphi(B_\varepsilon(0))$. Then V is an open neighborhood of 0 in $\mathfrak{h}$; Ω is an open neighborhood of I_n in $GL(n,\mathbb{R})$; and $\exp(V) = H\cap\Omega$. Because $\exp$ is a homeomorphism from $B(0,\log 2)$ onto its image, if we choose $\varepsilon < \log 2$ then the map $\exp: V \to H\cap\Omega$ is also a homeomorphism.

Proof of Theorem 4

Put $\mathfrak{h} = \mathrm{Lie}(H)$. By Theorem 3, for any $0<\varepsilon<\log 2$, if we put

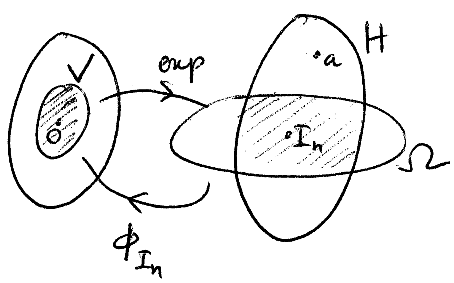
$$V = B(0,\varepsilon)\cap\mathfrak{h} \quad\text{and}\quad \Omega = \varphi(B(0,\varepsilon))$$

then Ω is open in $GL(n,\mathbb{R})$ and the map $\exp : V \to H \cap \Omega$ is a homeomorphism. We will first show that H, as a topological subspace of G, is a manifold. Since G is Hausdorff and second countable, so is H. We show that H is locally Euclidean.

We know that the map exp from $B(0, \log 2)$ to its image in $GL(n,\mathbb{R})$ is a homeomorphism. The inverse is the logarithm function. Put

$$\phi_{I_n} : H \cap \Omega \to V, \quad \phi_{I_n}(x) = \log(x).$$

For each $a \in H$, we define $\phi_a = \phi_{I_n} \circ L_{a^{-1}}$ from $a(H \cap \Omega)$ to V.



Then ϕ_a is also a homeomorphism. Thus, the open neighborhood $a(H \cap \Omega)$ of a in H is homeomorphic to V, which is an open neighborhood of 0 in $\mathfrak{h}$ (in fact $V \simeq \mathfrak{h}$ by the definition of V above). Thus, H is locally Euclidean. Therefore, H is a manifold with dimension equal to the dimension of $\mathfrak{h}$.

Next, we show that the family $(\phi_a)_{a \in H}$ is an analytic atlas on H. For $a, b \in H$, we show that the transition map $\phi_b \circ \phi_a^{-1}$ is analytic. Put $U = (a(H \cap \Omega)) \cap (b(H \cap \Omega))$. Then $\phi_b \circ \phi_a^{-1} : \phi_a(U) \to \phi_b(U)$. Take any $y \in \phi_a(U)$. There is $x \in U$ such that $y = \phi_a(x) = \phi_{I_n} \circ L_{a^{-1}}(x) = \phi_{I_n}(a^{-1}x)$

$$= \log(a^{-1}x).$$

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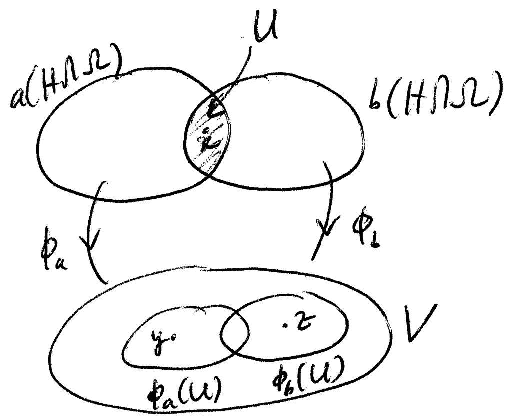


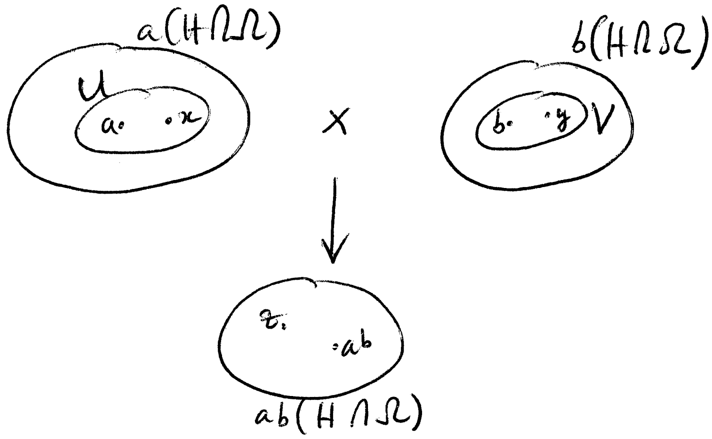
Thus, $a^{-1}x = \exp(y)$ and $x = a\exp(y)$.

We have $\phi_b \circ \phi_a^{-1}(y) = \phi_b(x) = \log(b^{-1}x)$
$$= \log(b^{-1}a\exp(y)).$$

This is an analytic function. Therefore, H together with the analytic atlas $(\phi_a)_{a\in H}$ is an analytic manifold.

Now we show that the multiplication map $H\times H \to H$ is analytic. Take $a, b \in H$ arbitrarily. Because the multiplication is continuous, the preimage of the set $ab(H\cap\Omega)$ is open in $H\times H$. Thus there exists an open nbhd U of a in $a(H\cap\Omega)$ and an open nbhd V of b in $b(H\cap\Omega)$ such that $UV \subset ab(H\cap\Omega)$.

For any $x\in U$, $y \in V$, we put $z = xy \in ab(H\cap\Omega)$. The local coordinate representation of the multiplication map is



$$\phi_a(U)\times\phi_b(V) \to \phi_{ab}(ab(H\cap\Omega))$$
$$(x', y') \mapsto z'$$
where $x' = \phi_a(x)$, $y' = \phi_b(y)$, $z' = \phi_{ab}(z)$.

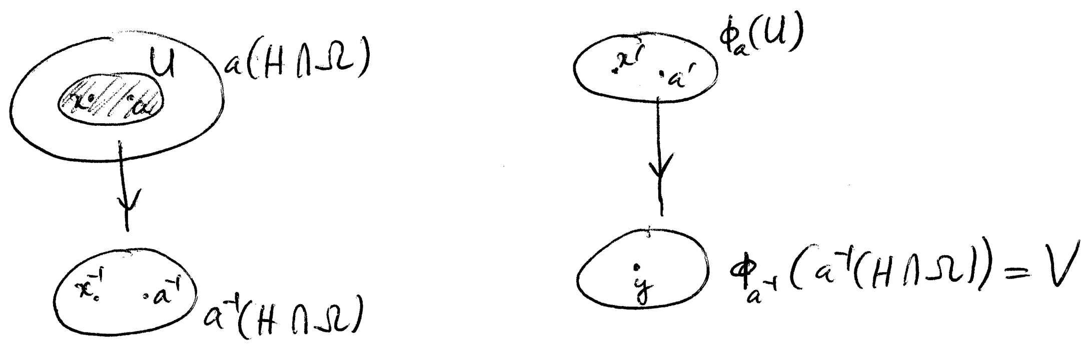
We have $x' = \phi_{I_n}\circ L_{a^{-1}}(x) = \log(a^{-1}x)$,
$$y' = \phi_{I_n}\circ L_{b^{-1}}(y) = \log(b^{-1}y),$$
$$z' = \phi_{I_n}\circ L_{(ab)^{-1}}(z) = \log((ab)^{-1}z) = \log(b^{-1}a^{-1}z).$$

Thus, $x = a\exp(x')$ and $y = b\exp(y')$ and

$$z' = \log(b^{-1}a^{-1}z) = \log(b^{-1}a^{-1}xy) = \log(b^{-1}a^{-1}a\exp(x')\,b\exp(y'))$$
$$= \log(b^{-1}\exp(x')\,b\exp(y')).$$

This implies that z' is an analytic function in variables x' and y'. Therefore, the multiplication is analytic.

Next, we show that the inversion map $H \to H$ is analytic. Let $a \in H$ arbitrarily. Since the inversion map is continuous, the preimage of $a^{-1}(H \cap \Omega)$ is an open nbhd of a in H. Thus, there exists an open nbhd U of a in $a(H \cap \Omega)$ such that $U^{-1} \subset a^{-1}(H \cap \Omega)$.



For any $x \in U$, put $x' = \phi_a(x)$. The coordinate representation of the inversion map is $\phi_a(U) \to \phi_{a^{-1}}(a^{-1}(H \cap \Omega))$, $x' \mapsto y$ where $y = \phi_{a^{-1}}(x^{-1})$. We have $x = a\exp(x')$. Thus, $x^{-1} = \exp(x')^{-1}a^{-1} = \exp(-x')a^{-1}$. Thus,

$$y = \phi_{a^{-1}}(x^{-1}) = \log(ax^{-1}) = \log(a\exp(-x')a^{-1}).$$

This is an analytic function in x'. Therefore, the inversion map is analytic.

$\boxed{4}$ Differential of topological-group morphisms

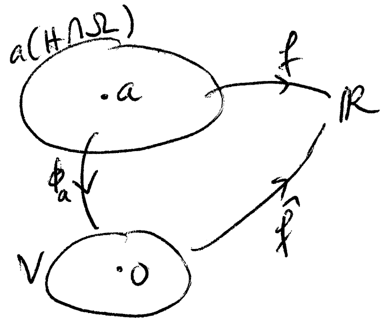
Let H and G be closed subgroups of $GL(n, \mathbb{C})$. They have the Lie group structure as defined in Section [3]. Thus, we can speak of the coordinate representations of a map $\varphi: H \to G$. If φ is a topological-group morphism, the coordinate representation on a neighborhood of an arbitrary point will be known if we know the coordinate representation on a neighborhood of the identity element. We realize from two following theorems (especially from the proof of Theorem 6) that this coordinate representation is the <u>differential of φ</u>.

<u>Theorem 5</u> Let G be a closed subgroup of $GL(n, \mathbb{R})$ and H be a closed subgroup of $GL(m, \mathbb{R})$. Let $\varphi: G \to H$ be a topological-group morphism. Then there exists a unique Lie-algebra morphism $\mu: Lie(G) \to Lie(H)$ satisfying $\exp(\mu(X)) = \varphi(\exp(X))$ for all $X \in Lie(G)$.

<u>Theorem 6</u> Let H and K be two closed subgroups of $GL(n, \mathbb{R})$. Let f be a topological-group morphism from H to K. Then f is real analytic.

Let H be a closed subgroup of $GL(n, \mathbb{R})$ and $\mathfrak{h} = Lie(H)$. We want to give explicitly a global frame on H. Take any $a \in H$. Consider the coordinate chart $\phi_a: a(H \cap \Omega) \to V$, where V is an open neighborhood of 0 in $\mathfrak{h}$. For each $f \in C^\omega(H)$, where $C^\omega(H)$ denotes the space of all real-analytic functions from H to $\mathbb{R}$, the coordinate representation of f on

$a(H\cap\Omega)$ is $\hat{f} = f\circ\phi_a^{-1}$. We define $D_v f(a) := D_v\hat{f}(0)$ for each tangent



vector $v\in\mathfrak{h}$. By the definition of directional derivatives,

$$D_v\hat{f}(0) = \frac{d}{dt}\Big|_{t=0}\hat{f}(0+tv) = \frac{d}{dt}\Big|_{t=0} f(\phi_a^{-1}(tv))$$

$$= \frac{d}{dt}\Big|_{t=0} f(a\exp(tv)).$$

Therefore, $D_v f(a) = \frac{d}{dt}\Big|_{t=0} f(a\exp(tv)) \quad \forall a\in H, \forall f\in C^\omega(H), \forall v\in\mathfrak{h}.$

Goodman and Wallach use a different style of notation for the vector field D_v. They write X_A^H instead, i.e.

$$X_A^H f(a) = \frac{d}{dt}\Big|_{t=0} f(a\exp(tA)) \quad \forall a\in H, \forall f\in C^\omega(H), \forall A\in\mathfrak{h}.$$

If $(e_1,\dots,e_m)$ is a basis of $\mathfrak{h}$ then $(X_{e_1}^H, X_{e_2}^H,\dots,X_{e_m}^H)$ is a <u>global frame</u> on H. In other words, at each point $a\in H$, the set $\{X_{e_1}^H, X_{e_2}^H,\dots,X_{e_m}^H\}$ is linearly independent, and the value at a of any vector field is a linear combination of this set.

The imbedding $i_H : H\to GL(n,\mathbb{R})$ is a topological-group morphism. It is a homeomorphism onto its image. By Theorem 5, there exists a unique Lie-algebra morphism $di_H : \mathfrak{h}\to M_n(\mathbb{R})$ such that $i_H(\exp(X)) = \exp(di_H(X))$ for all $X\in\mathfrak{h}$. This implies $di_H(X) = X$ and hence $di_H = id_{\mathfrak{h}}$. Thus, H is a smooth submanifold of $GL(n,\mathbb{R})$ according to the terminology in Lee

"Introduction to smooth manifolds", p. 98. Because H is a closed subset of $GL(n,\mathbb{R})$, it is properly embedded by Proposition 5.5 in the same reference. By Exercise 10-9, p. 270 in the same reference, every smooth vector field on H extends to a smooth vector field on $GL(n,\mathbb{R})$. We have $Lie(GL(n,\mathbb{R})) = M_n(\mathbb{R})$, whose basis is $\{e_{ij} : 1 \le i,j \le n\}$ where e_{ij} is the matrix with value 1 at position (i,j) and value 0 elsewhere. We know that each vector field on $GL(n,\mathbb{R})$ is a linear combination of $\left\{\frac{\partial}{\partial e_{ij}} : 1 \le i,j \le n\right\}$. Let X and Y be two vector fields on H. Denote by $\tilde{X}$ and $\tilde{Y}$ extensions of X and Y to $GL(n,\mathbb{R})$. We write

$$(\tilde{X}f)(a) = \alpha_{ij}(a)\frac{\partial f}{\partial e_{ij}}(a), \quad (\tilde{Y}f)(a) = \beta_{ij}(a)\frac{\partial f}{\partial e_{ij}}(a) \quad \forall f \in C^\infty(GL(n,\mathbb{R})), \forall a \in GL(n,\mathbb{R}).$$

Define a vector field XY on H by

$$((XY)f)(a) := ((\tilde{X}\tilde{Y})f)(a) := (X\beta_{ij})(a)\frac{\partial f}{\partial e_{ij}}(a) \quad \forall f \in C^\infty(H), \forall a \in H.$$

We can check that this definition doesn't depend on the specific choice of extensions $\tilde{X}$ and $\tilde{Y}$ for X and Y. Define another vector field on H:

$$[X,Y] := XY - YX.$$

[Theorem 7 Let H be a closed subgroup of $GL(n,\mathbb{R})$ and $\mathfrak{h} = Lie(H)$. Put $\mathfrak{X} = \{X_A^H : A \in \mathfrak{h}\}$. Then the map $A \in \mathfrak{h} \mapsto X_A^H \in \mathfrak{X}$ is a Lie-algebra morphism.]

For each $g \in H$, we define the left-translation $L(g)$ on $C^\infty(H)$ by

$$(L(g)f)(a) = f(g^{-1}a) \quad \forall f \in C^\infty(H), \forall a \in H.$$

A vector field X on H is said to be left-invariant if it commutes with the operator $L(g)$ for all $g \in H$. Left-invariant vector field has a remarkable property that its value at the identity element I_n determines its values elsewhere in H. This idea will be made clear in the proof of the following theorem.

$$\begin{tikzcd}
f \arrow[r, "L(g)"] \arrow[dd, "X"'] & L(g)f \arrow[d, "X"] \\
 & X(L(g)f) \arrow[d, equal] \\
Xf \arrow[r, "L(g)"] & L(g)(Xf)
\end{tikzcd}$$

Theorem 8 Every left-invariant vector field on H is of the form X_A^H for a unique $A \in \mathfrak{h}$.

Proof of Theorem 5

Fix $X \in Lie(G)$. Consider the map $f_X : \mathbb{R} \to H$, $f_X(t) = \varphi(\exp(tX))$. We show that f_X is a continuous group morphism from $(\mathbb{R}, +)$ to H. First, $f_X(0) =$ $= \varphi(\exp(0)) = \varphi(I_n) = I_m$. For $t_1, t_2 \in \mathbb{R}$, we have

$$\begin{aligned}
f_X(t_1+t_2) = \varphi(\exp((t_1+t_2)X)) &= \varphi(\exp(t_1X+t_2X)) \\
&= \varphi(\exp(t_1X)\exp(t_2X)) \quad \text{(since } t_1X \text{ and } t_2X \text{ commute)} \\
&= \varphi(\exp(t_1X))\,\varphi(\exp(t_2X)) \quad \text{(since } \varphi \text{ is a group morphism)} \\
&= f_X(t_1)\,f_X(t_2).
\end{aligned}$$

Thus, f_X is a group morphism. Next, we show that f_X is continuous. Because the left-translations are continuous, it suffices to show that f_X is continuous at 0. This is trivial because f_X is the composition of two continuous maps, which are φ and exp.

By Theorem 2, there exists a unique matrix $\mu(X) \in M_n(\mathbb{R})$ such that $f_X(t) = \exp(t\mu(X))$. Because $f_X(t) \in H$ for all $t \in \mathbb{R}$, $\mu(X) \in Lie(H)$. We have proved the uniqueness part. Next, we show μ is linear. Using the identity (10), we get

$$\exp(t(\mu(X_1) + \mu(X_2))) = \varphi(\exp(t(X_1 + X_2))) \quad \forall t \in \mathbb{R}, \forall X_1, X_2 \in Lie(G)$$

$$\exp(st\mu(X)) = \varphi(\exp(stX)) \quad \forall t, s \in \mathbb{R}, \forall X \in Lie(G)$$

By the uniqueness of $\mu(X_1 + X_2)$ and $\mu(tX)$, we get

$$\mu(X_1) + \mu(X_2) = \mu(X_1 + X_2),$$
$$t\mu(X) = \mu(tX).$$

Therefore, μ is a linear map. By the identity (11), we get

$$\exp(t[\mu(X_1), \mu(X_2)]) = \varphi(\exp(t[X_1, X_2])) \quad \forall t \in \mathbb{R}, \forall X_1, X_2 \in Lie(G).$$

By the uniqueness of $\mu([X_1, X_2])$, we have $[\mu(X_1), \mu(X_2)] = \mu([X_1, X_2])$. Thus, μ is a Lie-algebra morphism.

Proof of Theorem 6

We recall the analytic structure on H and K which was introduced in Section $\boxed{3}$.

Atlas on H: $(\phi_a)_{a\in H}$ where $\phi_a : a(H\cap\Omega_H)\to V_H$, $\phi_a(x) = \log(a^{-1}x)$,

$V_H = B(0,\varepsilon)\cap \mathrm{Lie}(H)$, $0<\varepsilon<\log 2$,

Ω_H is an open nbhd of I_n in $GL(n,\mathbb{R})$.

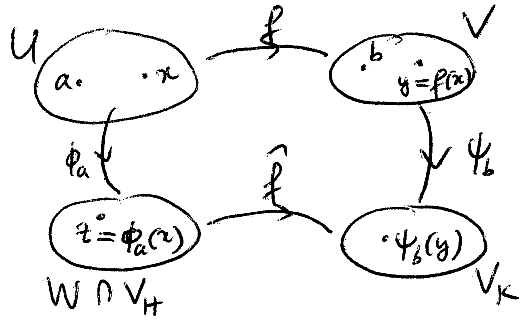
Atlas on K: $(\psi_b)_{b\in K}$ where $\psi_b : b(K\cap\Omega_K)\to V_K$, $\psi_b(y) = \log(b^{-1}y)$,

$V_K = B(0,\varepsilon)\cap \mathrm{Lie}(K)$, $0<\varepsilon<\log 2$,

Ω_K is an open nbhd of I_n in $GL(n,\mathbb{R})$.

We show that f is analytic. By Theorem 5, there exists a unique Lie-algebra morphism $df : \mathrm{Lie}(H)\to \mathrm{Lie}(K)$ such that $f(\exp(x)) = \exp(df(x))$ for all $x\in \mathrm{Lie}(H)$. Since df is a linear map between finite-dimensional spaces, it is continuous. Thus, the set $W = (df)^{-1}(B(0,\log 2)\cap \mathrm{Lie}(K))$ is an open nbhd of 0 in $\mathrm{Lie}(H)$.

Take any $a\in H$. Put $b = f(a)$ and $V = b(K\cap\Omega_K)$ and $U = \phi_a^{-1}(W\cap V_H)\cap f^{-1}(V)$. Then U is an open nbhd of a in H and V is an open nbhd of b in K. The coordinate representation of $f : U\to V$ is $\hat f : W\cap V_H \to V_K$, $\hat f = \psi_b\circ f\circ \phi_a^{-1}$.



For each $z\in W\cap V_H$, there is $x\in U$ such that $z = \phi_a(x)$. Then

$$\hat f(z) = \psi_b\circ f\circ \phi_a^{-1}(z) = \psi_b(f(x)) = \log(b^{-1}f(x)) = \log(b^{-1}f(a\exp(z)))$$

$$= \log(\underbrace{b^{-1}f(a)}_{=I_n} f(\exp(z))) = \log(f(\exp(z))) = \log(\exp(df(z))).$$

Since $z \in W$, $df(z) \in B(0, \log 2)$. Thus, $\log(\exp(df(z))) = df(z)$. Therefore, $\hat{f}(z) = df(z)$ which is a linear map (φ extended to a map from $\mathrm{Lie}(H)$ to $\mathrm{Lie}(K)$). In particular, $\hat{f}$ is analytic.

Proof of Theorem 7

Because each smooth vector field on H extends to a smooth vector field on $GL(n, \mathbb{R})$, we assume $H = GL(n, \mathbb{R})$. Now we write X_A instead of X_A^H.

First, we show

$$X_{A+cB} = X_A + cX_B \quad \forall c \in \mathbb{R}, \forall A, B \in M_n(\mathbb{R}).$$

Take any $f \in C^\omega(GL(n, \mathbb{R}))$ and $a \in H$. The coordinate representation of f on $B(a, \log 2)$ is $\hat{f}(g) = f(a \exp(g))$. By definition,

$$X_{A+cB} f(a) = D_{A+cB} \hat{f}(0),$$
$$X_A f(a) = D_A \hat{f}(0),$$
$$X_B f(a) = D_B \hat{f}(0).$$

The directional derivatives of $\hat{f}$ satisfy $D_{A+cB}\hat{f}(0) = D_A \hat{f}(0) + c D_B \hat{f}(0)$. Thus,

$$X_{A+cB} f(a) = X_A f(a) + c X_B f(a).$$

Next, we show that $X_{[A,B]} = [X_A, X_B]$ for all $A, B \in \mathfrak{g}$. Thanks to the linearity of the map $A \mapsto X_A$, it suffices to show

$$X_{[e_{ij}, e_{kl}]} = [X_{e_{ij}}, X_{e_{kl}}] \quad \forall 1 \le i, j, k, l \le n.$$

We have

$$(X_{e_{kl}}f)(a) = \frac{d}{dt}\Big|_{t=0} f(a\exp(te_{kl})) = \frac{\partial f}{\partial e_{rs}}(a)\, \frac{d}{dt}\Big|_{t=0}(a\exp(te_{kl}))$$

$$= \frac{\partial f}{\partial e_{rs}}(a)\,(ae_{kl})_{rs} = \frac{\partial f}{\partial e_{rs}}(a)\,\delta_{sl}\,a_{rk} = \frac{\partial f}{\partial e_{rl}}(a)\,a_{rk}.$$

Thus,
$$X_{e_{kl}} = x_{rk}\frac{\partial}{\partial e_{rl}}. \qquad (12)$$

By definition,
$$X_{e_{ij}}X_{e_{kl}} = X_{e_{ij}}(x_{rk})\frac{\partial}{\partial e_{rl}} = x_{si}\frac{\partial x_{rk}}{\partial e_{sj}}\frac{\partial}{\partial e_{rl}} = x_{ri}\,\delta_{kj}\frac{\partial}{\partial e_{rl}}$$
$$= \delta_{kj}X_{e_{il}}.$$

Similarly, $X_{e_{kl}}X_{e_{ij}} = \delta_{il}X_{e_{kj}}$. Thus,

$$[X_{e_{ij}}, X_{e_{kl}}] = X_{e_{ij}}X_{e_{kl}} - X_{e_{kl}}X_{e_{ij}} = \delta_{kj}X_{e_{il}} - \delta_{il}X_{e_{kj}}.$$

We have

$$X_{[e_{ij},e_{kl}]} = X_{e_{ij}e_{kl}-e_{kl}e_{ij}} = X_{\delta_{kj}e_{il}-\delta_{il}e_{kj}} \underset{\text{linearity}}{=\!=} \delta_{kj}X_{e_{il}} - \delta_{ij}X_{e_{kj}}.$$

Therefore, $[X_{e_{ij}}, X_{e_{kl}}] = X_{[e_{ij},e_{kl}]}$.

<u>Proof of Theorem 8</u>

Let X be a left-invariant vector field on H. By definition,

$$L(g)(Xf)(a) = X(L(g)f)(a) \quad \forall a, g \in H,\ \forall f \in C^\omega(H).$$

In other words,

$$(Xf)(g^{-1}a) = X(b \mapsto f(g^{-1}b))(a) \quad \forall a, g \in H,\ \forall f \in C^\omega(H).$$

Taking $g = a$, we get

$$(Xf)(I_n) = X(b \mapsto f(a^{-1}b))(a) \quad \forall a \in H,\ \forall f \in C^\omega(H).$$

Equivalently, $X(b \mapsto f(ab))(I_n) = (Xf)(a) \quad \forall a \in H,\ \forall f \in C^\omega(H). \qquad (*)$

Thus, the value of X at I_n, which is a derivation (a linear map from $C^\infty(H)$ to $\mathbb{R}$ satisfying the Leibnitz rule), determines the value of X elsewhere in H. Let $\{e_1, \dots, e_m\}$ be a basis of $\mathfrak{h} = \mathrm{Lie}(H)$. Then $\{X_{e_1}, \dots, X_{e_m}\}$ is a global frame on H. Thus, $X(I_n)$ is a linear combination of $\{X_{e_1}(I_n), \dots, X_{e_m}(I_n)\}$. Write $X(I_n) = \alpha_i X_{e_i}(I_n) \overset{\text{Theorem 7}}{=\!=\!=} X_A(I_n)$, where $A = \alpha_i e_i \in \mathfrak{h}$. For $a \in H$ and $f \in C^\infty(H)$, we have

$$(Xf)(a) \overset{(*)}{=\!=} X(b \mapsto f(ab))(I_n) = X_A(b \mapsto f(ab))(I_n)$$
$$= \frac{d}{dt}\Big|_{t=0} f(aI_n \exp(tA)) = (X_A f)(a).$$

Thus, $X = X_A$. Now we show the uniqueness of A. Suppose there is $B \in \mathfrak{h}$ such that $X_A = X_B$. Then $(X_A \, \mathrm{id}_H)(I_n) = (X_B \, \mathrm{id}_H)(I_n)$. Thus

$$\frac{d}{dt}\Big|_{t=0} \exp(tA) = \frac{d}{dt}\Big|_{t=0} \exp(tB),$$

which is $A = B$.

$\boxed{5}$ Algebraic groups and regular functions

There is a special way whereby a closed subgroup of $GL(n, \mathbb{C})$ can be given. That is as the zero set of one or more polynomials:

$$G = \{g \in GL(n, \mathbb{C}) : f(g) = 0 \quad \forall f \in \mathcal{A}\} \tag{13}$$

where $\mathcal{A}$ is a set of polynomials on $M_n(\mathbb{C})$. Each of such polynomials is in $\mathbb{C}[x_{11}, x_{12}, \dots, x_{1n}, x_{21}, \dots, x_{nn}]$. A group G given by (13) is called a <u>linear algebraic group</u>, or simply <u>algebraic group</u>. Of course, not every set of the form $\{g \in GL(n,\mathbb{C}) : f(g)=0 \ \forall f \in \mathcal{A}\}$ is a group. For example, if $\mathcal{A} = \{X \mapsto X - 2I_n\}$ then the zero set of $\mathcal{A}$ is $\{2I_n\}$ which is not a group.

Not all subgroups of $GL(n,\mathbb{C})$ are algebraic groups. For example, consider $n=1$ $(GL(1,\mathbb{C}) = \mathbb{C}^\times)$ and

$$G = \{z \in \mathbb{C}^\times : |z| = 1\}.$$

Any polynomial on $\mathbb{C}$ which vanishes on G must be identically zero on $\mathbb{C}$. The zero set of the zero polynomial is $\mathbb{C}$, not G. We also observe that G given by $z\bar{z} - 1 = 0$, which is not a polynomial equation.

The special linear group $SL(n,\mathbb{C}) = \{g \in GL(n,\mathbb{C}) : \det(g) = 1\}$ is an algebraic subgroup because $\det(g) = \det\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$ is a polynomial of $x_{11}, x_{12}, \dots, x_{nn}$. The isometry group

$$O(B) = \{g \in GL(n,\mathbb{C}) : B(gv, gw) = B(v,w) \ \ \forall v, w \in V\}$$

of a bilinear form $B : V \times V \to \mathbb{C}$, where $\dim_{\mathbb{C}} V = n$, is an algebraic group because the equation $B(gv, gw) = B(v,w)$ is a (quadratic) polynomial equation.

Now that we have defined algebraic groups, we need to define morphisms among them. Such a morphism should be related to polynomials because the algebraic groups are defined ~~by~~ based on polynomials.

The set of <u>regular functions</u> on $GL(n,\mathbb{C})$ is defined as

$$\mathcal{V}[GL(n,\mathbb{C})] := \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, \det(x)^{-1}]. \qquad (14)$$

Equivalently, this set can be considered as a quotient ring

$$\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, y] / (y \det(x) - 1).$$

Let G be an algebraic subgroup of $GL(n,\mathbb{C})$. The set of regular functions on G is defined as

$$\mathcal{V}[G] := \{ f|_G : f \in \mathcal{V}[GL(n,\mathbb{C})] \}. \qquad (15)$$

Of course, there may be a lot of regular functions on $GL(n,\mathbb{C})$ having the same restriction on G. To remove this redundancy, we have another way to represent $\mathcal{V}[G]$ as follows. We define an equivalence relation on $\mathcal{V}[GL(n,\mathbb{C})]$: $f_1 \sim f_2$ if and only if $f_1|_G = f_2|_G$. Then we can define $\mathcal{V}[G] := \mathcal{V}[GL(n,\mathbb{C})]/\sim$.

Equivalently, if we define $\cancel{\mathcal{V}[G]}$ the ideal

$$I_G = \{ f \in \mathcal{V}[GL(n,\mathbb{C})] : f(g) = 0 \ \forall g \in G \} \qquad (16)$$

then we can define $\mathcal{V}[G] := \mathcal{V}[GL(n,\mathbb{C})]/I_G$.

Let G and H two linear algebraic groups. A map $\varphi : G \to H$ is called a <u>regular map</u> if for each $f \in \mathcal{V}[H]$, $f \circ \varphi \in \mathcal{V}[G]$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, dashed] & H \arrow[d, "f \in \mathcal{V}[H]"] \\ & \mathbb{C} \end{tikzcd}$$

In other words, φ is a regular map if we can define the dual map

$$\varphi^* : \mathcal{V}[H] \to \mathcal{V}[G], \quad \varphi^*(f)(g) = f(\varphi(g)) \quad \forall f \in \mathcal{V}[H], \ \forall g \in G.$$

An <u>algebraic group morphism</u> between two algebraic groups G and H is a group morphism $\varphi: G \to H$ which is a regular map. If φ has an inverse and φ^{-1} is also a regular map then φ is called an algebraic group isomorphism. For example, if G is an algebraic group then the inversion map $\eta: G \to G$, $\eta(g) = g^{-1}$ is an algebraic group isomorphism. Indeed, $\eta(g) = g^{-1} = (\det g)^{-1}\, \mathrm{adj}(g)$, where $\mathrm{adj}(g)$ is the transpose of the cofactor matrix of g. Each coefficient of $\mathrm{adj}(g)$ is a polynomial of the coefficients of g. For each $f \in \mathcal{U}[G]$, $f\circ\eta(g)$ is a polynomial of entries of $\eta(g)$ and $(\det \eta(g))^{-1} = \det(g)$. Thus $f\circ\eta(g)$ is a polynomial of the entries of g and $(\det g)^{-1}$. Thus $f\circ\eta \in \mathcal{U}[G]$. Therefore, η is a regular map.

Likewise, the left-translation $L_g: G \to G$, $L_g(x) = gx$ and the right-translation $R_g: G \to G$, $R_g(x) = xg$ are algebraic group isomorphisms.

According to Theorem 8, a left-invariant vector field on $GL(n,\mathbb{C})$ is given by X_A for some $A \in M_n(\mathbb{C})$, where

$$X_A f(a) = \frac{d}{dt}\Big|_{t=0} f(a\exp(tA)) \quad \forall f \in C^{\omega}(GL(n,\mathbb{C}))\ \forall a \in GL(n,\mathbb{C}).$$

We have
$$f(a\exp(tA)) = f(a(I_n + tA + O(t^2))) = f(a(I_n + tA)) + O(t^2).$$

Thus,
$$X_A f(a) = \frac{d}{dt}\Big|_{t=0} f(a(I_n + tA)). \qquad (17)$$

This equation can be taken as the definition for a ~~vector field~~ left-invariant vector field on $GL(n,\mathbb{C})$. The advantage is that we have eliminated the exponential map, or more precisely the nonlinear terms in the Taylor expansion of the exponential map.

Let G be an algebraic subgroup of $GL(n, \mathbb{C})$ and I_G be the ideal of all regular functions that vanishes on G, as defined in (16). Define

$$\mathfrak{g} = \{A \in M_n(\mathbb{C}) : X_A f \in I_G, \forall f \in I_G\} \qquad (18)$$

It turns out that $\mathfrak{g} = \mathrm{Lie}(G)$. Indeed, for $A \in \mathfrak{g}$ and $f \in I_G$ we have

$$0 = X_A f(I_n) = \frac{d}{dt}\Big|_{t=0} f(\exp(tA)).$$

Put $\varphi(t) = f(\exp(tA))$ for all $t \in \mathbb{R}$. Then $\varphi'(0) = 0$. Because $X_A f \in I_G$, $X_A^2 f = X_A(X_A f) \in I_G$. Thus,

$$\begin{aligned} 0 = X_A^2 f(I_n) &= \frac{d}{dt}\Big|_{t=0} X_A f(\exp(tA)) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(\exp(tA)\exp(sA)) \\ &= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} f(\exp((t+s)A)) \\ &= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \varphi(t+s) \\ &= \varphi''(0). \end{aligned}$$

Similarly, $\varphi^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Moreover, $\varphi(0) = f(I_n) = 0$ since $f \in I_G$, $I_n \in G$. Because $\varphi \in C^\omega(\mathbb{R})$, we conclude $\varphi(t) = 0$ for all $t \in \mathbb{R}$. Thus $f(\exp(tA)) = 0$ for all $t \in \mathbb{R}$ and $f \in I_G$. Because G is an algebraic group, it is the zero set of a set $\mathcal{A}$ of polynomials. Thus, $\exp(tA) \in G$. This means $A \in \mathrm{Lie}(G)$.

Conversely, for $A \in \mathrm{Lie}(G)$, $\exp(tA) \in G$ for all $t \in \mathbb{R}$. Thus, $a\exp(tA) \in G$

for all $a \in G$ and $t \in \mathbb{R}$. For each $f \in I_G$, $f(a\exp(tA)) = 0$. Thus,

$$X_{A,1} f(a) = \frac{d}{dt}\Big|_{t=0} f(a\exp(tA)) = 0 \quad \forall a \in G, \forall f \in I_G.$$

Hence, $X_A f \in I_G$ and therefore $A \in \mathfrak{g}$.

We have showed that the Lie algebra of an algebraic group G can be given in an analytic way (9) or an algebraic way (18). Two ways give the same result.

$\boxed{6}$ Representations, regular representations and their differentials

Let G be an algebraic subgroup of $GL(n, \mathbb{C})$. A <u>representation</u> of G is a pair (ρ, V), where V is a vector space over $\mathbb{C}$ (possibly infinite-dimensional) and $\rho : G \to GL(V)$ is a group morphism. We see that this definition is valid for any group G, not just for algebraic groups. However, because we will need to consider the differential of ρ later, a topological-group structure on G is required.

In case $\dim V < \infty$, we take a basis $(e_1, \dots, e_d)$ of V. Then the group $GL(V)$ is isomorphic to the group $GL(d, \mathbb{C})$. The group morphism $\rho : G \to GL(V)$ can be identified with the map

$$g \in G \mapsto \begin{pmatrix} \rho_{11}(g) & \rho_{12}(g) & \cdots & \rho_{1d}(g) \\ \vdots & \vdots & & \vdots \\ \rho_{d1}(g) & \rho_{d2}(g) & \cdots & \rho_{dd}(g) \end{pmatrix} \in GL(d, \mathbb{C}).$$

We call (ρ, V) a <u>regular</u> or <u>rational</u> representation if $\dim_{\mathbb{C}} V < \infty$ and the coefficient maps $\rho_{ij} \in \mathcal{V}[G]$. Note that this definition doesn't depend on the choice of basis of V. Indeed, if $(\bar{e}_1, \dots, \bar{e}_d)$ is another basis of V then

$$\begin{pmatrix} \bar{\rho}_{11}(g) & \cdots & \bar{\rho}_{1d}(g) \\ \vdots & & \vdots \\ \bar{\rho}_{d1}(g) & \cdots & \bar{\rho}_{dd}(g) \end{pmatrix} = A^{-1} \begin{pmatrix} \rho_{11}(g) & \cdots & \rho_{1d}(g) \\ \vdots & & \vdots \\ \rho_{d1}(g) & \cdots & \rho_{dd}(g) \end{pmatrix} A \qquad \forall g \in G$$

where A is a constant matrix in $GL(d, \mathbb{C})$. Then each $\bar{\rho}_{ij}$ is also a ~~rational~~ regular function.

Let (ρ, V) be a regular representation of an algebraic group G. For each $B \in \mathrm{End}(V)$, we define a map $f_B^\rho : G \to \mathbb{C}$,

$$f_B^\rho(g) = tr_V(\underbrace{\rho(g) B}_{\text{Composite map}}) \qquad (19)$$

With respect to a basis $(e_1, \ldots, e_d)$ of V, we write $\rho(g) = (\rho_{ij}(g))_{1 \le i,j \le d}$ and $B = (b_{ij})_{1 \le i,j \le n}$. Then $f_B^\rho(g)$ is a linear combination of $\{\rho_{ij}(g) : 1 \le i,j \le d\}$. Thus, f_B^ρ is a regular function on G, i.e. $f_B^\rho \in V[G]$. The set

$$E^\rho = \{ f_B^\rho : B \in \mathrm{End}(V) \}$$

is a vector space generated by $\{\rho_{ij}(\cdot) : 1 \le i,j \le d\}$ and is called the space of <u>representative functions</u> associated with ρ.

Given a representation (ρ, V) of an algebraic group G, a subspace W of V is called <u>G-invariant</u> if $\rho(g) W \subset W$ for all $g \in G$. In other words, the matrix form of ρ is of the form $\begin{pmatrix} \sigma(g) & * \\ 0 & \tau(g) \end{pmatrix}$ for all $g \in G$, where the basis vectors of W were listed first. In this case, the representation $\rho : G \to GL(V)$ induces a representation $\rho : G \to GL(W)$, which tends to be "smaller" or "better". A representation (ρ, V) of G is called <u>irreducible</u> if it has no G-invariant subspace except for 0 and V itself. (ρ, V) is called <u>locally regular</u> if every finite-dimensional subspace E of V is contained in a finite-dimensional G-invariant subspace F such that $\rho: G \to F$ is a regular representation of G.

Given two representations (ρ, V) and (τ, W) of an algebraic group G, a linear map $T \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ is called a <u>G-intertwining</u> map if for every $g \in G$, the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\tau(g)"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

Note that if a G-intertwining map $T \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ is invertible as a set-theoretic map then T^{-1} is also a G-intertwining map. That is because the diagram also commutes.

$$\begin{tikzcd} V & W \arrow[l, "T^{-1}"'] \arrow[d, "\tau(g)"] \\ V \arrow[u, phantom, ""] & W \arrow[l, "T^{-1}"'] \end{tikzcd}$$

with $\rho(g): V \to V$ on the left.

If there exists an invertible G-intertwining map between two representations (ρ, V) and (τ, W) of G, then we call them to be <u>equivalent representations</u>.

Now we define the differential of a regular representation. Let (ρ, V) be a regular representation of an algebraic group G. By definition, $\rho: G \to GL(V)$ is a group morphism. Moreover, ρ is continuous because its coefficient maps $\rho_{ij}: G \to \mathbb{C}$ are regular. Thus, ρ is a topological-group morphism. By Section $\boxed{4}$, ρ has a differential $d\rho : \mathfrak{g} = \mathrm{Lie}(G) \to \mathrm{End}(V) = \mathrm{Lie}(GL(V))$ satisfying

$$\exp(d\rho(A)) = \rho(\exp A) \quad \forall A \in \mathfrak{g}.$$

This is an implicit formula for $d\rho(A)$. To get an explicit formula, we observe that $\exp(t\, d\rho(A)) = \exp(d\rho(tA)) = \rho(\exp(tA))$ for all $t \in \mathbb{R}$ and $A \in \mathfrak{g}$. Thus,

$$d\rho(A) = \frac{d}{dt}\Big|_{t=0} \exp(t\, d\rho(A)) = \frac{d}{dt}\Big|_{t=0} \rho(\exp(tA)) \quad \forall A \in \mathfrak{g}. \tag{20}$$

We call $d\rho$ the <u>differential</u> of the representation ρ. The differential version of (19) is

$$X_A f_B^\rho(I_n) = \mathrm{tr}_V(d\rho(A)B) \quad \forall A \in \mathfrak{g}\ \forall B \in \mathrm{End}(V) \tag{21}$$

where $X_A f(a) = \frac{d}{dt}\Big|_{t=0} f(a \exp(tA))$ as defined on page 25 (with $H = GL(n,\mathbb{C})$).

Indeed,
$$\begin{aligned} X_A f_B^\rho(I_n) &= \frac{d}{dt}\Big|_{t=0} f_B^\rho(\exp(tA)) \overset{(19)}{=\!=} \frac{d}{dt}\Big|_{t=0} \mathrm{tr}_V(\rho(\exp(tA))B) \\ &= \frac{d}{dt}\Big|_{t=0} \mathrm{tr}_V(\exp(d\rho(tA)B)) \\ &= \frac{d}{dt}\Big|_{t=0} \mathrm{tr}_V(\exp(t\, d\rho(A)B)) \\ &= \mathrm{tr}_V\Big(\cancel{d\rho(A)} \frac{d}{dt}\Big|_{t=0} \exp(t\, d\rho(A)B)\Big) \\ &= \mathrm{tr}_V(d\rho(A)B). \end{aligned}$$

We defined the regularity for a representation (π, V) using coefficient functions. There is a coordinate-free definition as follows: (π, V) is <u>regular</u> if $\dim_{\mathbb{C}} V < \infty$ and $g \in G \mapsto \langle v^*, \pi(g)v\rangle \in \mathbb{C}$ is a regular function for all $v^* \in V^*$ and $v \in V$. We call $\dim_{\mathbb{C}} V$ the <u>degree</u> of π.

7 Some examples of representations

In this section, we take some examples of representations to compute their differentials, representative functions, and study their regularity, irreducibility.

Example 1 Let G be an algebraic subgroup of $GL(V)$ with $\dim_{\mathbb{C}} V < \infty$. The map $\pi: G \to GL(V)$, $\pi(g)v = gv$ (simply written as $\pi(g) = g$) is a representation of G on V. It is called the defining representation of G. With a fixed choice of basis for V, $\pi(g)$ is represented by a matrix $(\pi_{ij}(g)) = (g_{ij})$. Each component function is a polynomial of $g_{11}, g_{12}, \cdots, g_{nn}$ and thus regular. Thus, π is a regular representation. The differential of (π, V) is $d\pi: \mathfrak{g} = \mathrm{Lie}(G) \longrightarrow \mathrm{End}(V)$ with

$$d\pi(A) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tA)) = \frac{d}{dt}\Big|_{t=0} \exp(tA) = A \quad \forall A \in \mathfrak{g}.$$

Example 2 Let (π, V) be a regular representation of an algebraic group G. Define the dual representation (π^*, V^*) of G as $\pi^*(g)v^* = v^* \circ \pi(g^{-1})$. This is indeed a representation because

$$\pi^*(g_1 g_2)v^* = v^* \circ \pi((g_1 g_2)^{-1}) = v^* \circ \pi(g_2^{-1} g_1^{-1}) = v^* \circ \pi(g_2^{-1}) \circ \pi(g_1^{-1})$$
$$= \pi^*(g_1)(v^* \circ \pi(g_2^{-1})) = \pi^*(g_1) \circ \pi^*(g_2) v^*$$
$$\forall g_1, g_2 \in G, \forall v^* \in V^*.$$

Take $v^{**} \in V^{**}$. There is $v \in V$ such that $\langle v^{**}, v^* \rangle = \langle v^*, v \rangle$ for all $v^* \in V^*$. Here the brackets $\langle \cdot, \cdot \rangle$ denotes duality. Then

$$\langle v^{**}, \pi^*(g)v^* \rangle = \langle \pi^*(g)v^*, v \rangle = \langle v^*, \pi(g^{-1})v \rangle.$$

The map $g \in G \mapsto \langle v^*, \pi(g^{-1})v \rangle \in \mathbb{C}$ is a regular function because π is regular. Thus, π^* is a regular representation. Let (e_i) be a basis of V and (e_i^*) be the dual basis. Then the linear map $V \to V^*$, $e_i \mapsto e_i^*$ identifies V with V^* as vector spaces. By that way, $\mathrm{End}(V)$ is also identified with $\mathrm{End}(V^*)$. Then $\pi^*(g)$ is identified with $\pi(g^{-1})$.

Next, we calculate the representative functions of (π^*, V^*) in terms of those of (π, V). Let $E^\pi = \{f_A : A \in \mathrm{End}(V)\}$, where $f_A(g) = \mathrm{tr}_V(\pi(g)A)$ for $A \in \mathrm{End}(V)$, $g \in G$, be the space of representative functions associated with π. The representative functions associated with π^* are $\{f_C^* : C \in \mathrm{End}(V^*)\}$ where

$$f_C^*(g) = \mathrm{tr}_{V^*}(\pi^*(g)C) = \mathrm{tr}_V(\pi(g^{-1})\bar{C}) = f_{\bar{C}}(g^{-1}) \quad \forall g \in G, \forall C \in \mathrm{End}(V^*)$$

where C is identified with $\bar{C} \in \mathrm{End}(V)$. By identifying V^* with V, we can write $f_C^*(g) = f_C(g^{-1})$.

Now we consider the relation between the irreducibilities of (π, V) and (π^*, V^*). If W is a G-invariant subspace of (π, V), then the space

$$W^\perp = \{v^* \in V^* : \langle v^*, v \rangle = 0 \ \forall v \in W\}$$

is also a G-invariant subspace of (π^*, V^*). Indeed, for all $v^* \in W^\perp$ and $v \in W$

$$(\pi^*(g)v^*)(v) = (v^* \circ \pi(g^{-1}))(v) = \langle v^*, \underbrace{\pi(g^{-1})v}_{\in W} \rangle = 0.$$

This means $\pi^*(g)v^*$ vanishes on W. Thus $\pi^*(g)v^* \in W^*$. We observe that if $W \neq \{0\}, V$ then $W^\perp \neq \{0\}, V^*$. Thus, if (π, V) is reducible then (π^*, V^*) is also reducible. Equivalently, if (π^*, V^*) is irreducible, so is (π, V).

Let (π^{**}, V^{**}) be the dual representation of (π^*, V^*). The map $V^{**} \to V$, $v^{**} \mapsto v$ such that $\langle v^{**}, v^* \rangle = \langle v^*, v \rangle$ for all $v^* \in V^*$ is a linear isomorphism between V^{**} and V. It turns out to be a G-intertwining map. Indeed,

$$\langle \pi^{**}(g)(v^{**}), v^* \rangle = (v^{**} \circ \pi^*(g^{-1}))(v^*) = \langle v^{**}, \pi^*(g^{-1})(v^*) \rangle$$

$$= \langle \pi^*(g^{-1})(v^*), v \rangle$$

$$= (v^* \circ \pi(g))(v)$$

$$= \langle v^*, \pi(g) v \rangle \qquad \forall v^* \in V^*.$$

Hence, (π^{**}, V^{**}) and (π, V) are equivalent represetations on G. This implies that if (π, V) is irreducible then so is (π^*, V^*). Therefore, (π, V) is irreducible if and only if (π^*, V^*) is irreducible.

We now find the connection between the differentials $d\pi^*$ and $d\pi$. Recall

$$d\pi^* : \mathfrak{g} = \mathrm{Lie}(G) \to \mathrm{End}(V^*), \qquad d\pi : \mathfrak{g} = \mathrm{Lie}(G) \to \mathrm{End}(V).$$

For $A \in \mathfrak{g}$ and $B \in \mathrm{End}(V^*)$,

$$d\pi^*(A)B \overset{(20)}{=\!=} \frac{d}{dt}\Big|_{t=0} \pi^*(\exp(tA))B = \frac{d}{dt}\Big|_{t=0} B \circ \pi(\exp(tA)^{-1})$$

$$= \frac{d}{dt}\Big|_{t=0} B \circ \pi(\exp(-tA)) = B \circ \left(\frac{d}{dt}\Big|_{t=0} \pi(\exp(-tA)) \right)$$

$$= B \circ d\pi(-A) = -B \circ d\pi(A).$$

Thus, $d\pi^*(A) = -d\pi(A)^t$ for all $A \in \mathfrak{g}$.

Example 3 Let (π_1, V_1) and (π_2, V_2) be two regular representations of an algebraic group G. Define the direct sum representation $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ by

$$(\pi_1 \oplus \pi_2)(g)(v_1 \oplus v_2) = \pi_1(g)v_1 \oplus \pi_2(g)v_2 \quad \forall g \in G,\ v_1 \in V_1,\ v_2 \in V_2.$$

Then $\pi = \pi_1 \oplus \pi_2$ is a representation of G on $V = V_1 \oplus V_2$ because

$$\pi(g_1 g_2)(v_1 \oplus v_2) = \pi_1(g_1 g_2)v_1 \oplus \pi_2(g_1 g_2)v_2 = \pi_1(g_1)\circ \pi_1(g_2)v_1 \oplus \pi_2(g_1)\circ\pi_2(g_2)v_1$$

$$= \pi(g_1)\left(\pi_1(g_2)v_1 \oplus \pi_2(g_2)v_2\right)$$

$$= \pi(g_1)\circ\pi(g_2)(v_1 \oplus v_2).$$

For $v_1^* \in V_1^*$, $v_2^* \in V_2^*$, $v_1 \in V_1$, $v_2 \in V_2$,

$$\langle v_1^* \oplus v_2^*, \pi(g)(v_1 \oplus v_2)\rangle = \langle v_1^* \oplus v_2^*, \pi_1(g)v_1 \oplus \pi_2(g)v_2\rangle$$

$$= \underbrace{\langle v_1^*, \pi_1(g)v_1\rangle}_{\text{regular function}} + \underbrace{\langle v_2^*, \pi_2(g)v_2\rangle}_{\text{regular function}} \qquad (22)$$

Thus, (π, V) is a regular representation of G.

Recall that the space of representative functions associate with π_1 is given by $E^{\pi_1} = \{ f_A^{\pi_1}(g) = tr_{V_1}(\pi_1(g)A) : A \in End(V_1)\}$. It is generated by the functions of the form $g \in G \mapsto \langle v_1^*, \pi_1(g)v_1\rangle \in \mathbb{C}$ for $v_1 \in V_1$, $v_1^* \in V_1^*$. Thus, by (22) we have $E^{\pi} = E^{\pi_1} + E^{\pi_2}$.

Now let us compute the differential of π. By definition,

$$d\pi(A) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tA)) \quad \forall A \in \mathfrak{g} = Lie(G).$$

For $v_1 \in V_1$, $v_2 \in V_2$, $A \in \mathfrak{g}$,

$$d\pi(A)(v_1 \oplus v_2) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tA))(v_1 \oplus v_2)$$

$$= \frac{d}{dt}\Big|_{t=0} \left[\pi_1(\exp(tA))v_1 \oplus \pi_2(\exp(tA))v_2\right]$$

$$= \frac{d}{dt}\Big|_{t=0} \pi_1(\exp(tA))v_1 \oplus \frac{d}{dt}\Big|_{t=0} \pi_2(\exp(tA))v_2$$

$$= d\pi_1(A)v_1 \oplus d\pi_2(A)v_2.$$

Therefore, $d\pi(A) = d\pi_1(A) \oplus d\pi_2(A)$.

Example 4 Let (π_1, V_1) and (π_2, V_2) be two regular representations of an algebraic group G. Define the tensor product representation $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ by

$$(\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2 \quad \forall g \in G, v_1 \in V_1, v_2 \in V_2.$$

Then $\pi = \pi_1 \otimes \pi_2$ is a representation of G on $V = V_1 \otimes V_2$ because

$$\pi(g_1g_2)(v_1 \otimes v_2) = \pi_1(g_1g_2)v_1 \otimes \pi_2(g_1g_2)v_2 = \pi_1(g_1)\circ\pi_1(g_2)v_1 \otimes \pi_2(g_1)\circ\pi_2(g_2)v_2$$

$$= \pi(g_1)(\pi_1(g_2)v_1 \otimes \pi_2(g_2)v_2)$$

$$= \pi(g_1)\circ\pi(g_2)(v_1 \otimes v_2).$$

For $v_1^* \in V_1^*$, $v_2^* \in V_2^*$, $v_1 \in V_1$, $v_2 \in V_2$,

$$\langle v_1^* \otimes v_2^*, \pi(g)(v_1 \otimes v_2)\rangle = \langle v_1^* \otimes v_2^*, \pi_1(g)v_1 \otimes \pi_2(g)v_2\rangle$$

$$= \underbrace{\langle v_1^*, \pi_1(g)v_1\rangle}_{\text{regular}} \underbrace{\langle v_2^*, \pi_2(g)v_2\rangle}_{\text{regular}}. \qquad (23)$$

Thus, (π, V) is a regular representation of G. Moreover, (23) implies that E^{π} is the linear span of $E^{\pi_1} \cdot E^{\pi_2}$.

Now let us compute the differential of π. By definition, for $v_1 \in V_1$, $v_2 \in V_2$, $A \in \mathfrak{g}$,

$$d\pi(A)(v_1 \otimes v_2) = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tA))(v_1 \otimes v_2)$$

$$= \frac{d}{dt}\Big|_{t=0} [\pi_1(\exp(tA))v_1 \otimes \pi_2(\exp(tA))v_2]$$

$$= \frac{d}{dt}\Big|_{t=0} \pi_1(\exp(tA))v_1 \otimes \pi_2(\exp(0A))v_2 + \pi_1(\exp(0A))v_1 \otimes \frac{d}{dt}\Big|_{t=0} \pi_2(\exp(tA))v_2$$

$$= d\pi_1(A)v_1 \otimes v_2 + v_1 \otimes d\pi_2(A)v_2$$

Thus, $d\pi(A) = d\pi_1(A) \otimes id_{V_2} + id_{V_1} \otimes d\pi_2(A)$.

Example 5 (Adjoint representation)

Let G be an algebraic group and $\mathfrak{g} = Lie(G)$. Because $\mathfrak{g}$ is a vector space, we can speak of representations of G on $\mathfrak{g}$. There is such a representation, called the adjoint representation. Define

$$Ad: G \to GL(\mathfrak{g}), \quad Ad(g)X = gXg^{-1} \quad \forall g \in G, \forall X \in \mathfrak{g}. \qquad (24)$$

First we check if Ad is well-defined. For $g \in G$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$,

$$\exp(tgXg^{-1}) = \exp(gtXg^{-1}) \overset{(7)}{=\!=} \sum_{k=0}^{\infty} \frac{(gtXg^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{g(tX)^k g^{-1}}{k!}$$

$$= g\left(\sum_{k=1}^{\infty} \frac{(tX)^k}{k!}\right)g^{-1} = g\underbrace{\exp(tX)}_{\in G}g^{-1} \in G.$$

Therefore, $gXg^{-1} \in \mathfrak{g}$. For a fixed $g \in G$, the map $X \in \mathfrak{g} \mapsto gXg^{-1} \in \mathfrak{g}$ is a linear automorphism of $\mathfrak{g}$. Thus, $Ad(g) \in GL(\mathfrak{g})$.

Next, we check if Ad is a group morphism. For $g_1, g_2 \in G$,

$$Ad(g_1g_2)X = g_1g_2X(g_1g_2)^{-1} = g_1g_2Xg_2^{-1}g_1^{-1} = g_1(Ad(g_2)X)g_1^{-1} = Ad(g_1)\circ Ad(g_2)X.$$

Thus, $Ad(g_1g_2) = Ad(g_1)Ad(g_2)$. Hence, $(Ad, \mathfrak{g})$ is a representation of G. The matrix representation of $Ad(g) = (X \mapsto gXg^{-1})$ has coefficients which are polynomials of the coefficients of g and g^{-1}. Thus, they belong to

$\mathbb{C}[g_{11}, g_{12}, \dots, g_{nn}, \det(g)^{-1}]$. Hence, $(\mathrm{Ad}, \mathfrak{g})$ is a regular representation. The ~~matrix~~ representative functions associate with $(\mathrm{Ad}, \mathfrak{g})$ don't have simple forms, so we don't compute them here.

We now compute the differential of $(\mathrm{Ad}, \mathfrak{g})$. Denote $\mathrm{ad} = d(\mathrm{Ad})$. By definition, for $X, Y \in \mathfrak{g}$,

$$
\begin{aligned}
\mathrm{ad}(X)Y = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX))Y &= \frac{d}{dt}\Big|_{t=0} \exp(tX)\, Y\, (\exp(tX))^{-1} \\
&= \frac{d}{dt}\Big|_{t=0} \exp(tX)\, Y \exp(-tX) \\
&= \left(\frac{d}{dt}\Big|_{t=0} \exp(tX)\right) Y \exp(0) + \exp(0)\, Y \frac{d}{dt}\Big|_{t=0} \exp(-tX) \\
&= XY - YX \\
&= [X, Y].
\end{aligned}
$$

Therefore, $\mathrm{ad}(X)Y = [X, Y] \quad \forall X, Y \in \mathfrak{g}$. $\qquad (25)$

Example 6 Let G be an algebraic group and $\mathcal{O}[G]$ be the space of regular functions on G. (as introduced on page 34). There are two well-known representations of G on $\mathcal{O}[G]$, namely the <u>left translation</u> and <u>right translation</u>.

$$
L: G \to GL(\mathcal{O}[G]), \qquad R: G \to GL(\mathcal{O}[G]),
$$

$$
L(g)f(x) = f(g^{-1}x). \qquad R(g)f(x) = f(xg).
$$

Unless G is the trivial group, $\mathcal{O}[G]$ is infinite-dimensional. Thus, both L and R are not regular representations. However, we will point out that they are locally regular.

Let E be a finite-dimensional subspace of $\mathcal{O}[G]$. Let $\{f_1|_G, f_2|_G, \dots, f_m|_G\}$, with $f_i \in \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, \det(x)^{-1}]$, be a basis of E. Put

$$F = \text{linear span } \{ L(g) f_i|_G : g \in G, 1 \le i \le m \}.$$

Each element of F is of the form $f|_G = \sum_{i=1}^{m} c_i L(g_i) f_i|_G$. Thus,

$$f(x) = \sum_{i=1}^{m} c_i f_i(g_i^{-1} x). \qquad (*)$$

If x_{ij} denotes the map on $M_n(\mathbb{C})$ which maps a matrix A to the entry A_{ij}, then

$$x_{kj}(hg) = \sum_{r=1}^{n} x_{kr}(h)\, x_{rj}(g).$$

Thus, $x_{kj}(g_i^{-1} x) = \sum_{r=1}^{n} d_{kjir}\, x_{rj}(x) \quad \forall x \in GL(n, \mathbb{C})$,

where d_{kjir} is a constant. Because $f_i(g_i^{-1}x)$ is a polynomial of $x_{kj}(g_i^{-1}x)$, $1 \le k, j \le n$, ~~and~~ we can write $f_i(g_i^{-1}x) = \sum_{j=1}^{m_i} d_{ij} f_{ij}(x)$, where $f_{ij} \in \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, \det(x^{-1})]$.

Then $(*)$ can be written as $\quad f(x) = \sum_{i=1}^{m} \sum_{j=1}^{m_i} c_i d_{ij} f_{ij}(x) \quad \forall x \in GL(n, \mathbb{C})$.

Thus, F is finite-dimensional. Because each $f_i = L(I_n) f_i \in F$, $E \subset F$. Moreover, $L(g_1)(L(g_2) f_i) = L(g_1 g_2) f_i \in F$ for all $g_1, g_2 \in G$, $1 \le i \le m$. Thus, F is a G-invariant subspace of $(L, \mathcal{O}[G])$. Hence, $(L, \mathcal{O}[G])$ is locally regular. Similarly, $(R, \mathcal{O}[G])$ is also ~~regularly~~ locally regular.

Example 7 (Regular representations of $\mathbb{C}$)

Consider the additive group $(\mathbb{C}, +)$. In order to speak of representations of $(\mathbb{C}, +)$, we must by some way embed it into a matrix group $GL(n, \mathbb{C})$. The

following map is a group morphism $(\mathbb{C},+) \to GL(2,\mathbb{C})$, $z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$.
It is injective, so $(\mathbb{C},+)$ can be identified with the subgroup $G = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$ of $GL(2,\mathbb{C})$. Moreover, G is an algebraic group because it is the zero set of the polynomials $x_{11}-1$, x_{21}, $x_{22}-1$. We now determine all regular representations of G. Let (π, V) be a regular representation of G. It is not obvious how to characterize π even though G looks simple.
As a manifold, G is of dimension 2. Its Lie algebra $\mathfrak{g}$ is of the same topological dimension (Theorem 3, page 11). Thus, as a vector space over $\mathbb{C}$, $\mathfrak{g}$ is 1-dimensional. The differential $d\pi : \mathfrak{g} \to \mathrm{End}(V)$ is a linear map, so it is particularly simple:

$$d\pi(zB) = z \underbrace{d\pi(B)}_{= A \in \mathrm{End}(V)} = zA \qquad \forall z \in \mathbb{C} \qquad (*)$$

where B is any nonzero element in $\mathfrak{g}$. Once $d\pi$ is given by $(*)$, we can determine π from the relation $\pi(\exp X) = \exp d\pi(X)$.

$$\pi(\exp(zB)) = \exp d\pi(zB) = \exp(zA) \qquad \forall z \in \mathbb{C} \quad (**)$$

Now we want to find explicitly a nonzero element $B \in \mathfrak{g}$. That is to find $B \neq 0$ such that $\exp(tB) \in G$ for all $t \in \mathbb{R}$.

We can take $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ because $B^2 = 0$ and thus,

$$\exp(tB) = I_2 + tB + \sum_{k=2}^{\infty} \frac{(tB)^k}{k!} = I_2 + tB = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \forall t \in \mathbb{R}.$$

Actually, we even have $\exp(zB) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ for all $z \in \mathbb{C}$. Then $(**)$ becomes

$$\pi\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp(zA) \quad \forall z \in \mathbb{C}.$$

With the identification of $(\mathbb{C},+)$ with G, we can write

$$\pi(z) = \exp(zA) \quad \forall z \in \mathbb{C} \qquad (26)$$

for some $A \in \operatorname{End}(V)$. Conversely, let π be a map given by (26). Then π is a representation of G. Because $G = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}\right\}$, the determinant of every element of G is equal to 1. Thus, $\mathcal{V}[G] = \mathbb{C}[z]$. Thus, π is a regular representation if and only if every coefficient of the matrix $\exp(zA)$ is in $\mathbb{C}[z]$. We know that this happens if and only if $\left.\frac{d^k}{dz^k}\right|_{z=0} \exp(zA)$ is identically zero for some $k \in \mathbb{N}$. Since

$$\left.\frac{d^k}{dz^k}\right|_{z=0} \exp(zA) = A^k \quad \forall k \in \mathbb{N},$$

it is required that $A^k = 0$ for some $k \in \mathbb{N}$. Therefore, all regular representations of $(\mathbb{C},+)$ are given by (26) where A is a nilpotent element of $\operatorname{End}(V)$.

Example 8 (Regular representations of $\mathbb{C}^\times$)

We want to characterize all regular representations of the multiplicative group $\mathbb{C}^\times = GL(1,\mathbb{C})$. Let $(\varphi, \mathbb{C}^n)$ be such a representation. Then $\varphi : \mathbb{C}^\times \to GL(n,\mathbb{C})$ is a group morphism. Because $\mathcal{V}[\mathbb{C}^\times] = \mathbb{C}[z, z^{-1}]$, every coefficient of the matrix $\varphi(z)$ is in $\mathbb{C}[z,z^{-1}]$. Equivalently, it is a linear combination of z^k, $k \in \mathbb{Z}$. Thus, we can write

$$\varphi(z) = \sum_{k \in \mathbb{Z}} z^k T_k \quad \forall z \in \mathbb{C}^\times \qquad (*)$$

where $T_k \in M_n(\mathbb{C})$ is a constant matrix. The identity $\varphi(z_1 z_2) = \varphi(z_1)\varphi(z_2)$

becomes $\sum_{k\in\mathbb{Z}} z_1^k z_2^k T_k = \sum_{i,j\in\mathbb{Z}} z_1^i z_2^j T_i T_j \quad \forall z_1, z_2 \in \mathbb{C}^\times$.

This is equivalent to $\begin{cases} T_i T_j = 0 & \forall i,j\in\mathbb{Z},\ i\neq j \\ T_k^2 = T_k & \forall k\in\mathbb{Z}. \end{cases}$

By $(*)$, $I_n = \varphi(1) = \sum_{k\in\mathbb{Z}} T_k$. Thus, $v = \sum_{k\in\mathbb{Z}} T_k v \quad \forall v\in\mathbb{C}^n$, with the understanding that only finitely many summands are nonzero.

For $v\in\mathbb{C}^n$ and $k\in\mathbb{Z}$, put $v_k = T_k v$. Then $T_k v_k = T_k^2 v = T_k v = v_k$. Thus, v_k belongs to the set $E_k = \{v\in\mathbb{C}^n : T_k v = v\}$. Thus, $\mathbb{C}^n$ is the sum of the vector spaces E_k, $k\in\mathbb{Z}$. Because $T_i T_j = 0$ for $i\neq j$, $E_i\cap E_j = \cancel{\emptyset}\,\{0\}$. Thus,

$$\mathbb{C}^n = \bigoplus_{k\in\mathbb{Z}} E_k. \qquad (**)$$

For each $v\in E_k$, $\varphi(z)v \overset{(*)}{=\!=} \sum_{i\in\mathbb{Z}} z^i T_i v = z^k T_k v = z^k v$ for all $z\in\mathbb{C}^\times$.

Conversely, if $v\in\mathbb{C}^n$ and $\varphi(z)v = z^k v$ for all $z\in\mathbb{C}^\times$ then $(*)$ implies $T_k v = v$ and $T_i v = 0$ for all $i\neq k$. Therefore,

$$E_k = \{v\in\mathbb{C}^n : \varphi(z)v = z^k v \ \ \forall z\in\mathbb{C}^\times\}. \qquad (***)$$

$(**)$ and $(***)$ give us the structure of the representation $(\varphi, \mathbb{C}^n)$. If $\mathbb{C}^n = \bigoplus_{k\in\mathbb{Z}} F_k$ is a decomposition of vector spaces then we can recover a regular representation $(\varphi, \mathbb{C}^n)$ of $\mathbb{C}^\times$ in which $E_k = F_k$. Indeed, define

$$\varphi(z)v := z^k v \quad \forall z\in\mathbb{C}^\times\ \forall v\in F_k.$$

This furnishes a regular representation of $\mathbb{C}^\times$ on $\mathbb{C}^n$. We have obtained a characterization of regular representations of $\mathbb{C}^\times$.

As a final remark, we see from (***) that E_k is a $\mathbb{C}^\times$-invariant subspace of $(\varphi, \mathbb{C}^n)$. In fact, every subspace of E_k is $\mathbb{C}^\times$-invariant. Thus, $(\varphi, \mathbb{C}^n)$ is irreducible if and only if $n=1$. Also, all regular representations of $\mathbb{C}^\times$ are <u>completely reducible</u>, a notion to be introduced in Part 2.

8 Jordan decomposition

In linear algebra, there is a well-known result saying that every matrix in $M_n(\mathbb{C})$ can be written uniquely as the sum of a diagonalizable matrix and a nilpotent matrix that commute with each other. ~~Moreover, these two matrices~~ Such a representation is called <u>additive Jordan decomposition</u>.

Here is a way to see why this is true. Let $A \in M_n(\mathbb{C})$. The map $\mathbb{C}[z] \to \mathrm{End}(\mathbb{C}^n)$, $f \mapsto f(A)$ is a ring-representation of $\mathbb{C}[z]$ on $\mathbb{C}^n$. Thus, $\mathbb{C}^n$ is a $\mathbb{C}[z]$-module. Since $\mathbb{C}[z]$ is a principal ring, $\mathbb{C}^n$ is a finitely generated module over a principal ring. Let $q(z) = (z-\lambda_1)^{r_1} \cdots (z-\lambda_m)^{r_m}$ be the characteristic polynomial of A. We know that $q(A)=0$ (Cayley-Hamilton theorem). Thus, $q(z)$ is an exponent of $\mathbb{C}^n$. By the structure theorem of finitely generated modules over a principal ring (Theorem 7.5, Lang "Algebra" p. 149),

$$\mathbb{C}^n = E(\lambda_1) \oplus E(\lambda_2) \oplus \cdots \oplus E(\lambda_m)$$

where $E(\lambda_k) = \ker(A-\lambda_k I)^{r_k}$ is an invariant submodule of $\mathbb{C}^n$. By Chinese Remainder theorem, there exists a polynomial $p(z) \in \mathbb{C}[z]$ such that

$p(z) \equiv \lambda_k \mod (z-\lambda_k)^{r_k}$ for all $1 \le k \le m$. Put $S = p(A) \in M_n(\mathbb{C})$.
Because $(z-\lambda_k)^{r_k} \mid (p(z)-\lambda_k)$, $\ker(A-\lambda_k I)^{r_k} \subset \ker(p(A)-\lambda_k I)$. Thus, $E(\lambda_k) \subset \ker(S-\lambda_k I)$, which implies that S acts by scalar in each space $E(\lambda_k)$. Thus, S is diagonalizable. Put $N = A-S$. Then

$$Nv = Av - Sv = (A-\lambda_k I)v \quad \forall v \in E_k \ \forall 1 \le k \le n.$$

Thus, $N^n v = (A-\lambda_k I)^n v = 0$ for all $v \in E_k$ and $1 \le k \le n$. Thus, $N^n = 0$. In other words, N is nilpotent. Moreover,

$$SN = p(A) \circ (1-p)(A) = (p(1-p))(A) = ((1-p)p)(A) = (1-p)(A) \circ p(A) = NS.$$

Now suppose that $A = S_1 + N_1$ where S_1 is diagonalizable, N_1 is nilpotent and $S_1 N_1 = N_1 S_1$. Write

$$\mathbb{C}^n = F(\lambda_1') \oplus \cdots \oplus F(\lambda_r')$$

as the decomposition of $\mathbb{C}^n$ into eigenspaces of S_1. Then each of them is also invariant under A because $S_1 A = A S_1$. By the uniqueness part of the structure theorem, $r = m$ and $F(\lambda_1') = E(\lambda_1), \ldots, F(\lambda_m') = E(\lambda_m)$. Hence, S acts by scalar-multiplication on each $E(\lambda_i)$. Write $Sv = \lambda_i' v$ for every $v \in E(\lambda_i)$.

Then $$0 = N_1^n v = (A-S_1)^n v = (A-\lambda_i' I)^n v \quad \forall v \in E(\lambda_i).$$

Hence, $(x-\lambda_i')^n$ is an exponent of $E(\lambda_i)$. Since $(x-\lambda_i)^{r_i}$ is also an exponent of $E(\lambda_i)$ and $E(\lambda_i) \neq \{0\}$, $\lambda_i' = \lambda_i$. Thus, S_1 acts by multiplying by λ_i on each space $E(\lambda_i)$. This implies $S_1 = S$.

The additive Jordan decomposition in $M_n(\mathbb{C})$ implies the <u>multiplicative Jordan decomposition</u> in $GL(n,\mathbb{C})$, which says that every matrix in $GL(n,\mathbb{C})$ can be written uniquely as a product of a diagonalizable matrix and a unipotent matrix that commute with each other. Indeed, let $A \in GL(n,\mathbb{C})$ and $A = S+N$ be its additive Jordan decomposition. Because all eigenvalues of S are those of A, none of which is zero because A is invertible, S is also invertible. Thus, $A = S(I_n + S^{-1}N)$. Since S and N commute with each other, $(S^{-1}N)^n = S^{-n}N^n = 0$. Thus, $U = I_n + S^{-1}N$ is unipotent. Moreover, $SU = US$. The uniqueness follows from the uniqueness of the additive Jordan decomposition.

It is a very interesting fact that these two types of Jordan decompositions are still true when we replace $GL(n,\mathbb{C})$ by an algebraic subgroup of G, and $M_n(\mathbb{C})$ by $\mathfrak{g} = \mathrm{Lie}(G)$.

<u>Theorem 9</u> Let G be an algebraic subgroup of $GL(n,\mathbb{C})$ and $\mathfrak{g} = \mathrm{Lie}(G)$. Then we have two following statements.

(i) For each $A \in \mathfrak{g}$, the additive Jordan decomposition of A as a matrix in $M_n(\mathbb{C})$, say $A = S+N$, satisfies $S, N \in \mathfrak{g}$.

(ii) For each $A \in G$, the multiplicative Jordan decomposition of A as a matrix in $GL(n,\mathbb{C})$, say $A = SU$, satisfies $S, U \in G$.

More suggestive notations for S, N, U are A_s, A_N, A_u. The following theorem says that Jordan decompositions are preserved by algebraic-group morphisms and differentials.

Theorem 10 Let $\varphi : G \to H$ be a morphism between two algebraic groups G and H. Then

(i) $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$ for all $g \in G$.

(ii) $d\varphi(A_s) = (d\varphi(A))_s$ and $d\varphi(A_N) = (d\varphi(A))_N$.

Before going into the proof of Theorem 9, it is important for us to be aware of a nontrivial connection between unipotents and nilpotents in $M_n(\mathbb{C})$. Namely, if U is a unipotent then there exists a nilpotent N such that $U = \exp(N)$. Indeed, there is a number $z \in \mathbb{C} \setminus \{w \in \mathbb{C} : Re(w) \leq 0, Im(w) = 0\}$ such that $z^{-1}U \in B(I_n, 1)$. By a property of the exponential map on page 11, there is a matrix $A \in B(0, \log 2)$ such that $z^{-1}U = \exp(A)$. Write $z = \exp(w)$ with some $w \in \mathbb{C}$, $-\pi < Im(w) \leq \pi$. Then $U = z\exp(A) = \exp(w)\exp(A) = \exp(wI_n + A)$. Put $A' = wI_n + A$. Let λ be an eigenvalue of A'. Then e^{λ} is an eigenvalue of U, which is 1. Thus $\lambda = k2\pi i$ for some $k \in \mathbb{Z}$. Because $\lambda - w$ is an eigenvalue of A and $A \in B(0, \log 2)$, $|\lambda - w| < \log 2$. Thus,

$$|Im\,\lambda| \leq |Im\,w| + |Im(\lambda - w)| \leq \pi + |\lambda - w| < \pi + \log 2 < 2\pi.$$

Thus, $\lambda = 0$. Since all eigenvalues of A' are zero, A' is nilpotent.

Proof of Theorem 9

(i) Since $SN = NS$, $\exp(tS)\exp(tN) = \exp(t(S+N)) = \exp(tA) \in G$. Thus, it suffices to show that $\exp(tS) \in G$ for all $t \in \mathbb{R}$. Because G is an algebraic group, it is the zero set of a family of polynomials I_G in $\mathbb{C}[g_{11}, g_{12}, \dots, g_{nn}]$. To show that $\exp(tS) \in G$ for all $t \in \mathbb{R}$ is to show that $\exp(tS)$ vanishes all polynomials in I_G.

Let $R: G \to GL(\mathcal{V}[GL(n,\mathbb{C})])$ be the right-translation representation of G (see Section $\boxed{7}$, Example 6). Put $g = \exp(tS) \in GL(n,\mathbb{C})$.

$$R(g)f(x) = f(xg) \qquad \forall f \in \mathcal{V}[GL(n,\mathbb{C})] \quad \forall x \in GL(n,\mathbb{C}).$$

Thus, we need to show that $R(g)f(x) = 0$ for all $f \in I_G$ and $x \in G$.

Because S is diagonalizable, g is too. Indeed, if $S = P^{-1}\operatorname{diag}(\lambda_1, \dots, \lambda_n)P$ for some $P \in GL(n,\mathbb{C})$ then $g = P^{-1}\operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})P$. Assume that the basis of $\mathbb{C}^n$ is chosen so that g is diagonal. Write $g = \operatorname{diag}(g_1, \dots, g_n)$. Each function $f \in \mathcal{V}[GL(n,\mathbb{C})]$ is a polynomial in $\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, \det(x)^{-1}]$. Thus, $\mathcal{V}[GL(n,\mathbb{C})]$ as a ring is generated by the functions const, $x_{11}, x_{12}, \dots, x_{nn}$, $\det(x)^{-1}$. We have

$$R(g)x_{ij}(a) = x_{ij}(ag) = g_j a_{ij} = g_j x_{ij}(a)$$

$$R(g)\det^{-1}(a) = \det^{-1}(ag) = \cancel{\deg}\ \det(g)^{-1}\det^{-1}(a)$$

$$R(g)\underbrace{c(a)}_{\text{const}} = c(ag) = c = c(a).$$

Thus, $R(g)$ acts by scalar-multiplication on the basis of $\mathcal{V}[GL(n,\mathbb{C})]$. As a vector space over $\mathbb{C}$, $\mathcal{V}[GL(n,\mathbb{C})]$ is generated by the functions

$\det(x)^{-r} x_{11}^{r_{11}} x_{12}^{r_{12}} \cdots x_{nn}^{r_{nn}}$ where $r, r_{ij} \geq 0$. Because $R(g)$ is not only a linear automorphism of $\mathcal{V}[GL(n,\mathbb{C})]$ but also a ring morphism, it acts by scalar-multiplication on each of these functions. If $\mathcal{V}[GL(n,\mathbb{C})]$ were a finite-dimensional vector space over $\mathbb{C}$, $R(g)$ would be a diagonalizable transformation. However, we are close to that because $(R, \mathcal{V}[GL(n,\mathbb{C})])$ is a locally regular representation according to Section $\boxed{7}$, Example 6.

Now fix $f \in I_G$. We want to show that $R(g)f(x) = 0$ for all $x \in G$. There is a finite-dimensional G-invariant subspace W of $(R, \mathcal{V}[GL(n,\mathbb{C})])$ that contains f. Thus, $R(g)|_W : W \to W$ is a diagonalizable transformation. We have showed that $R(\exp(tS))|_W$ is a diagonalizable matrix in $GL(W)$. Next we show that $R(\exp(tN))|_W$ is ~~nil~~unipotent in $GL(W)$. Suppose that this is proved. Since $R : G \to GL(\mathcal{V}[GL(n,\mathbb{C})])$ is a group morphism,

$$R(\exp(tA))|_W = R(\exp(tS)\exp(tN))|_W = R(\exp(tS))|_W R(\exp(tN))|_W$$
$$= R(\exp(tN))|_W R(\exp(tS))|_W.$$

This is the multiplicative Jordan decomposition of $R(\exp(tA))|_W$ in $GL(W)$. Thus, $R(\exp(tS))|_W$ is a polynomial of $R(\exp(tA))|_W$. In other words, there is a polynomial $p(z) \in \mathbb{C}[z]$ such that $R(\exp(tS))|_W = p(R(\exp(tA)))|_W$. Thus,

$$R(\exp(tS))f = p(R(\exp(tA)))f = \sum_{k=1}^{r} \alpha_k R(\exp(tA))^k f.$$

For each $x \in G$, $R(\exp(tS))f(x) = \sum_{k=1}^{r} \alpha_k f(\underbrace{x \underbrace{\exp(ktA)}_{\in G}}_{\in G}) = 0.$

Thus, all we need is to show that $R(\exp(tN))|_W$ is unipotent in $GL(W)$. ~~We will show even more generally that $R(\exp(tN)) - id_{V[GL(n,\mathbb{C})]}$ is nilpotent in $End(V[GL(n,\mathbb{C})])$~~. For each $h \in V[GL(n,\mathbb{C})]$,

$$\left(R(\exp(tN)) - id_{V[GL(n,\mathbb{C})]}\right) h(x) = h(x\exp(tN)) - h(x).$$

Put $\varphi(t) = h(x\exp(tN))$. Then $\varphi'(0) = \frac{d}{dt}\Big|_{t=0} h(x\exp(tN)) = X_N h(x)$ (review the notation X_A on page 25). Similarly, $\varphi^{(k)}(0) = X_N^k h(x)$ for all $k \in \mathbb{N}$. Since φ is analytic,

$$\varphi(t) = \varphi(0) + \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{k!} t^k. \qquad (*)$$

Because $h \in V[GL(n,\mathbb{C})]$, $h(x\exp(tN))$ is a polynomial of the coefficients of $x\exp(tN) = x\left(I_n + \sum_{k=1}^{n-1} \frac{(tN)^k}{k!}\right)$ and $\det(x\exp(tN))^{-1} = \det(x)^{-1}\exp(\underbrace{tr(tN)}_{=0})$

$$= \det(x)^{-1}.$$

Thus, $\varphi(t) = h(x\exp(tN))$ is a polynomial of t. We then infer from $(*)$ that there is $n_0 \in \mathbb{N}$ such that $\varphi^{(k)}(0) = 0$ for all $k > n_0$. Thus, $X_N^k h(x) = 0$ for all $k > n_0$. We can write $(*)$ as

$$h(x\exp(tN)) - h(x) = \sum_{k=1}^{n_0} \frac{X_N^k h(x)}{k!} t^k.$$

Thus, $\left(R(\exp(tN)) - id_{V[GL(n,\mathbb{C})]}\right) h = \left(\sum_{k=1}^{n_0} X_N^k \frac{t^k}{k!}\right) h.$

Note that n_0 depends on h. Thus, we could have denoted n_0 by n_h instead. Let $(h_1, \dots, h_m)$ be a basis of W and $n' = \max\{n_{h_1}, \dots, n_{h_m}\}$. Then

$$\left(R(\exp(tN)) - id_{\mathcal{V}[GL(n,\mathbb{C})]}\right) h = \left(\sum_{k=1}^{n'} X_N^k \frac{t^k}{k!}\right) h \quad \forall h \in W.$$

Thus, $$R(\exp(tN))|_W - id_W = \sum_{k=1}^{n'} X_N^k \frac{t^k}{k!}.$$

As an element in $End(W)$, X_A is nilpotent. Thus, $R(\exp(tN))|_W - id_W$ is also nilpotent.

(ii) For $A \in G \subset GL(n,\mathbb{C})$, we have the Jordan multiplication $A = SU$ with $S, U \in GL(n,\mathbb{C})$. Note that S is the same as the diagonal summand in the additive Jordan decomposition of A. To show that $S, U \in G$, it suffices to show $S \in G$. That is to show S vanishes all polynomials in I_G. Equivalently, we need to show that ~~$R(S)$ vanishes on I_G~~ $R(S)f(x) = 0$ for all $f \in I_G$, $x \in G$.

Suppose that the basis of $\mathbb{C}^n$ is chosen so that S is diagonal. By repeating the arguments in Part (i), replacing g by S, we can show that $R(S)$ acts by scalar-multiplication on the basis of $\mathcal{V}[GL(n,\mathbb{C})]$. Fix $f \in I_G$ and let W be a finite-dimensional G-invariant subspace of $(R, \mathcal{V}[GL(n,\mathbb{C})])$ that contains f. Then $R(S)|_W$ is a diagonalizable matrix in $GL(W)$. Next, we show that $R(U)|_W$ is unipotent in $GL(W)$. Suppose that this is proved. Then

$$R(A)|_W = R(S)|_W \, R(U)|_W = R(U)|_W \, R(S)|_W$$

is the Jordan decomposition of $R(A)|_W$ in $GL(W)$. Thus, $R(S)|_W$ is a polynomial of $R(A)|_W$. Write $R(S)|_W = p(R(A))|_W$ where $p(z) \in \mathbb{C}[z]$. Then

$$R(S)f(x) = p(R(A))f(x) = \sum_{k=0}^{r} \alpha_k R(A)^k f(x) = \sum_{k=0}^{r} \alpha_k R(A^k) f(x)$$

$$= \sum_{k=0}^{r} \alpha_k f(\underbrace{xA^k}_{\in G}) = 0 \quad \forall x \in G.$$

Thus, $R(S)f(x) = 0$ for all $x \in G$.

Hence, all we need to do is showing that $R(U)|_W$ is unipotent in $GL(W)$. Since U is unipotent, there exists an nilpotent matrix $\tilde{N} \in M_n(\mathbb{C})$ such that $U = \exp(\tilde{N})$. From now, the proof is just a repetition of the proof that $R(\exp(tN))|_W - id_W$ is nilpotent in Part (i), provided that tN is replaced by $\tilde{N}$.

Update on 10/3/2014.

Right after Theorem 5 on page 24, we would like to insert an important property of differentials, namely the functorial property. It implies that isomorphic closed groups have isomorphic Lie algebras.

Theorem 5' Let $G \subset GL(n,\mathbb{C})$, $H \subset GL(m,\mathbb{C})$, $K \subset GL(l,\mathbb{C})$ be closed subgroups, and $\pi: G \to H$, $\rho: H \to K$ be topological-group morphisms. Then $d(\rho\circ\pi) = d\rho \circ d\pi$.

For the proof, we denote $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ the Lie algebras of G, H, K respectively. By the definition of differential of a topological-group morphism, $d(\rho\circ\pi)$ is the unique Lie-algebra morphism $X: \mathfrak{g} \to \mathfrak{k}$ such that $\rho\circ\pi(\exp(A)) = \exp(X(A))$ for all $A \in \mathfrak{g}$. Because $d\rho: \mathfrak{h} \to \mathfrak{k}$ and $d\pi: \mathfrak{g} \to \mathfrak{h}$ are Lie-algebra morphisms, the composition $d\rho \circ d\pi$ is also a Lie-algebra morphism. Thus, all we need to show is that

$$\rho(\pi(\exp(A))) = \exp(d\rho(d\pi(A))) \quad \forall A \in \mathfrak{g}.$$

By the definition of $d\rho$, we have $\rho(\exp(d\pi(A))) = \exp(d\rho(d\pi(A)))$. Thus, we only need to show $\pi(\exp(A)) = \exp(d\pi(A))$. This is true by the definition of $d\pi$.