Part 2: Structure of Classical Groups

This is an exposition for Chapter 2 of the book Goodman-Wallach "Symmetry, Representations and Invariants". Our goal is to study the connectedness of the classical groups $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$, and the reducibility of their regular representations. To do so, some prerequisites are introduced. One of them is the notion of maximal torus, by which we can show that classical groups are generated by unipotent elements. Then we can show an interesting relation between the group and topology structures in $GL(n,\mathbb{C})$, namely every subgroup generated by unipotent elements is connected.

The reducibility of a regular representation of a connected group is equivalent to that of its Lie algebra. Thus, we can shift from studying the group to studying its Lie algebra. Working with a Lie algebra is of great advantage because it is a vector space wherein we can speak of subspace decompositions, eigenvalues, eigenvectors, etc. Indeed, we will introduce the notion of roots (somewhat similar to eigenvalues) and the root space decomposition of a Lie algebra (somewhat similar to the eigenspace decomposition of a ~~classical~~ vector space). This strategy is applied to show that the adjoint representation of a classical group is irreducible. Another application is to

~~show~~ explicitly construct all irreducible regular representations of $SL(2,\mathbb{C})$. We can also show that all regular representations of $SL(2,\mathbb{C})$ are completely reducible. The techniques we use to study $SL(2,\mathbb{C})$, more precisely the reducibility of its regular representations, and the adjoint representation of the classical groups are typical. In Part 3, we will use similar techniques to study the reducibility of arbitrary regular representations of the classical groups. In this write-up, we will discuss:

- Maximal torus (including toral groups, characters of a group, calculation of the maximal tori and characters of classical groups).
- Connectedness of classical groups.
- Reducibility of classical groups, a typical example (including more detail of the strategy of studying a classical group through its Lie algebra, and how it is done for $SL(2,\mathbb{C})$).
- Root space decomposition of the Lie algebras of classical groups (including the notion of roots, calculation of roots, two structure theorems) ~~and an app Application~~
- Irreducibility of the adjoint representation.

$\boxed{1}$ Maximal torus in a classical group

A type of algebraic groups that is of particular importance is the toral groups. For $l \in \mathbb{N}$, the group $(\mathbb{C}^\times)^l$ is isomorphic to the diagonal subgroup of $GL(l,\mathbb{C})$. With this identification, $(\mathbb{C}^\times)^l$ is an algebraic group. A toral group is an algebraic group isomorphic to $(\mathbb{C}^\times)^l$. The integer l is called the rank. For example, the groups

$$H_1 = \{\mathrm{diag}(x_1, x_2, \dots, x_n) : x_i \in \mathbb{C}^\times \ \forall 1 \le i \le n\}$$
$$H_2 = \{\mathrm{diag}(1, x_2, \dots, x_n) : x_i \in \mathbb{C}^\times \ \forall 2 \le i \le n\}$$

are toral subgroups of $GL(n, \mathbb{C})$ with $\mathrm{rank}(H_1) = n$, $\mathrm{rank}(H_2) = n-1$. As a consequence of the definition, toral groups are abelian.

Regular representations of a toral group are quite simple. We will describe them in terms of regular characters. A character of a group G is a group morphism $\chi : G \to \mathbb{C}^\times$. When G is an algebraic group, we call χ a <u>regular character</u> if it is a regular function. For examples the functions $\chi_1, \chi_2 : \mathbb{C}^\times \to \mathbb{C}^\times$, $\chi_1(z) = z^{-1}$, $\chi_2(z) = \bar{z}$ are characters. χ_1 is regular but χ_2 is not.

It is helpful to give some description of regular characters. Suppose H is a toral subgroup of rank ℓ of $GL(n, \mathbb{C})$. By definition, there is an algebraic-group morphism from H to $(\mathbb{C}^\times)^\ell$. Denote the component functions by $\chi_1, \chi_2, \dots, \chi_\ell$. They are regular characters. Indeed, because

$$(\mathbb{C}^\times)^\ell \equiv \{\mathrm{diag}(z_1, \dots, z_\ell) : z_i \in \mathbb{C}^\times\} \subset GL(\ell, \mathbb{C}),$$

$\mathcal{V}[(\mathbb{C}^\times)^\ell] = \mathbb{C}[z_1, \dots, z_\ell, (z_1 \cdots z_\ell)^{-1}] = \mathbb{C}[z_1, z_1^{-1}, \dots, z_\ell, z_\ell^{-1}]$. Let $\varphi = (\chi_1, \dots, \chi_\ell) : H \to (\mathbb{C}^\times)^\ell$ be the regular map. By the definition of regularity, $f \circ \varphi$ is a regular function on H for every $f \in \mathbb{C}[z_1, z_1^{-1}, \dots, z_\ell, z_\ell^{-1}]$. Each $g \in \mathcal{V}[\mathbb{C}^\times] = \mathbb{C}[z, z^{-1}]$ can be considered as a function in $\mathbb{C}[z_1, z_1^{-1}, \dots, z_\ell, z_\ell^{-1}]$ by viewing z as z_1. Then $g \circ \chi_1 = g \circ \varphi$ is a regular function. Thus, χ_1 is regular.

Let $\mathfrak{X}(H)$ be the set of all regular characters of H. We can define an operation on $\mathfrak{X}(H)$ by pointwise multiplication, namely $(f_1 f_2)(h) := f_1(h) f_2(h)$ for all $h \in H$. This operation turns $\mathfrak{X}(H)$ into an abelian group, which is called the

<u>Character group</u> of H. As a consequence, $\chi_1^{r_1}\chi_2^{r_2}\cdots\chi_\ell^{r_\ell} \in \mathfrak{X}(H)$ for all $r_1,\ldots,r_\ell \in \mathbb{Z}$. These are, in fact, all elements of $\mathfrak{X}(H)$. To see that, it is convenient to identify H with $(\mathbb{C}^\times)^\ell$ and $\chi_1,\ldots,\chi_\ell$ with the corresponding coordinate maps, i.e. $\chi_k(z_1,\ldots,z_\ell) = z_k$. Let $\bar{\chi}$ be a regular character on $(\mathbb{C}^\times)^\ell$. Put

$$\bar{\chi}_k(z) = \bar{\chi}(1,\ldots,\underset{\hat{k}}{z},\ldots,1) \qquad \forall z \in \mathbb{C}^\times.$$

Then $\bar{\chi}_k$ is a regular function on $\mathbb{C}^\times$ and $\bar{\chi}(z_1,\ldots,z_\ell) = \bar{\chi}_1(z_1)\cdots\bar{\chi}_\ell(z_\ell)$. Because $\bar{\chi}_k : \mathbb{C}^\times \to \mathbb{C}^\times = GL(1,\mathbb{C})$ is regular, it is a regular representation of $\mathbb{C}^\times$. By Example 8 of Part 1, pages 50-52, $\bar{\chi}_k(z) = z^{r_k}$ for some $r_k \in \mathbb{Z}$. Therefore,

$$\bar{\chi}(z_1,\ldots,z_\ell) = z_1^{r_1}\cdots z_\ell^{r_\ell} = \chi_1(z_1,\ldots,z_\ell)^{r_1}\cdots\chi_\ell(z_1,\ldots,z_\ell)^{r_\ell}.$$

These observations give a group morphism $\mathfrak{X}(H) \simeq (\mathbb{Z}^\ell,+)$. Because $\mathcal{V}[H] \equiv \mathcal{V}[(\mathbb{C}^\times)^\ell] = \mathbb{C}[z_1,z_1^{-1},\ldots,z_\ell,z_\ell^{-1}]$, $\mathcal{V}[H]$ is a linear combination of $\mathfrak{X}(H)$. In addition, $\mathfrak{X}(H)$ is a linearly independent set over $\mathbb{C}$. Thus, $\mathfrak{X}(H)$ is a basis over $\mathbb{C}$ of the vector space $\mathcal{V}[H]$.

Now we describe regular representations of toral groups. Roughly speaking, the vector space is decomposed into invariant subspaces, the action of the group on each of which is by scalar-multiplication.

<u>Theorem 1</u> Let H be a toral group and (ρ, V) be a regular representation of H. Then $V = \bigoplus_{\chi\in\mathfrak{X}(H)} V(\chi)$ where $V(\chi) = \{v \in V : \rho(h)v = \chi(h)v \ \forall h \in H\}$.

As a subset of $M_\ell(\mathbb{C})$, $(\mathbb{C}^\times)^\ell$ is a submanifold of dimension 2ℓ. Thus, its Lie algebra is a 2ℓ-dimensional Euclidean space (review the remark on page 13, before

Theorem 3, of Part 1), and ~~because~~ hence is an l-dimensional vector space over $\mathbb{C}$. This implies $(\mathbb{C}^\times)^l$ and $(\mathbb{C}^\times)^m$ have Lie algebras of different dimensions as vector spaces over $\mathbb{C}$ if $l \neq m$. By the functorial property of differential maps (Theorem 5, page 61, Part 1), $(\mathbb{C}^\times)^l$ and $(\mathbb{C}^\times)^m$ are not isomorphic closed groups if $l \neq m$. Therefore, the class of isomorphic toral groups is determined by the rank.

Another observation is that the inclusion order of toral groups implies the same order of their ranks. Namely, if H and G are toral groups and $H \subset G$ then $\mathrm{rank}(H) \leq \mathrm{rank}(G)$. Indeed, the inclusion map $i: H \to G$ is a topological group morphism which is injective. Its differential is thus also a monomorphism by the functorial property. Thus,

$$\underbrace{\dim_{\mathbb{C}} \mathrm{Lie}(H)}_{= \mathrm{rank}(H)} \leq \underbrace{\dim_{\mathbb{C}} \mathrm{Lie}(G)}_{= \mathrm{rank}(G)}$$

Also by the inclusion map and its differential, we can show that a toral subgroup of $GL(n, \mathbb{C})$ has rank less than or equal to $\dim M_n(\mathbb{C}) = n^2$. Given a subgroup G of $GL(n, \mathbb{C})$, we call a subgroup H of G a ~~to~~ <u>maximal torus</u> of G if H is a toral subgroup and is not contained in a strictly larger toral subgroup of G. From the above observations, we see that if G has at least one toral subgroup, it has a maximal subgroup which is one with maximal rank.

Here is a handy criterion to check if a toral subgroup is a maximal torus.

[<u>Theorem 2</u> Let G be a subgroup of $GL(n, \mathbb{C})$ and H be a toral subgroup of G. Then H is a maximal torus iff $\forall g \in G, (gh = hg \ \forall h \in H \Rightarrow g \in H)$.]

This theorem implies that a maximal torus of G is a maximal abelian subgroup of G. Indeed, suppose H is a maximal torus of G, K is an abelian subgroup of G and there exists $\cancel{x \in H \setminus K}$ $x \in K \setminus H$, then the group generated by H and x is another toral group of G. This is a contradiction.

Now we will calculate maximal tori of the classical groups $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$, their characters and Lie algebras. Recall the definition of $SO(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$ given on pages 4 and 5 in Part 1.

$$SO(n,\mathbb{C}) = O(n,\mathbb{C}) \cap SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) : A^T A = I_n, \det(A) = 1\},$$

$$Sp(n,\mathbb{C}) = \{A \in GL(2n,\mathbb{C}) : A^T J A = J\},$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Now we consider each case.

- $\underline{G = GL(n,\mathbb{C})}$

Denote $H = \{\mathrm{diag}(x_1,\dots,x_n) : x_i \in \mathbb{C}^\times, \forall 1 \le i \le n\}$. H is an algebraic group in $GL(n,\mathbb{C})$ because it is determined by the polynomial equations $x_{ij} = 0$ for $1 \le i \ne j \le n$. Moreover, $H \simeq (\mathbb{C}^\times)^n$. Thus H is a toral subgroup of G, and $\mathrm{rank}(H) = n$. Take $\alpha = (\alpha_{ij}) \in G$ such that

$$(\alpha_{ij})\,\mathrm{diag}(x_1,\dots,x_n) = \mathrm{diag}(x_1,\dots,x_n)(\alpha_{ij}) \quad \forall x_1,\dots,x_n \in \mathbb{C}^\times.$$

The equation at position (i,j) is $\alpha_{ij} x_j = \alpha_{ij} x_i$. Thus, $\alpha_{ij} = 0$ for all $1 \le i \ne j \le n$. Then $\alpha \in H$. By Theorem 2, H is a maximal torus of G. The characters $\chi_i : H \to \mathbb{C}^\times$, $\chi_i(\mathrm{diag}(x_1,\dots,x_n)) = x_i$, $1 \le i \le n$, generate the group $\mathcal{X}(H)$.

We guess the Lie algebra of H is $\mathfrak{h} = \{\mathrm{diag}(a_1,\dots,a_n) : a_i \in \mathbb{C}, \forall 1 \le i \le n\}$.

As we discussed earlier, $\dim_{\mathbb{C}} \mathrm{Lie}(H) = \mathrm{rank}(H) = n = \dim \mathfrak{h}$. Because $\exp(tA) \in H$ for all $t \in \mathbb{R}$, $A \in \mathfrak{h}$, $\mathfrak{h} \subset \mathrm{Lie}(H)$. Thus, $\mathrm{Lie}(H) = \mathfrak{h}$.

- $G = SL(n, \mathbb{C})$

Denote $H = \{ \mathrm{diag}(x_1, \dots, x_{n-1}, (x_1 \cdots x_{n-1})^{-1}) : x_i \in \mathbb{C}^{\times}\ \forall 1 \leq i \leq n \}$. H is an algebraic subgroup of $GL(n, \mathbb{C})$ because it is determined by the polynomial equations $\det(x) = 1$, $x_{ij} = 0$ for $1 \leq i \neq j \leq n$. Moreover, $H \simeq (\mathbb{C}^{\times})^{n-1}$ via the map

$$\mathrm{diag}(x_1, \dots, x_{n-1}, (x_1 \cdots x_{n-1})^{-1}) \longmapsto \mathrm{diag}(x_1, \dots, x_{n-1}).$$

Thus, H is a toral subgroup of G and $\mathrm{rank}(H) = n-1$. Let $\alpha = (\alpha_{ij}) \in G$ be an element commuting with all elements in H.

$$(\alpha_{ij})\, \mathrm{diag}(x_1, \dots, x_n) = \mathrm{diag}(x_1, \dots, x_n)\, (\alpha_{ij}) \quad \forall x_1, \dots, x_n \in \mathbb{C}^{\times},\ x_1 \cdots x_n = 1.$$

The equation at position (i,j) is $\alpha_{ij} x_j = \alpha_{ij} x_i$. Thus, $\alpha_{ij} = 0$ for all $1 \leq i \neq j \leq n$. Then $\alpha \in H$. Thus, H is a maximal torus of G. The character group $\mathcal{X}(H)$ is generated by $\chi_i : H \to \mathbb{C}^{\times}$, $\chi_i(\mathrm{diag}(x_1, \dots, x_n)) = x_i$ for $1 \leq i \leq n$. We guess the Lie algebra of H is $\mathfrak{h} = \{ \mathrm{diag}(a_1, \dots, a_n) : a_i \in \mathbb{C}, \sum_{i=1}^{n} a_i = 0 \}$. By the definition of $\mathrm{Lie}(H)$, $\mathfrak{h} \subset \mathrm{Lie}(H)$. Because $\dim_{\mathbb{C}} \mathrm{Lie}(H) = \mathrm{rank}(H) = n-1 = \dim_{\mathbb{C}} \mathfrak{h}$, we have indeed $\mathrm{Lie}(H) = \mathfrak{h}$.

- $G = SO(n, \mathbb{C})$

The diagonal subgroup, i.e. the largest subgroup consisting of diagonal elements, of $SO(n, \mathbb{C})$ is $\{ \mathrm{diag}(x_1, \dots, x_n) : x_i = \pm 1$, an even number of them are equal to $1 \}$. This is, however, not a toral group because it is a finite group. Thus, the fashion that the maximal torus is the diagonal subgroup is not applicable for this case.

Recall that the group $SO(n,\mathbb{C})$ was first introduced when we tried to give a matrix representation of the group $SO(B)$ where B is a symmetric nondegenerate bilinear form on $\mathbb{C}^n$ (review page 5 of Part 1). Thus, we should have focused on finding a maximal torus for $SO(B)$ rather than $SO(n,\mathbb{C})$. This point of view leads to the hope that with a suitable choice of basis for $\mathbb{C}^n$ or of B, $SO(B)$ has a matrix representation such that the maximal torus is the diagonal group. We continue to choose the standard basis $(e_1, e_2, \dots, e_n)$ for $\mathbb{C}^n$. But instead of choosing $B(e_i, e_j) = \delta_{ij}$, which results in the representation matrix $SO(n,\mathbb{C})$, we choose $B(e_i, e_j) = \delta_{i,n+1-j}$.

The representation matrix of B is
$$\Gamma = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in M_n(\mathbb{C}).$$

Thus, the matrix form of $SO(B)$ is
$$SO(\mathbb{C}^n, B) := \{A \in GL(n,\mathbb{C}) : A^T \Gamma A = \Gamma,\ \det(A) = 1\} \qquad (1)$$

We point out that the diagonal subgroup of $SO(\mathbb{C}^n, B)$ is indeed a maximal torus.

$A = \operatorname{diag}(x_1, \dots, x_n) \in SO(\mathbb{C}^n, B)$ if and only if $x_1 x_2 \cdots x_n = 1$ and
$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

The latter equation is equivalent to $x_i x_{n+1-i} = 1$ for all $1 \le i \le n$. To be more concrete, we consider two cases.

- $n = 2l$: $x_i = x_{2l+1-i}^{-1} \quad \forall\, 1 \le i \le l$
- $n = 2l+1$: $x_i = x_{2l+2-i}^{-1} \quad \forall\, 1 \le i \le l$, $x_{l+1} = 1$.

The case $n = 2l$

Denote $H = \{\operatorname{diag}(x_1, \dots, x_l, x_l^{-1}, \dots, x_1^{-1}) : x_i \in \mathbb{C}^\times\ \forall\, 1 \le i \le l\}$. Then $H \simeq (\mathbb{C}^\times)^l$.

Thus, H is a toral subgroup of G and $\mathrm{rank}(H) = \ell$. Let $\alpha = (\alpha_{ij})$ be an element in $SO(\mathbb{C}^{2\ell}, B)$ that commutes with all elements in H. Then the equation

$$(\alpha_{ij})\,\mathrm{diag}(x_1,\dots,x_\ell,x_\ell^{-1},\dots,x_1^{-1}) = \mathrm{diag}(x_1,\dots,x_\ell,x_\ell^{-1},\dots,x_1^{-1})(\alpha_{ij}) \quad \forall x_1,\dots,x_\ell \in \mathbb{C}^\times$$

yields $\alpha_{ij} x_i = \alpha_{ij} x_j$ for all $1 \le i,j \le 2\ell$. Thus $\alpha_{ij} = 0$ if $i \ne j$. Thus, $\alpha \in H$. This implies H is a maximal torus of G. The characters $\chi_i : H \to \mathbb{C}^\times$, $\chi_i(\mathrm{diag}(x_1,\dots,x_\ell,x_\ell^{-1},\dots,x_1^{-1})) = x_i$, $1 \le i \le \ell$, generate $\mathfrak{X}(H)$. We guess the Lie algebra of H is $\mathfrak{h} = \{\mathrm{diag}(a_1,\dots,a_\ell,-a_\ell,\dots,-a_1) : a_i \in \mathbb{C},\ \forall 1 \le i \le \ell\}$. Indeed, $\mathfrak{h} \subset \mathrm{Lie}(H)$ by the definition of $\mathrm{Lie}(H)$, and $\dim_{\mathbb{C}} \mathfrak{h} = \ell = \mathrm{rank}(H) = \dim_{\mathbb{C}} \mathrm{Lie}(H)$.

The case $n = 2\ell+1$

Denote $H = \{\mathrm{diag}(x_1,\dots,x_\ell,1,x_\ell^{-1},\dots,x_1^{-1}) : x_i \in \mathbb{C}^\times\ \forall 1 \le i \le \ell\}$. By the same arguments as in the previous case, we can show that H is a maximal torus of G, $\mathrm{rank}(H) = \ell$, that $\mathfrak{X}(H)$ is generated by $\chi_i : H \to \mathbb{C}^\times$, $\chi_i(\mathrm{diag}(x_1,\dots,x_\ell,1,x_\ell^{-1},\dots,x_1^{-1})) = x_i$, $1 \le i \le \ell$, and that the Lie algebra of H is

$$\mathfrak{h} = \{\mathrm{diag}(a_1,\dots,a_\ell,0,-a_\ell,\dots,-a_1) : a_i \in \mathbb{C}\ \forall 1 \le i \le \ell\}.$$

- $G = Sp(n,\mathbb{C})$

The diagonal subgroup of $Sp(n,\mathbb{C})$ consists of matrices $A = \mathrm{diag}(x_1,\dots,x_{2n})$ such that

$$\begin{pmatrix} \begin{matrix} x_1 & & \\ & \ddots & \\ & & x_n \end{matrix} & \\ & \begin{matrix} x_{n+1} & & \\ & \ddots & \\ & & x_{2n} \end{matrix} \end{pmatrix} \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \begin{pmatrix} \begin{matrix} x_1 & & \\ & \ddots & \\ & & x_n \end{matrix} & \\ & \begin{matrix} x_{n+1} & & \\ & \ddots & \\ & & x_{2n} \end{matrix} \end{pmatrix} = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}.$$

It is equal to $\{\mathrm{diag}(x_1,\dots,x_{2n}) : x_i x_{i+n} = 1\ \forall 1 \le i \le n\}$. This looks similar to the maximal torus of $SO(\mathbb{C}^{2n}, B)$ which was described earlier. Recall that $Sp(n,\mathbb{C})$ is a matrix representation of $SO(\Omega)$ where Ω is a skew-symmetric nondegenerate bilinear form on $\mathbb{C}^{2n}$. Note that the name Ω is used instead of B to avoid possible confusion with the bilinear form $B(e_i, e_j) = \delta_{i,2n+1-j}$.

We would like to choose Ω such that the diagonal subgroup of $SO(\Omega)$ is

$$H = \{\operatorname{diag}(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}) : x_i \in \mathbb{C}^\times \ \forall 1 \le i \le n\}.$$

For that, we choose

$$\Omega(e_i, e_j) = \begin{cases} 1 & \text{if } i \le j,\ i+j = 2n+1, \\ 0 & \text{if } i \le j,\ i+j \ne 2n+1, \\ -\Omega(e_j, e_i) & \text{if } i > j. \end{cases}$$

The matrix representation of $SO(\Omega)$ is denoted by

$$Sp(\mathbb{C}^{2n}, \Omega) = \left\{ A \in GL(2n, \mathbb{C}) : A^T \begin{pmatrix} & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & \end{pmatrix} A = \begin{pmatrix} & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & \end{pmatrix} \right\}. \quad (2)$$

By the same arguments as in the case $G = SO(\mathbb{C}^{2\ell}, B)$, we get the same results therein.

As a matter of terminology, we call groups $SL(\ell+1, \mathbb{C})$ to be of type A_ℓ, groups $SO(\mathbb{C}^{2\ell+1}, B)$ to be of type B_ℓ, groups $Sp(\mathbb{C}^{2\ell}, \Omega)$ to be of type C_ℓ, and groups $SO(\mathbb{C}^{2\ell}, B)$ to be of type D_ℓ. We get a survey of maximal torus of each type:

- Type A_ℓ: $H = \{\operatorname{diag}(x_1, \dots, x_\ell, (x_1 \cdots x_\ell)^{-1}) : x_i \in \mathbb{C}^\times \ \forall 1 \le i \le \ell\}$.

  $\mathfrak{h} = \{\operatorname{diag}(a_1, \dots, a_\ell, -a_1 - \dots - a_\ell) : a_i \in \mathbb{C} \ \forall 1 \le i \le \ell\}$
- Type B_ℓ: $H = \{\operatorname{diag}(x_1, \dots, x_\ell, 1, x_\ell^{-1}, \dots, x_1^{-1}) : x_i \in \mathbb{C}^\times \ \forall 1 \le i \le \ell\}$

  $\mathfrak{h} = \{\operatorname{diag}(a_1, \dots, a_\ell, 0, -a_\ell, \dots, -a_1) : a_i \in \mathbb{C} \ \forall 1 \le i \le \ell\}$

• Type C_ℓ and D_ℓ: $H = \{\operatorname{diag}(x_1,\dots,x_\ell,x_\ell^{-1},\dots,x_1^{-1}) : x_i \in \mathbb{C}^\times\ \forall 1\le i\le \ell\}$

$\mathfrak{h} = \{\operatorname{diag}(a_1,\dots,a_\ell,-a_\ell,\dots,-a_1) : a_i \in \mathbb{C}\ \ \forall 1\le i\le \ell\}$.

In each of these cases, the character group $\mathfrak{X}(H)$ is generated by the coordinate functions $x_1,\dots,x_\ell$. Thus $\mathcal{O}[H] = \mathbb{C}[x_1,\dots,x_\ell,x_1^{-1},\dots,x_\ell^{-1}]$. Also, $\operatorname{rank}(H)=\ell$.

We know that every diagonalizable matrix $A\in GL(n,\mathbb{C})$ is of the form $A = P^{-1}DP$ where $P\in GL(n,\mathbb{C})$ and D is a diagonal matrix. It is surprising that the same fashion is true for groups of $A_\ell, B_\ell, C_\ell, D_\ell$ type. Namely, every diagonalizable element of a classical group G is conjugate to an element in the maximal torus of G (according to the survey) via an element in G.

<u>Theorem 3</u> Let G be a classical group, H be its maximal torus given in the survey, and $g\in G$ be a diagonalizable matrix. Then there exists $\gamma\in G$ such that $\gamma^{-1}g\gamma\in H$.

<u>Proof of Theorem 1</u>

We show by induction in $\ell = \operatorname{rank}(H)$. It is convenient to identify H with the subgroup $\{\operatorname{diag}(z_1,\dots,z_\ell) : z_i\in\mathbb{C}^\times\ \forall 1\le i\le \ell\}$ of $GL(\ell,\mathbb{C})$. For $\ell = 1$, (ρ, V) is a regular representation of $\mathbb{C}^\times$. In Example 8 of Part 1, pages 50-51, we pointed out that $V = \bigoplus_{k\in\mathbb{Z}} E_k$ where

$$E_k = \{v\in V : \rho(z)v = z^k v \quad \forall z\in\mathbb{C}^\times\}.$$

Note that $E_k = V(z\mapsto z^k)$. Because all regular characters of $\mathbb{C}^\times$ are of the form $z\in\mathbb{C}^\times\mapsto z^k\in\mathbb{C}^\times$ for some $k\in\mathbb{Z}$, the decomposition of V above is exactly $V = \bigoplus_{\chi\in\mathfrak{X}(\mathbb{C}^\times)} V(\chi)$.

Suppose that the conclusion is true for $l-1$ with $l\geq 2$. Let H be a toral group of rank l and (ρ, V) be a regular representation of H. We have $(z_1, z_2, \dots, z_l) = (z_1, 1, \dots, 1)(1, z_2, \dots, z_l)$. Define $\tilde{\rho}: \mathbb{C}^\times \to GL(V)$, $\tilde{\rho}(z_1) = \rho(z_1, 1, \dots, 1)$. The regularity of ρ is equivalent to that the function

$$(z_1, \dots, z_l) \in H \mapsto \langle v^*, \rho(z_1, \dots, z_l) v\rangle \in \mathbb{C}$$

is reglular, i.e. belonging to $\mathbb{C}[z_1, z_1^{-1}, \dots, z_l, z_l^{-1}]$, for all $v^* \in V^*$ and $v \in V$. Thus, the function $z_1 \in \mathbb{C}^\times \mapsto \langle v^*, \tilde{\rho}(z_1) v\rangle \in \mathbb{C}$ is also regular for all $v^* \in V^*$ and $v \in V$. This means $(\tilde{\rho}, V)$ is a regular representation of $\mathbb{C}^\times$. Then we get the decomposition $V = \bigoplus_{k\in\mathbb{Z}} E_k$, where

$$E_k = \{v \in V : \rho(z_1, 1, \dots, 1) v = z_1^k v \quad \forall z_1 \in \mathbb{C}^\times\}.$$

For each $k \in \mathbb{Z}$, $\rho(1, z_2, \dots, z_l) \in GL(E_k)$. Define $\rho_k : (\mathbb{C}^\times)^{l-1} \to GL(E_k)$,

$$\rho_k(z_2, \dots, z_l) v = \rho(1, z_2, \dots, z_l) v \quad \forall v \in E_k.$$

Then (ρ_k, E_k) is a regular representation of $(\mathbb{C}^\times)^{l-1}$. By the induction hypothesis,

$$E_k = \bigoplus_{\theta \in \mathfrak{X}((\mathbb{C}^\times)^{l-1})} E_k(\theta),$$

where $E_k(\theta) = \{v \in E_k : \rho(1, z_2, \dots, z_l) v = \theta(z_2, \dots, z_l) v \ \forall (z_2, \dots, z_l) \in (\mathbb{C}^\times)^{l-1}\}$.

Combining with the definition of E_k, we can write

$$\begin{aligned} E_k(\theta) &= \{w \in V : \rho(1, z_2, \dots, z_l)\rho(z_1, 1, \dots, 1) w = \theta(z_2, \dots, z_l) z_1^k w \quad \forall (z_1, \dots, z_l) \in (\mathbb{C}^\times)^l\} \\ &= \{w \in V : \rho(z_1, \dots, z_l) w = z_1^k \theta(z_2, \dots, z_l) w \quad \forall (z_1, \dots, z_l) \in (\mathbb{C}^\times)^l\}. \end{aligned}$$

Because each regular character χ of $(\mathbb{C}^\times)^l$ is of the form $z_1^k \theta(z_2, \dots, z_l)$ for some $k \in \mathbb{Z}$ and $\theta \in \mathfrak{X}((\mathbb{C}^\times)^{l-1})$, the decomposition $V = \bigoplus_{\substack{k \in \mathbb{Z} \\ \theta \in \mathfrak{X}((\mathbb{C}^\times)^{l-1})}} E_k(\theta)$ is exactly $V = \bigoplus_{\chi \in \mathfrak{X}(H)} V(\chi)$.

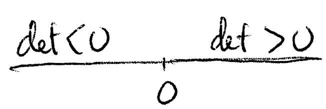
Proof of Theorem 2

Suppose that H is contained in a toral subgroup K of G. We want to show $H=K$. Because K is abelian, every element g of K commutes with every element of H. By the hypothesis, $g \in H$. Thus, $K \subset H$.

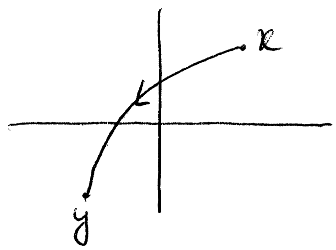
$\boxed{2}$ Connectedness of classical groups

In this section, we show that the classical groups $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $SO(\mathbb{C}^n, B)$, $Sp(\mathbb{C}^{2n},\Omega)$ are ~~connet~~ connected. Because they are Lie groups, they are locally path-connected. Thus, the connectedness and path-connectedness are equivalent. The connectedness then gives valuable relations between the group and its Lie algebra. This issue will be discussed in the next section.

We start with the observation that the real general linear group $GL(n,\mathbb{R})$ is not connected. Indeed, the continuous map $\det : GL(n,\mathbb{R}) \to \mathbb{R}$ partitions $GL(n,\mathbb{R})$ into two open disjoint nonempty subsets, namely $\{A : \det(A) > 0\}$ and $\{B : \det(B) < 0\}$. The fact that $GL(n,\mathbb{R})$ is not connected can be seen as that every continuous path from a matrix A with $\det(A) > 0$ to a matrix B with $\det(B) < 0$ must pass through a matrix C with $\det(C) = 0$.



Such argument, however, do not work for $GL(n,\mathbb{C})$ because any two points in $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$ can be connected by a continuous path therein. Thus, we can hope that $GL(n,\mathbb{C})$ is connected.



In Problem ①, Homework #2, we showed that the exponential map $\exp : M_n(\mathbb{C}) \to GL(n,\mathbb{C})$ is surjective. Since $\exp$ is continuous and $M_n(\mathbb{C})$ is connected, so is $GL(n,\mathbb{C})$.

Let $A, B \in SL(n,\mathbb{C})$. There is a continuous path $\gamma : [0,1] \to GL(n,\mathbb{C})$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Define $\tilde{\gamma} : [0,1] \to SL(n,\mathbb{C})$,

$$\tilde{\gamma}(t) = \left(\frac{\gamma_1(t)}{\det \gamma(t)}, \gamma_2(t), \dots, \gamma_n(t) \right),$$

where $\gamma_i(t)$ is the i'th column of matrix $\gamma(t)$. Then $\tilde{\gamma}$ is well-defined, continuous and $\tilde{\gamma}(0) = A$, $\tilde{\gamma}(1) = B$. Hence, $SL(n,\mathbb{C})$ is connected.

By the definition of the Lie algebra associated with a bilinear form (see Eq. (3) in Part 1), the Lie algebra of $SO(\mathbb{C}^n, B)$ and $Sp(\mathbb{C}^{2l}, \Omega)$ are respectively

$$so(\mathbb{C}^n, B) := \left\{ A \in M_n(\mathbb{C}) : A^T \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} + \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} A = 0 \right\}$$

$$sp(\mathbb{C}^{2l}, \Omega) := \left\{ A \in M_{2l}(\mathbb{C}) : A^T \begin{pmatrix} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & 0 \end{pmatrix} A = 0 \right\}$$

It is not clear if $\exp(so(\mathbb{C}^n, B)) = SO(\mathbb{C}^n, B)$ and $\exp(sp(\mathbb{C}^{2l}, \Omega)) = Sp(\mathbb{C}^{2l}, \Omega)$. We need a different method to study if they are connected. The following theorem provides such a one.

<u>Theorem 4</u> Let G be a linear algebraic group that is generated by unipotent elements. Then G is connected.

If $SO(\mathbb{C}^n, B)$ and $Sp(\mathbb{C}^{2l}, \Omega)$ are generated by unipotent elements, they are connected. The problem, however, is still hard because those are "big" groups*. Recall the Jordan decomposition (Theorem 9 in Part 1): every element g in an algebraic group G is of the form $g = su$ where $s, u \in G$, s is diagonalizable and u is unipotent. Thus, the question reduces to whether diagonalizable matrices in $SO(\mathbb{C}^n, B)$ and $Sp(\mathbb{C}^{2l}, \Omega)$ are generated by unipotents therein. Because we are interested in these two groups, let us call either of them G and will distinguish them when necessary. Theorem 3 is now of great help.

Recall that a matrix conjugate to a unipotent matrix is also unipotent. Indeed, let $U = I_n + A$ where A is nilpotent. Then $P^{-1}UP = P^{-1}(I_n + A)P$ $= I_n + P^{-1}AP$, where $P^{-1}AP$ is nilpotent. Thus, $P^{-1}UP$ is unipotent. Suppose that the maximal torus H of G (given on pages 10-11) is generated by unipotents in G in sense that each element in H is a product of unipotents in G. By Theorem 3, every diagonalizable matrix $g \in G$ is of the form $g = \gamma^{-1} h \gamma$, where $\gamma \in G$ and $h \in H$. Write $H = u_1 u_2 \cdots u_k$ where u_i's are unipotents in G. Then $g = (\gamma^{-1} u_1 \gamma)(\gamma^{-1} u_2 \gamma) \cdots (\gamma^{-1} u_k \gamma)$. Each block is unipotent, so g is generated by unipotent matrices. Therefore, what we need to know is whether each element in H is a product of unipotent elements in G. This question is much simpler than the original one because matrices in H are diagonal.

Let us start with $SO(\mathbb{C}^2, B)$. By (1), $SO(\mathbb{C}^2, B) = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^\times \right\}$.

The only unipotent element thereof is I_2. Thus $SO(\mathbb{C}^2, B)$ is not generated by unipotent elements. However, $SO(\mathbb{C}^2, B)$ is still connected because $\mathbb{C}^\times$ is connected and the map $z \in \mathbb{C}^\times \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in SO(\mathbb{C}^2, B)$ is surjective and continuous.

For the group $SO(\mathbb{C}^{2\ell}, B)$ with $\ell \geq 2$, we want to know if a diagonal matrix $h = \mathrm{diag}(x_1, \dots, x_\ell, x_\ell^{-1}, \dots, x_1^{-1})$ is the product of unipotent matrices in $SO(\mathbb{C}^{2\ell}, B)$. Write $h = \underbrace{\mathrm{diag}(x_1, 1, \dots, 1, x_1^{-1})}_{= h_1} \underbrace{\mathrm{diag}(1, x_2, \dots, x_\ell, x_\ell^{-1}, \dots, x_2^{-1}, 1)}_{= h_2}$.

By Theorem 10 of Part 1, the image of a unipotent under an algebraic-group morphism is also a unipotent. By the group embedding $SO(\mathbb{C}^{2(\ell-1)}, B) \hookrightarrow SO(\mathbb{C}^{2\ell}, B)$, $A \mapsto \begin{pmatrix} 1 & & \\ & \boxed{A} & \\ & & 1 \end{pmatrix}$, we can view h_2 as an element in $SO(\mathbb{C}^{2(\ell-1)}, B)$. To be able to use the induction in $\ell \geq 2$ to say that h_2 is the product of unipotents in $SO(\mathbb{C}^{2(\ell-1)}, B)$, we need $\ell \geq 3$ (because $SO(\mathbb{C}^2, B)$ is not generated by unipotent!) Thus, the case $\ell = 2$ must be considered separately. Suppose that $SO(\mathbb{C}^4, B)$ is generated by unipotents. This group can be embedded into $SO(\mathbb{C}^{2\ell}, B)$ by

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \mapsto \begin{pmatrix} A_1 & & A_2 \\ & \boxed{I_{2\ell-4}} & \\ A_3 & & A_4 \end{pmatrix}.$$

Then h_1 can be regarded as an element in $SO(\mathbb{C}^4, B)$, which is the product of unipotents. Thus, $SO(\mathbb{C}^{2\ell}, B)$ for $\ell \geq 3$ is generated by unipotents if $SO(\mathbb{C}^4, B)$

is so. Goodman-Wallach constructed on page 79 a group morphism $\pi: SL(2,\mathbb{C})\times SL(2,\mathbb{C}) \to SO(\mathbb{C}^4, B)$ whose image contains the maximal torus of $SO(\mathbb{C}^4,B)$. In addition, if $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ is identified with a subgroup of $SL(4,\mathbb{C})$ via $(A_1,A_2)\mapsto \begin{pmatrix} A_1 & 0 \\ 0 & A_2\end{pmatrix}$ then π is an algebraic-group morphism. The following result is very useful in our situation.

Let $\pi: K\to G$ be an algebraic-group morphism from an algebraic group K to a classical group G. Suppose that K is generated by unipotent elements and the maximal torus of G is contained in the image of π. Then G is generated by unipotent elements.

Indeed, because K is generated by unipotents, $\pi(K)$ is too. Let $H\subset \pi(K)$ be the maximal torus of G. Then each element of H is a product of unipotents in $\pi(K)\subset G$.

Return to our problem. $SO(\mathbb{C}^4,B)$ is generated by unipotents if $SL(2,\mathbb{C})$ is so. The maximal torus of $SL(2,\mathbb{C})$ consists of diagonal matrices $\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}$ for $z\in\mathbb{C}^\times$.

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix} = \begin{pmatrix} 1 & -z \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ z^{-1}-1 & 1\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ z-1 & 1\end{pmatrix}.$$

Thus, $SL(2,\mathbb{C})$ is generated by unipotent elements. We conclude that $SO(\mathbb{C}^{2l},B)$ is connected for all $l\in\mathbb{N}$.

Next, let us consider $SO(\mathbb{C}^{2l+1},B)$. Note that $SO(\mathbb{C},B)=\{1\}$ is connected. For $l\geq 1$, we have a group embedding $SO(\mathbb{C}^{2l},B)\hookrightarrow SO(\mathbb{C}^{2l+1},B)$,

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longmapsto \begin{pmatrix} A_1 & 0 & A_2 \\ 0 & 1 & 0 \\ A_3 & 0 & A_4 \end{pmatrix},$$

whose image contains the maximal torus of $SO(\mathbb{C}^{2l+1}, B)$. Because $SO(\mathbb{C}^{2l}, B)$ is generated by unipotents for $l \geq 2$, so is $SO(\mathbb{C}^{2l+1}, B)$. We need to consider the case $l = 1$. Goodman-Wallach constructed on page 78 an algebraic-group morphism $SL(2, \mathbb{C}) \to SO(\mathbb{C}^3, B)$ whose image contains the maximal torus of $SO(\mathbb{C}^3, B)$. Because $SL(2, \mathbb{C})$ is generated by unipotents, so is $SO(\mathbb{C}^3, B)$. We conclude that $SO(\mathbb{C}^{2l+1}, B)$ is connected for all $l \geq 0$.

Now let us consider $Sp(\mathbb{C}^{2l}, \Omega)$. By direct calculation from (2), we get $Sp(\mathbb{C}^2, \Omega) = SL(2, \mathbb{C})$, which is generated by unipotent elements. Consider $l \geq 2$. Each matrix $h = \mathrm{diag}(x_1, x_2, \ldots, x_l, x_l^{-1}, \ldots, x_1^{-1})$ in the maximal torus of $Sp(\mathbb{C}^{2l}, \Omega)$ can be written as

$$h = \underbrace{\mathrm{diag}(x_1, 1, \ldots, 1, x_1^{-1})}_{= h_1} \underbrace{\mathrm{diag}(1, x_2, \ldots, x_l, x_l^{-1}, \ldots, x_2^{-1}, 1)}_{= h_2}$$

We can view h_1 as belonging to $Sp(\mathbb{C}^2, \Omega)$ by the group embedding $Sp(\mathbb{C}^2, \Omega) \to Sp(\mathbb{C}^{2l}, \Omega)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b \\ & \boxed{I_{2l-2}} & \\ c & & d \end{pmatrix}$. Thus, h_1 is a product of unipotent elements.

Similarly, h_2 can be viewed as belonging to $Sp(\mathbb{C}^{2(l-1)}, \Omega)$ by the group embedding $Sp(\mathbb{C}^{2(l-1)}, \Omega) \to Sp(\mathbb{C}^{2l}, \Omega)$, $A \mapsto \begin{pmatrix} 1 & & \\ & \boxed{A} & \\ & & 1 \end{pmatrix}$. By the induction on l, we know that h_2 is the product of unipotents in $Sp(\mathbb{C}^{2l}, \Omega)$. We conclude that $Sp(\mathbb{C}^{2l}, \Omega)$ is connected for all $l \geq 1$.

Here are some applications of the connectedness of $Sp(\mathbb{C}^{2l}, \Omega)$ and $SO(\mathbb{C}^n, \Omega)$

Because $Sp(l,\mathbb{C})$ is a representation matrix of Ω in some basis (review Theorem 1 in Part 1), it is related to $Sp(\mathbb{C}^{2l},\Omega)$ by

$$Sp(l,\mathbb{C}) = A^T Sp(\mathbb{C}^{2l},\Omega) A$$

for some $A \in M_{2l}(\mathbb{C})$. Thus, $Sp(l,\mathbb{C})$ is also connected. Similarly, $SO(n,\mathbb{C})$ is connected. On page 5 of Part 1, we posed a question whether $Sp(l,\mathbb{C}) \subset SL(2l,\mathbb{C})$. Recall that

$$Sp(l,\mathbb{C}) = \{A \in GL(2l,\mathbb{C}) : A^T J A = J\}$$

where $J = \begin{pmatrix} & I_l \\ -I_l & \end{pmatrix}$. Thus, every element of $Sp(l,\mathbb{C})$ has determinant equal to 1 or -1. Because $\det(I_n) = 1$ and $Sp(l,\mathbb{C})$ is connected, every element of its has determinant 1. Thus, $Sp(l,\mathbb{C}) \subset SL(2l,\mathbb{C})$.

Proof of Theorem 4

Take $g \in G$. We want to find a continuous path in G from I_n to g. Because G is generated by unipotents, $g = u_1 u_2 \dots u_k$ where each u_i is a unipotent element in G. By a remark on page 55 of Part 1, there are nilpotent elements $\gamma_1, \gamma_2, \dots, \gamma_k$ in $M_n(\mathbb{C})$ such that $u_i = \exp(\gamma_i)$ for all $1 \le i \le k$. We want to show that each $\gamma_i \in \mathfrak{g}$, the Lie algebra of G. To simplify the notation, let us denote u_i by u and γ_i by γ.

We show that $\exp(t\gamma) \in G$ for all $t \in \mathbb{R}$. Because G is an algebraic group, it is the zero set of some $J \subset \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}]$. Take $f \in J$. It suffices to show $f(\exp(t\gamma)) = 0$ for all $t \in \mathbb{R}$. Put $\varphi(t) = f(\exp(t\gamma))$. Because γ is nilpotent, $\exp(t\gamma)$ is a matrix with coefficients in $\mathbb{C}[t]$. Thus,

$\varphi(t)$ is a matrix with coefficients in $\mathbb{C}[t]$. For every $m \in \mathbb{N}$,

$$\varphi(m) = f(\exp(m\gamma)) = f((\exp\gamma)^m) = f(u^m) = 0$$

because f vanishes on G. Since φ vanishes on an infinite set, it is identically zero.

We have showed that $\gamma_1, \dots, \gamma_k \in \mathfrak{g}$. Consider the map $\tau : [0,1] \to G$, $\tau(t) = \exp(t\gamma_1) \cdots \exp(t\gamma_k)$. Then τ is continuous, $\tau(0) = I_n$ and $\tau(1) = u_1 u_2 \cdots u_k = g$.