Some review in Lie Algebra

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The write-up answers the following questions. They can be considered as steps to explain the idea of using root systems to characterize complex simple Lie algebras.

- Prove a well-known result in linear algebra: a set of mutually commuting diagonalizable matrices can be diagonalized simultaneously.
- Compute the root systems of $sl(2,\mathbb{C})$, $sl(3,\mathbb{C})$, $su(2)$, $su(3)$.
- Draw those root systems.
- Explain the characterization of complex simple Lie algebras.

1

We prove the following lemma.

> Let $\mathscr{F}$ be a set of mutually commuting diagonalizable matrices in $M_n(\mathbb{C})$. Then there exists a basis of $\mathbb{C}^n$ that diagonalizes every member of $\mathscr{F}$.

Proof. For two commuting matrices A and B, each eigenspace of A is invariant under B. Indeed, if $(A-\lambda I_n)v = 0$ then $(A-\lambda I_n)Bv = B(A-\lambda I_n)v = B(0) = 0$. For $n = 1$, each element of $\mathscr{F}$ is a complex number; thus, the statement of the lemma is true. Suppose the statement of the lemma is true for all $n < m$ for some $m \geq 2$. Let $\mathscr{F}$ be a set of mutually commuting diagonalizable matrices in $M_m(\mathbb{C})$. Take $A \in \mathscr{F}$. Let $E(\lambda_1)$, $E(\lambda_2)$,...,$E(\lambda_k)$ be the eigenspaces of A.

$$\begin{aligned} E(\lambda_j) &= \{v \in \mathbb{C}^n : \ (A - \lambda_j I_n)v = 0\}, \\ \mathbb{C}^n &= E(\lambda_1) \oplus E(\lambda_2) \oplus ... \oplus E(\lambda_k). \end{aligned}$$

Each $B \in \mathscr{F}$ can be viewed as a linear operator on $E(\lambda_j)$. Since $\dim E(\lambda_j) < m$, by the induction hypothesis, $E(\lambda_j)$ has a basis $\mathcal{B}_j$ that diagonalizes every member of $\mathscr{F}$. Then the basis of $\mathbb{C}^n$ obtained by concatenating $\mathcal{B}_1$, $\mathcal{B}_2$,...,$\mathcal{B}_k$ diagonalizes every member of $\mathscr{F}$. □

Lie theory provides another way for us to see why the lemma is true. It comes from the fact that every finite-dimensional representation of a reductive complex Lie algebra admits a weight space decomposition. We view $M_n(\mathbb{C})$ as a Lie algebra $\mathfrak{g}$ with Lie bracket $[A, B] = AB - BA$, and diagonalizable matrices as semisimple elements of $\mathfrak{g}$. Because $\mathscr{F}$ is a mutually commuting set, so is the vector subspace of $\mathfrak{g}$ generated by $\mathscr{F}$. Without loss of generality, we can assume that $\mathscr{F}$ is a vector subspace of $\mathfrak{g}$. Then $\mathscr{F}$ is also a Lie subalgebra of $\mathfrak{g}$ because $[A, B] = 0 \in \mathscr{F}$ for all $A, B \in \mathscr{F}$.

We assume the fact that the sum of two commuting semisimple elements of $\mathfrak{g}$ is a semisimple element. In terms of matrices, this assumption says that the sum of two commuting diagonalizable matrices is also a diagonalizable matrix. Then every element of $\mathscr{F}$ is semisimple. We know that the map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $\mathrm{ad}(X)Y = [X, Y]$ is a Lie algebra representation of $\mathfrak{g}$. Thus, it maps semisimple elements to semisimple elements. This implies $\mathrm{ad}(A)$ is semisimple for every $A \in \mathscr{F}$. Then $\mathscr{F}$ is a toral subalgebra of $\mathfrak{g}$. Thus, $\mathscr{F}$ is contained in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Consider the following Lie algebra representation of $\mathfrak{g}$, so called the defining representation

$$\pi : \mathfrak{g} \to \mathrm{End}(\mathbb{C}^n), \quad \pi(X)v = Xv \quad \forall X \in \mathfrak{g},\ v \in \mathbb{C}^n.$$

For each $\mu \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$), we define a vector subspace of $\mathbb{C}^n$

$$\mathbb{C}^n(\mu) = \{v \in \mathbb{C}^n :\ \pi(A)v = \langle \mu, A\rangle\, v \ \ \forall A \in \mathfrak{h}\}.$$

Then $(\pi, \mathbb{C}^n)$ admits a weight space decomposition

$$\mathbb{C}^n = \bigoplus_{\mu \in \mathfrak{h}^*} \mathbb{C}^n(\mu).$$

Since $\mathbb{C}^n$ is finite dimensional, only finitely many summands are nonzero. Each element of $\mathfrak{h}$, if viewed as a linear operator on $\mathbb{C}^n$, acts by scalar on each subspace $\mathbb{C}^n(\mu)$. Therefore, the basis of $\mathbb{C}^n$ obtained by concatenating (arbitrarily chosen) bases of $\mathbb{C}^n(\mu)$, $\mu \in \mathfrak{h}^*$, diagonalizes every element of $\mathscr{F}$.

2

Compute the root system of $sl(2, \mathbb{C})$.

$$sl(2, \mathbb{C}) = \{A \in M_2(\mathbb{C}) : \mathrm{tr}(A) = 0\}.$$

Denote $\mathfrak{g} = sl(2, \mathbb{C})$. It has a Cartan subalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} s & \\ & -s \end{pmatrix} : s \in \mathbb{C} \right\}.$$

Because $\mathfrak{h}$ is a one-dimensional vector space, its dual $\mathfrak{h}^*$ is also one-dimensional. We can identify $\mathfrak{h}^*$ with $\mathbb{C}$ by defining

$$\left\langle \alpha, \begin{pmatrix} s & \\ & -s \end{pmatrix} \right\rangle = \alpha s \quad \forall \alpha, s \in \mathbb{C}.$$

A root of $\mathfrak{g}$ is defined as an element $\alpha \in \mathbb{C}\backslash\{0\}$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [Y, X] = \langle \alpha, Y \rangle X \quad \forall Y \in \mathfrak{h}\} \neq \{0\}.$$

We have

$$\underbrace{\begin{pmatrix} s & \\ & -s \end{pmatrix}}_{Y} \underbrace{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}}_{X} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} s & \\ & -s \end{pmatrix} = \left\langle \alpha, \begin{pmatrix} s & \\ & -s \end{pmatrix} \right\rangle \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad \forall s \in \mathbb{C}$$

$$\Leftrightarrow \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix} = \alpha \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} a = 0, \\ (\alpha + 2)c = 0, \\ (\alpha - 2)b = 0. \end{cases}$$

The system has nontrivial solution $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ if and only if $\alpha = \pm 2$. Therefore, $sl(2, \mathbb{C})$ has 2 roots. The root space decomposition is

$$sl(2, \mathbb{C}) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2},$$

where

$$\mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{g}_2 = \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-2} = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Compute the root system of $sl(3, \mathbb{C})$.

$$sl(3, \mathbb{C}) = \{A \in M_3(\mathbb{C}) : \operatorname{tr}(A) = 0\}.$$

Denote $\mathfrak{g} = sl(3, \mathbb{C})$. It has a Cartan subalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} : \; a, b \in \mathbb{C} \right\}.$$

Because $\mathfrak{h}$ is a two-dimensional vector space, its dual $\mathfrak{h}^*$ is also two-dimensional. We can identify $\mathfrak{h}^*$ with $\mathbb{C}^2$ by defining

$$\left\langle (x, y), \begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} \right\rangle = xa + yb \quad \forall x, y, a, b \in \mathbb{C}.$$

A root of $\mathfrak{g}$ is defined as an element $(x, y) \in \mathbb{C}^2\backslash\{(0, 0)\}$ such that

$$\mathfrak{g}_{(x,y)} = \{X \in \mathfrak{g} : [Y, X] = \langle (x, y), Y \rangle X \quad \forall Y \in \mathfrak{h}\} \neq \{0\}.$$

We have

$$\underbrace{\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix}}_{Y}\underbrace{\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix}}_{X} - \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix}\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix}$$

$$= \left\langle (x,y), \begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix}\right\rangle \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix} \quad \forall a,b \in \mathbb{C}$$

$$\Leftrightarrow \begin{pmatrix} 0 & c_2(a-b) & c_3(2a+b) \\ c_4(b-a) & 0 & c_6(a+2b) \\ c_7(-2a-b) & c_8(-a-2b) & 0 \end{pmatrix} = (ax+by)\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix} \quad \forall a,b \in \mathbb{C}$$

$$\Leftrightarrow \begin{cases} c_1 = c_5 = 0 \\ c_2\left[a(1-x)+b(-1-y)\right] = 0 \\ c_3\left[a(2-x)+b(1-y)\right] = 0 \\ c_4\left[a(-1-x)+b(1-y)\right] = 0 \\ c_6\left[a(1-x)+b(2-y)\right] = 0 \\ c_7\left[a(-2-x)+b(-1-y)\right] = 0 \\ c_8\left[a(-1-x)+b(-2-y)\right] = 0 \end{cases} \quad \forall a,b \in \mathbb{C}.$$

This system has nontrivial solution $\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix}$ if and only if

$$(x,y) \in \{\pm(1,-1), \pm(2,1), \pm(1,2)\}.$$

Therefore, $sl(3,\mathbb{C})$ has 6 roots. The root space decomposition is

$$sl(3,\mathbb{C}) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{(1,-1)} \oplus \mathfrak{g}_{(-1,1)} \oplus \mathfrak{g}_{(2,1)} \oplus \mathfrak{g}_{(-2,-1)} \oplus \mathfrak{g}_{(1,2)} \oplus \mathfrak{g}_{(-1,-2)},$$

where

$$\mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\mathfrak{g}_{(1,-1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(1,-1)}}, \quad \mathfrak{g}_{(-1,1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(-1,1)}},$$

$$\mathfrak{g}_{(2,1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(2,1)}}, \quad \mathfrak{g}_{(-2,-1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{X_{(-2,-1)}},$$

$$\mathfrak{g}_{(1,2)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(1,2)}}, \quad \mathfrak{g}_{(-1,-2)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{X_{(-1,-2)}}.$$

Compute the root system of $su(2)$.

$$\begin{aligned} su(2) &= \{X \in M_2(\mathbb{C}) : X + X^* = 0,\ \mathrm{tr}(X) = 0\} \\ &= \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} :\ a \in \mathbb{R},\ b \in \mathbb{C} \right\}. \end{aligned}$$

Denote $\mathfrak{g} = su(2)$. It is a real Lie algebra (of dimension 3), so does not automatically admit a root space decomposition which is available for every complex simple Lie algebra. In fact, by direct computation we see that there is no $\alpha \in \mathfrak{h}^*$, where

$$\mathfrak{h} = \left\{ \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix} :\ a \in \mathbb{R} \right\}$$

is a maximal abelian subspace of $su(2)$, such that

$$\{X \in \mathfrak{g} :\ [Y, X] = \langle \alpha, Y \rangle\, X \quad \forall Y \in \mathfrak{h}\} \neq \{0\}.$$

However, $\mathfrak{g}$ still has a good decomposition along the idea of root space decomposition. The idea of root space decomposition is that if $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is its Cartan subalgebra then $\mathfrak{g}$ is a direct sum of vector subspaces, each of which is invariant under every member of the set $\{\mathrm{ad}(Y)\}_{Y \in \mathfrak{h}} \subset \mathrm{End}(\mathfrak{g})$, where $\mathrm{ad}(Y)X = [Y, X]$. Each of these subspaces is one-dimensional over $\mathbb{C}$. In other words, $\mathfrak{g}$ has a basis that diagonalizes $\{\mathrm{ad}(Y)\}_{Y \in \mathfrak{h}}$ simultaneously.

In our situation, $\mathfrak{g}$ is a real Lie algebra, so the demand that every invariant subspace is one-dimensional over $\mathbb{R}$ could not be satisfied. But if we allow them to be of dimension less than or equal to 2, such a decomposition exists. We now derive that decomposition thanks to our earlier computation for $sl(2, \mathbb{C})$, the complexification of $su(2)$.

$$sl(2, \mathbb{C}) = \mathbb{C} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{A} \oplus \mathbb{C} \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{B} \oplus \mathbb{C} \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{C}.$$

We see that

$$\begin{aligned} A_1 &:= iA = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{g}, \\ A_2 &:= B - C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{g}, \\ A_3 &:= i(B + C) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{g}. \end{aligned}$$

Because A_1, A_2, A_3 are linearly independent over $\mathbb{R}$ and $\dim \mathfrak{g} = 3$, we have

$$\mathfrak{g} = \underbrace{\mathbb{R}A_1}_{V} \oplus \underbrace{\mathbb{R}A_2 \oplus \mathbb{R}A_3}_{W}.$$

For each $Y = \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix} \in \mathfrak{h}$, Y also belongs to the Cartan subalgebra of $sl(2,\mathbb{C})$. Thus,

$$\begin{aligned} \mathrm{ad}(Y)A_1 &= [Y,A_1] = 0 \in V, \\ \mathrm{ad}(Y)A_2 &= [Y,A_2] = [Y,B]-[Y,C] = 2iaB+2iaC = 2aA_3 \in W, \\ \mathrm{ad}(Y)A_3 &= [Y,A_3] = i[Y,B]+i[Y,C] = i(2iaB-2iaC) = -2aA_2 \in W. \end{aligned}$$

Therefore, V and W are invariant under every member of the set $\{\mathrm{ad}(Y)\}_{Y\in\mathfrak{h}} \subset \mathrm{End}(\mathfrak{g})$. For this reason, the decomposition $\mathfrak{g} = su(2) = V \oplus W$ is analogous to root space decomposition.

Compute the root system of $su(3)$.

$$\begin{aligned} su(3) &= \{X \in M_3(\mathbb{C}) : X + X^* = 0,\ \mathrm{tr}(X) = 0\} \\ &= \left\{ \begin{pmatrix} ia & c_1 & c_2 \\ -\bar{c}_1 & ib & c_3 \\ -\bar{c}_2 & -\bar{c}_3 & -ia-ib \end{pmatrix} : a,b \in \mathbb{R}, c_1,c_2,c_3 \in \mathbb{C} \right\}. \end{aligned}$$

Denote $\mathfrak{g} = su(3)$. It is a real Lie algebra of dimension 8. One of its maximal abelian subalgebra is

$$\mathfrak{h} = \left\{ \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & -ia-ib \end{pmatrix} : a,b \in \mathbb{R} \right\}.$$

Similarly to the situation of $su(2)$, we cannot expect $\mathfrak{g}$ to be a direct sum of one-dimensional subspaces, each of which is invariant under every member of the set $\{\mathrm{ad}(Y)\}_{Y\in\mathfrak{h}} \subset \mathrm{End}(\mathfrak{g})$. But if we allow them to be of dimension less than or equal to 2, such a decomposition exists. We now derive that decomposition thanks to our earlier computation for $sl(3,\mathbb{C})$, the complexification of $su(3)$.

$$sl(3,\mathbb{C}) = \underbrace{\mathbb{C}A \oplus \mathbb{C}B}_{Cartan\ subalgebra} \oplus \mathbb{C}X_{(1,-1)} \oplus \mathbb{C}X_{(-1,1)} \oplus \mathbb{C}X_{(2,1)} \oplus \mathbb{C}X_{(-2,-1)} \oplus \mathbb{C}X_{(1,2)} \oplus \mathbb{C}X_{(-1,-2)},$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We see that

$$A_1 := iA = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \in \mathfrak{g}, \quad A_2 := iB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \in \mathfrak{g},$$

$$A_3 := X_{(1,-1)} - X_{(-1,1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_4 := i\left(X_{(1,-1)} + X_{(-1,1)}\right) = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_5 := X_{(2,1)} - X_{(-2,-1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_6 := i\left(X_{(2,1)} + X_{(-2,-1)}\right) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_7 := X_{(1,2)} - X_{(-1,-2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_8 := i\left(X_{(1,2)} + X_{(-1,-2)}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \in \mathfrak{g}.$$

Because $A_1, A_2, ..., A_8$ are linearly independent over $\mathbb{R}$ and $\dim\mathfrak{g} = 8$,

$$\mathfrak{g} = \underbrace{\mathbb{R}A_1}_{V_1} \oplus \underbrace{\mathbb{R}A_2}_{V_2} \oplus \underbrace{\mathbb{R}A_3 \oplus \mathbb{R}A_4}_{V_3} \oplus \underbrace{\mathbb{R}A_5 \oplus \mathbb{R}A_6}_{V_4} \oplus \underbrace{\mathbb{R}A_7 \oplus \mathbb{R}A_8}_{V_5}.$$

For each $Y = \begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & -ia-ib \end{pmatrix} \in \mathfrak{h}$, Y also belongs to the Cartan subalgebra of $sl(3,\mathbb{C})$. Thus,

$$\mathrm{ad}(Y)A_1 = [Y, A_1] = 0 \in V_1,$$
$$\mathrm{ad}(Y)A_2 = [Y, A_2] = 0 \in V_2,$$

$$\begin{aligned} \mathrm{ad}(Y)A_3 = [Y, A_3] &= \left[Y, X_{(1,-1)}\right] - \left[Y, X_{(-1,1)}\right] \\ &= \langle(1,-1), Y\rangle X_{(1,-1)} - \langle(-1,1), Y\rangle X_{(-1,1)} \\ &= (ia - ib)X_{(1,-1)} - (-ia + ib)X_{(-1,1)} \\ &= (a-b)A_4 \in V_3, \end{aligned}$$

$$\begin{aligned} \mathrm{ad}(Y)A_4 = [Y, A_4] &= i\left[Y, X_{(1,-1)}\right] + i\left[Y, X_{(-1,1)}\right] \\ &= i\langle(1,-1), Y\rangle X_{(1,-1)} + i\langle(-1,1), Y\rangle X_{(-1,1)} \\ &= -(a-b)A_3 \in V_3. \end{aligned}$$

Similarly,

$$\begin{aligned} &\mathrm{ad}(Y)A_5 = (2a+b)A_6 \in V_4, \\ &\mathrm{ad}(Y)A_6 = -(2a+b)A_5 \in V_4, \\ &\mathrm{ad}(Y)A_7 = (a+2b)A_8 \in V_5, \\ &\mathrm{ad}(Y)A_8 = -(a+2b)A_7 \in V_5. \end{aligned}$$

Therefore, V_1, V_2, V_3, V_4, V_5 are invariant under every member of the set $\{\mathrm{ad}(Y)\}_{Y\in\mathfrak{h}} \subset \mathrm{End}(\mathfrak{g})$. For this reason, the decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$ is analogous to root space decomposition.

3

We draw the set of roots, called *root system*, for $sl(2,\mathbb{C})$ and $sl(3,\mathbb{C})$.
For $sl(2,\mathbb{C})$, let $\epsilon \in \mathfrak{h}^*$ be the coordinate map on $\mathfrak{h}$

$$\left\langle \varepsilon, \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\rangle = a.$$

For $sl(3,\mathbb{C})$, let $\epsilon_1, \epsilon_2 \in \mathfrak{h}^*$ be the coordinate maps on $\mathfrak{h}$

$$\left\langle \epsilon_1, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \right\rangle = a, \quad \left\langle \epsilon_2, \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} \right\rangle = b.$$

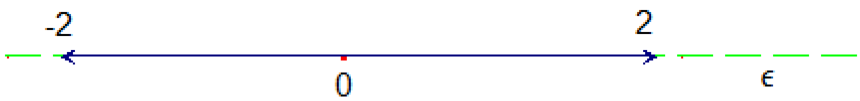

Figure 1: Root system of $sl(2,\mathbb{C})$.

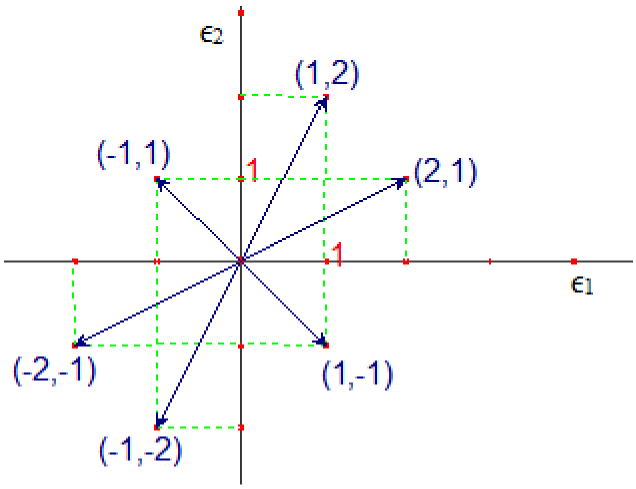

Figure 2: Root system of $sl(3,\mathbb{C})$.

4

We explain the how root systems can be used to classify complex simple Lie algebras.

Two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are said to be *isomorphic* if there is a map $T : \mathfrak{g}_1 \to \mathfrak{g}_2$ that is a linear isomorphism and preserves the Lie bracket.

A *simple* Lie algebra is a nonabelian Lie algebra that has no ideal other than $\{0\}$ and itself. In other words, a Lie algebra $\mathfrak{g}$ is simple if

- $[\mathfrak{g}, \mathfrak{g}] \not\equiv 0$,
- if $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$ then $\mathfrak{h} = \{0\}$ or $\mathfrak{h} = \mathfrak{g}$.

Isomorphic simple Lie algebras have 'equivalent' root space decompositions. Conversely, a list of a few features of the root system is enough to determine the Lie algebra up to an isomorphism. These features are encoded in a graph, called *Dynkin diagram*. We describe it in detail as follows.

Let Φ be the root system of a complex simple Lie algebra $\mathfrak{g}$ (relative to a choice of Cartan algebra $\mathfrak{h}$). A subset $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_r\}$ of Φ is called a set of *simple roots* if every $\gamma \in \Phi$ can be written uniquely as $\gamma = n_1\alpha_1 + ... + n_r\alpha_r$ with $n_1, ..., n_r \in \mathbb{Z}$, all of the same sign. For example, according to our earlier computation, a set of simple root of $sl(2, \mathbb{C})$ is $\{2\}$; that of $sl(3, \mathbb{C})$ is $\{(1, -1), (1, 2)\}$.

Let $\{\epsilon_1, \epsilon_2, ..., \epsilon_m\}$ be a basis of $\mathfrak{h}^*$. We know that each root of $\mathfrak{g}$ is a complex linear combination of $\epsilon_1, \epsilon_2, ..., \epsilon_m$. We can define a scalar product between two roots as

$$(\beta, \gamma) = \sum_{j=1}^{r} \beta_j \bar{\gamma}_j,$$

where $\beta = \sum \beta_j \epsilon_j$ and $\gamma = \sum \gamma_j \epsilon_j$. The length of root β is defined as $|\beta| = \sqrt{(\beta, \beta)}$.

The Dynkin diagram for Φ (relative to a choice of simple roots Δ) is a graph having vertices $v_1, v_2, ..., v_r$. Consider two distinct indices i and j. If α_i and α_j are orthogonal, we put no edge between v_i and v_j. If α_i and α_j are not orthogonal, we put one edge between v_i and v_j if $|\alpha_i| = |\alpha_j|$, two edges with an arrow from v_i to v_j if $|\alpha_i| = \sqrt{2}|\alpha_j|$, three edges with an arrow from v_i to v_j if $|\alpha_i| = \sqrt{3}|\alpha_j|$.

Two Dynkin diagrams are said to be *equivalent* if there is a bijective map of the vertices that preserves the edges and the direction of arrows. It is known that two simple Lie algebra are equivalent if and only if they have equivalent Dynkin diagrams. That makes the characterization of simple Lie algebras possible. In 1894, E. Cartan characterized all complex simple Lie algebras. Accordingly, every complex simple Lie algebra is isomorphic to exactly one algebra in the following list.

- $sl(l + 1, \mathbb{C})$ $(l \geq 1)$, where

$$sl(l + 1, \mathbb{C}) = \{A \in M_{l+1}(\mathbb{C}) : \ \mathrm{tr}(A) = 0\}.$$

- $so(2l + 1, \mathbb{C})$ $(l \geq 2)$, where

$$so(2l + 1, \mathbb{C}) = \left\{A \in M_{2l+1}(\mathbb{C}) : A + A^{\mathrm{t}} = 0\right\}.$$

- $sp(l, \mathbb{C})$ $(l \geq 3)$, where

$$sp(l, \mathbb{C}) = \left\{A \in M_{2l}(\mathbb{C}) : A^{\mathrm{t}} \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix} A = 0\right\}.$$

- $so(2l, \mathbb{C})$ ($l \geq 4$), where

$$so(2l, \mathbb{C}) = \left\{ A \in M_{2l}(\mathbb{C}) : A + A^{\mathrm{t}} = 0 \right\}.$$

- The exceptional Lie algebras G_2, F_4, E_6, E_7, E_8 whose dimensions are respectively 14, 52, 78, 133, 248.

Figure 3: The Dynkin diagram of $sl(l+1, \mathbb{C})$.

Figure 4: The Dynkin diagram of $so(2l+1, \mathbb{C})$.

Figure 5: The Dynkin diagram of $sp(l, \mathbb{C})$.

Figure 6: The Dynkin diagram of $so(2l, \mathbb{C})$.

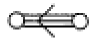

Figure 7: The Dynkin diagram of G_2.

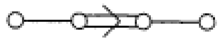

Figure 8: The Dynkin diagram of F_4.

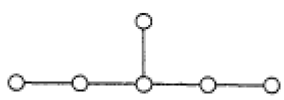

Figure 9: The Dynkin diagram of E_6.

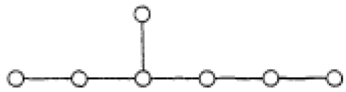

Figure 10: The Dynkin diagram of E_7.

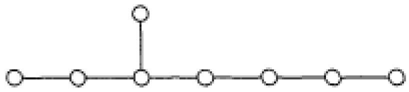

Figure 11: The Dynkin diagram of E_8.

References

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