Some review in Lie theory

The write-up answers the following questions.

- Explain the principle that a set of mutually commuting diagonalizable matrices can be diagonalized by the same basis.
- Compute the root systems for $sl(2,\mathbb{C})$, $sl(3,\mathbb{C})$, $su(2)$, $su(3)$.
- Draw those root systems.
- Explain the characterization of complex simple Lie algebras.

(1) Explain the principle that mutually commuting diagonalizable matrices can be diagonalized by the same basis.

Let $\mathfrak{a}$ be a set of mutually commuting diagonalizable matrices in $M_n(\mathbb{C})$. In linear algebra, the principle can be explained by first noticing that for two commuting matrices A and B, each eigenspace of A is invariant under B. Fixing $A \in \mathfrak{a}$, we see that every element of $\mathfrak{a}$, when viewed as a linear operator on an ~~eigenvalue~~ eigenspace of A, commutes with A. The problem then reduces to m similar problems, where m is the number of eigenspaces of A, in dimension less than n. This observation can be made into a formal proof by induction on n.

Lie theory provides another way for us to see why the principle is true. It comes from the fact that every finite-dimensional representation of a Lie algebra admits a weight space decomposition. We view $M_n(\mathbb{C})$ as a complex Lie algebra

$\mathfrak{g}$ with Lie bracket $[A,B] = AB - BA$, and diagonalizable matrices as semisimple elements of $\mathfrak{g}$. Because $\mathfrak{a}$ is a mutually commuting set, so is the vector subspace of $\mathfrak{g}$ generated by $\mathfrak{a}$. Thus, we can assume WLOG that $\mathfrak{a}$ is a vector subspace of $\mathfrak{g}$. Moreover, $\mathfrak{a}$ is also a Lie subalgebra of $\mathfrak{g}$ because $[A,B] = AB - BA = 0 \in \mathfrak{a}$ for all $A, B \in \mathfrak{a}$.

We assume the fact that the sum of two commuting semisimple elements of $\mathfrak{g}$ is also a semisimple element. In terms of matrices, this assumption says that the sum of two commuting diagonalizable matrices is a diagonalizable matrix. Then every element of $\mathfrak{a}$ is semisimple. We know that the map $\mathrm{ad}: \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, $\mathrm{ad}(X)Y = [X,Y]$ is a Lie algebra representation of $\mathfrak{g}$. Thus, it maps semisimple elements to semisimple elements. This implies $\mathrm{ad}(A)$ is semisimple for every $A \in \mathfrak{a}$. Then $\mathfrak{a}$ is a toral subalgebra of $\mathfrak{g}$. Thus, $\mathfrak{a}$ is contained in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Consider the following Lie algebra representation of $\mathfrak{g}$, so called the defining representation $\pi: \mathfrak{g} \to \mathrm{End}(\mathbb{C}^n)$, $\pi(X)v = Xv$ for all $X \in \mathfrak{g}$, $v \in \mathbb{C}^n$. For each $\mu \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$), we define a vector subspace of $\mathbb{C}^n$.

$$\mathbb{C}^n(\mu) := \{ v \in \mathbb{C}^n : \pi(A)v = \langle \mu, A \rangle v \quad \forall A \in \mathfrak{h} \}.$$

Then $(\pi, \mathbb{C}^n)$ admits a weight space decomposition

$$\mathbb{C}^n = \bigoplus_{\mu \in \mathfrak{h}^*} \mathbb{C}^n(\mu).$$

Since $\mathbb{C}^n$ is finite dimensional, only finitely many summands are nonzero.

Each element of $\mathfrak{h}$, if viewed as an operator from $\mathbb{C}^n$ to $\mathbb{C}^n$, acts by scalar on each subspace $\mathbb{C}^n(\mu)$. Therefore, the basis of $\mathbb{C}^n$ obtained by concatenating arbitrarily chosen bases of $\mathbb{C}^n(\mu)$, $\mu \in \mathfrak{h}^*$, diagonalizes every element of $\mathfrak{a}$.

② Compute the root system of $sl(2,\mathbb{C})$

$$sl(2,\mathbb{C}) = \{A \in M_2(\mathbb{C}) : \mathrm{tr}(A) = 0\}.$$

Denote $\mathfrak{g} = sl(2,\mathbb{C})$. It has a Cartan subalgebra $\mathfrak{h} = \left\{ \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} : s \in \mathbb{C} \right\}$.

Because $\mathfrak{h}$ is a one-dimensional vector space, its dual $\mathfrak{h}^*$ is also one-dimensional. We can identify $\mathfrak{h}^*$ with $\mathbb{C}$ by defining

$$\left\langle \alpha, \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \right\rangle := \alpha s \quad \forall \alpha, s \in \mathbb{C}.$$

A root of $\mathfrak{g}$ is defined as an element $\alpha \in \mathbb{C}\setminus\{0\}$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [Y, X] = \langle \alpha, Y \rangle X \quad \forall Y \in \mathfrak{h}\} \neq \{0\}.$$

We have

$$\underbrace{\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}}_{Y} \underbrace{\begin{pmatrix} a & b \\ c & -a \end{pmatrix}}_{X} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} = \left\langle \alpha, \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \right\rangle \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad \forall s \in \mathbb{C}$$

$$\Longleftrightarrow \quad \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix} = \alpha \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{cases} a = 0, \\ (\alpha+2)c = 0, \\ (\alpha-2)b = 0. \end{cases}$$

The system has nontrivial solution $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ if and only if $\alpha = \pm 2$.

Therefore, $sl(2,\mathbb{C})$ has 2 roots. The root space decomposition is

$$sl(2,\mathbb{C}) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_{-2},$$

where

$$\mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{g}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

<u>Compute the root system of $sl(3,\mathbb{C})$</u>

$$sl(3,\mathbb{C}) = \{A \in M_3(\mathbb{C}) : \mathrm{tr}(A) = 0\}.$$

Denote $\mathfrak{g} = sl(3,\mathbb{C})$. It has a Cartan subalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

Because $\mathfrak{h}$ is two-dimensional, its dual $\mathfrak{h}^*$ is also two dimensional. We can identify $\mathfrak{h}^*$ with $\mathbb{C}^2$ by defining

$$\left\langle (x,y), \begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} \right\rangle := xa + yb \quad \forall a,b,x,y \in \mathbb{C}.$$

A root of $\mathfrak{g}$ is an element $(x,y) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that the space

$$\mathfrak{g}_{(x,y)} = \{X \in \mathfrak{g} : [Y,X] = \langle (x,y), Y\rangle X \quad \forall Y \in \mathfrak{h}\} \neq \{0\}.$$

We have

$$\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix}\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix} - \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix}\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} = \left\langle (x,y), \begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} \right\rangle \times$$

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix} \forall a,b \in \mathbb{C}$$

$$\Longleftrightarrow \begin{pmatrix} 0 & c_2(a-b) & c_3(2a+b) \\ c_4(b-a) & 0 & c_6(a+2b) \\ c_7(-2a-b) & c_8(-a-2b) & 0 \end{pmatrix} = (ax+by)\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix}$$

$$\forall a, b \in \mathbb{C}$$

$$\Longleftrightarrow \begin{cases} c_1 = c_5 = 0 \\ c_2[a(1-x)+b(-1-y)] = 0 \\ c_3[a(2-x)+b(1-y)] = 0 \\ c_4[a(-1-x)+b(1-y)] = 0 \\ c_6[a(1-x)+b(2-y)] = 0 \\ c_7[a(-2-x)+b(-1-y)] = 0 \\ c_8[a(-1-x)+b(-2-y)] = 0 \end{cases} \qquad \forall a, b \in \mathbb{C}.$$

This system has nontrivial solution $\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & -c_1-c_5 \end{pmatrix}$ if and only if

$$(x,y) \in \{\pm(1,-1), \pm(2,1), \pm(1,2)\}.$$

Therefore, $sl(3,\mathbb{C})$ has 6 roots. The root space decomposition is

$$sl(3,\mathbb{C}) = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{(1,-1)} \oplus \mathfrak{g}_{(-1,1)} \oplus \mathfrak{g}_{(2,1)} \oplus \mathfrak{g}_{(-2,-1)} \oplus \mathfrak{g}_{(1,2)} \oplus \mathfrak{g}_{(-1,-2)}$$

where $\mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$,

$$\mathfrak{g}_{(1,-1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(1,-1)}}, \quad \mathfrak{g}_{(-1,1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(-1,1)}},$$

$$\mathfrak{g}_{(2,1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(2,1)}}, \qquad \mathfrak{g}_{(-2,-1)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{X_{(-2,-1)}},$$

$$\mathfrak{g}_{(1,2)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{X_{(1,2)}}, \qquad \mathfrak{g}_{(-1,-2)} = \mathbb{C}\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{X_{(-1,-2)}}.$$

<u>Compute the root system of $su(2)$</u>

$$su(2) = \{ X \in M_2(\mathbb{C}) : X + X^* = 0,\ tr(X) = 0 \}.$$

$$= \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} : a \in \mathbb{R},\ b \in \mathbb{C} \right\}.$$

Denote $\mathfrak{g} = su(2)$. It is a real Lie algebra (of dimension 3), so does not automatically possess a root space decomposition which is available for complex Lie algebras. In fact, by direct computation we see that there is no $\alpha \in \mathfrak{h}^*$, where $\mathfrak{h} = \left\{ \begin{pmatrix} ia & \\ & -ia \end{pmatrix} : a \in \mathbb{R} \right\}$ is the maximal abelian subspace of $su(2)$, such that

$$\{ X \in \mathfrak{g} : [Y, X] = \langle \alpha, Y \rangle X \quad \forall Y \in \mathfrak{h} \} \neq \{0\}.$$

However, $\mathfrak{g}$ still has a good decomposition along the idea of root space decomposition. The idea of root space decomposition is that if $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is its Cartan subalgebra then $\mathfrak{g}$ is a direct sum of vector subspaces, each of which is invariant under every member of the set $\{ad(Y)\}_{Y\in\mathfrak{h}} \subset End(\mathfrak{g})$, where $ad(Y)X := [Y,X]$. All of these subspaces are one-dimensional over $\mathbb{C}$. (In other words, $\mathfrak{g}$ has a basis that diagonalizes $\{ad(Y)\}_{Y\in\mathfrak{h}}$ simultaneously.)

In our situation, $\mathfrak{g}$ is real Lie algebra, so the demand that the invariant subspaces are one-dimensional over $\mathbb{R}$ could not be satisfied. But if we allow them to be of dimension ≤ 2, such a decomposition exists. We now derive that decomposition thanks to our earlier computation for $sl(2,\mathbb{C})$, the complexification of $su(2)$.

$$sl(2,\mathbb{C}) = \mathbb{C}\underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{A} \oplus \mathbb{C}\underbrace{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}_{B} \oplus \mathbb{C}\underbrace{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}}_{C}.$$

We see that
$$A_1 := iA = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \in \mathfrak{g},$$
$$A_2 := B - C = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \in \mathfrak{g},$$
$$A_3 := i(B+C) = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \in \mathfrak{g}.$$

Because A_1, A_2, A_3 are linearly independent over $\mathbb{R}$ and $\dim \mathfrak{g} = 3$, we have
$$\mathfrak{g} = \underbrace{\mathbb{R}A_1}_{V} \oplus \underbrace{\mathbb{R}A_2 \oplus \mathbb{R}A_3}_{W}.$$

For each $Y \in \mathfrak{h}$, Y also belongs to the Cartan subalgebra of $sl(2,\mathbb{C})$. Thus,
$$ad(Y)A_1 = [Y, A_1] = 0 \in V,$$
$$ad(Y)A_2 = [Y, A_2] = [Y,B] - [Y,C] = 2iaB + 2iaC = 2aA_3 \in W$$

(here $Y = \begin{pmatrix} ia & \\ & -ia \end{pmatrix}$ for some $a \in \mathbb{R}$)

$$ad(Y)A_3 = [Y, A_3] = i[Y,B] + i[Y,C] = 2i^2 aB + 2i^2 aC = -2aA_2 \in W.$$

Therefore, V and W are invariant under every member of the set $\{ad(Y)\}_{Y \in \mathfrak{h}} \subset \mathrm{End}(\mathfrak{g})$. For this reason, the decomposition $\mathfrak{g} = su(2) = V \oplus W$ is analoguous to root space decomposition.

<u>Compute the root system of $su(3)$</u>

$$su(3) = \{ X \in M_3(\mathbb{C}) : X + X^* = 0,\ tr(X) = 0 \}$$

$$= \left\{ \begin{pmatrix} ia & c_1 & c_2 \\ -\bar{c}_1 & ib & c_3 \\ -\bar{c}_2 & -\bar{c}_3 & -ia-ib \end{pmatrix} : a, b \in \mathbb{R},\ c_1, c_2, c_3 \in \mathbb{C} \right\}.$$

Denote $\mathfrak{g} = su(3)$. It is a real Lie algebra of dimension 8. One of its maximal abelian subalgebra is

$$\mathfrak{h} = \left\{ \begin{pmatrix} ia & & \\ & ib & \\ & & -ia-ib \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Similarly to the situation of $su(2)$, we cannot expect $\mathfrak{g}$ to be a direct sum of one-dimensional subspaces, each of which is invariant under every member of the family $\{ad(Y)\}_{Y \in \mathfrak{h}} \subset \mathrm{End}(\mathfrak{g})$. But if we allow them to be of dimension ≤ 2, such a decomposition exists. We now derive that decomposition thanks to our earlier computation for $sl(3,\mathbb{C})$, the complexification of $su(3)$.

$$\mathfrak{sl}(3,\mathbb{C}) = \underbrace{\mathbb{C}A \oplus \mathbb{C}B}_{\substack{\text{Cartan subalgebra} \\ \text{of } \mathfrak{sl}(3,\mathbb{C})}} \oplus \mathbb{C}X_{(1,-1)} \oplus \mathbb{C}X_{(-1,1)} \oplus \mathbb{C}X_{(2,1)} \oplus \mathbb{C}X_{(-2,-1)} \oplus \mathbb{C}X_{(1,2)} \oplus \mathbb{C}X_{(-1,-2)}.$$

where
$$A = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

We see that
$$A_1 := iA = \begin{pmatrix} i & & \\ & 0 & \\ & & -i \end{pmatrix} \in \mathfrak{g},$$

$$A_2 := iB = \begin{pmatrix} 0 & & \\ & i & \\ & & -i \end{pmatrix} \in \mathfrak{g},$$

$$A_3 := X_{(1,-1)} - X_{(-1,1)} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_4 := i(X_{(1,-1)} + X_{(-1,1)}) = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_5 := X_{(2,1)} - X_{(-2,-1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_6 := i(X_{(2,1)} + X_{(-2,-1)}) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_7 := X_{(1,2)} - X_{(-1,-2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in \mathfrak{g},$$

$$A_8 := i(X_{(1,2)} + X_{(-1,-2)}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \in \mathfrak{g}.$$

Because $A_1, A_2, \dots, A_8$ are linearly independent over $\mathbb{R}$ and $\dim \mathfrak{g} = 8$,

$$\mathfrak{g} = \underbrace{\mathbb{R}A_1}_{V_1} \oplus \underbrace{\mathbb{R}A_2}_{V_2} \oplus \underbrace{\mathbb{R}A_3 \oplus \mathbb{R}A_4}_{V_3} \oplus \underbrace{\mathbb{R}A_5 \oplus \mathbb{R}A_6}_{V_4} \oplus \underbrace{\mathbb{R}A_7 \oplus \mathbb{R}A_8}_{V_5}.$$

For each $Y = \begin{pmatrix} ia & & \\ & ib & \\ & & -ia-ib \end{pmatrix} \in \mathfrak{h}$, Y also belongs to the Cartan subalgebra of $sl(3,\mathbb{C})$. Thus,

$$ad(Y)A_1 = [Y, A_1] = 0 \in V_1,$$

$$ad(Y)A_2 = [Y, A_2] = 0 \in V_2,$$

$$\begin{aligned} ad(Y)A_3 = [Y, A_3] &= [Y, X_{(1,-1)}] - [Y, X_{(-1,1)}] \\ &= \langle (1,-1), Y \rangle X_{(1,-1)} - \langle (-1,1), Y \rangle X_{(-1,1)} \\ &= (ia - ib) X_{(1,-1)} - (-ia + ib) X_{(-1,1)} \\ &= (a-b) A_4 \in V_3, \end{aligned}$$

$$\begin{aligned} ad(Y)A_4 = [Y, A_4] &= i[Y, X_{(1,-1)}] + i[Y, X_{(-1,1)}] \\ &= i\langle (1,-1), Y \rangle X_{(1,-1)} + i\langle (1,-1), Y \rangle X_{(-1,1)} \\ &= -(a-b) A_3 \in V_3. \end{aligned}$$

Similarly,

$$ad(Y)A_5 = (2a+b) A_6 \in V_4,$$

$$ad(Y)A_6 = -(2a+b) A_5 \in V_4,$$

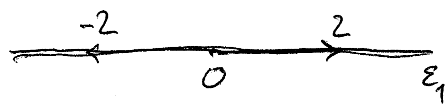
$$ad(Y)A_7 = (a+2b) A_8 \in V_5,$$

$$ad(Y)A_8 = -(a+2b) A_7 \in V_5.$$

Therefore, V_1, V_2, V_3, V_4, V_5 are invariant under every member of the set $\{ad(Y)\}_{Y\in\mathfrak{h}} \subset End(\mathfrak{g})$. For this reason, the decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus .. \oplus V_5$ is analoguous to root space decomposition.

③ Draw the root systems of $sl(2,\mathbb{C})$ and $sl(3,\mathbb{C})$

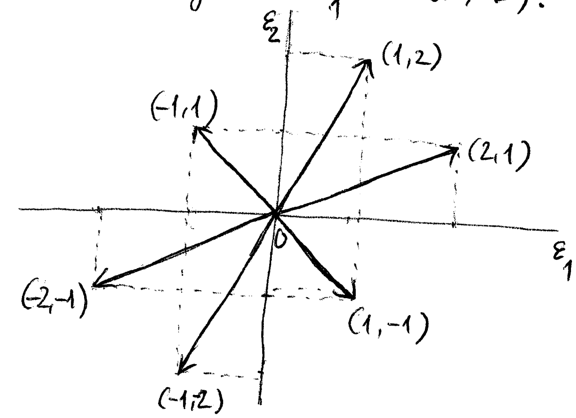
Root system of $sl(2,\mathbb{C})$:



ε_1 is the coordinate map on $\mathfrak{h}$

$$\varepsilon\begin{pmatrix} a & \\ & -a \end{pmatrix} = a.$$

Root system of $sl(3,\mathbb{C})$:



$\varepsilon_1, \varepsilon_2$ are the coordinate maps on $\mathfrak{h}$.

$$\varepsilon_1\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} = a,$$

$$\varepsilon_2\begin{pmatrix} a & & \\ & b & \\ & & -a-b \end{pmatrix} = b.$$

④ Explain the classification of complex simple Lie algebras

Two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are said to be isomorphic if there is a map $T: \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

- T is a linear isomorphism,
- T preserves Lie bracket.

A simple Lie algebra is a nonabelian Lie algebra that has no ideal other than $\{0\}$ and itself. In other words, a Lie algebra $\mathfrak{g}$ is simple if

- $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$,
- if $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$ then $\mathfrak{h} = \{0\}$ or $\mathfrak{g}$.

Isomorphic simple Lie algebras have equivalent root space decompositions. Conversely, a list of a few features of the root system (including the linear dependency among the roots, order of lengths of the roots, ...) is enough to determine the Lie algebra up to an isomorphism. That makes the characterization of simple Lie algebras possible. In 1894, E. Cartan characterized all complex simple Lie algebras. Accordingly, every complex simple Lie algebra is isomorphic to exactly one algebra in the following list.

1) $sl(l+1, \mathbb{C})$, $l \geq 1$.
2) $so(2l+1, \mathbb{C})$, $l \geq 2$.
3) $sp(l, \mathbb{C})$, $l \geq 3$.
4) $so(2l, \mathbb{C})$, $l \geq 4$.
5) The exceptional Lie algebras G_2, F_4, E_6, E_7, E_8 whose dimensions are respectively $14, 52, 78, 133, 248$.