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Math 8271 : Lie Algebras
Homework #1

① Problem 2, Goodman-Wallach, p. 11

Define a bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $B(x,y) = \sum_{i=1}^{n} x_i y_{n+1-i}$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. We'll determine the signature of B.

Denote by $\mathcal{B}_0 = (e_1, \dots, e_n)$ the standard basis of $\mathbb{R}^n$. Because B is a symmetric form, we can define a ~~form~~ quadratic form $Q(v) = B(v,v)$ $\forall v \in \mathbb{R}^n$.

Suppose that $(v_1, \dots, v_n)$ is a pseudo-orthogonal basis of V relative to B. Then there exist $0 \le p, q \le n$, $p+q = n$ such that

$$\left(B(v_i, v_j)\right)_{1 \le i,j \le n} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

For each $v = \sum_{i=1}^{n} a_i v_i$, $a_i \in \mathbb{R}$, we get $Q(v) = \sum_{i,j=1}^{n} a_i a_j B(v_i, v_j)$

$$= a_1^2 + \dots + a_p^2 - a_{p+1}^2 - \dots - a_n^2 .$$

Thus $p-q$ is also the signature of Q. Thus the problem now is to find the signature of Q. By the Law of Inertia, this quantity does not depend on the choice of basis. We have

$$Q(x) = B(x,x) = \sum_{i=1}^{n} x_i x_{n+1-i} = x_1 x_n + x_2 x_{n-1} + \dots + x_n x_1 \quad \text{where } [x]_{\mathcal{B}_0} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

We consider two cases:

- $\underline{n \text{ is even}}$, e.g. $n = 2m$

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Then
$$\begin{aligned}Q(x) &= 2x_1x_{2m} + 2x_2x_{2m-1} + \cdots + 2x_mx_{m+1}\\ &= \left[\left(\frac{x_1+x_{2m}}{\sqrt{2}}\right)^2 - \left(\frac{x_1-x_{2m}}{\sqrt{2}}\right)^2\right] + \cdots + \left[\left(\frac{x_m+x_{m+1}}{\sqrt{2}}\right)^2 - \left(\frac{x_m-x_{m+1}}{\sqrt{2}}\right)^2\right]\\ &= y_1^2 + \cdots + y_m^2 - y_{m+1}^2 - \cdots - y_{2m}^2,\end{aligned}$$

where
$$y_1 = \frac{x_1+x_{2m}}{\sqrt{2}},\quad y_2 = \frac{x_2+x_{2m-1}}{\sqrt{2}},\ \dots,\ y_m = \frac{x_m+x_{m+1}}{\sqrt{2}},$$
$$y_{m+1} = \frac{x_1-x_{2m}}{\sqrt{2}},\quad y_{m+2} = \frac{x_2-x_{2m-1}}{\sqrt{2}},\ \dots,\ y_{2m} = \frac{x_m-x_{m+1}}{\sqrt{2}}.$$

Recall that we have the formula for changing coordinates as foll

$$[x]_{\mathcal{B}} = P(\mathcal{B} \to \mathcal{B}_0)[x]_{\mathcal{B}_0}$$

We have
$$\begin{pmatrix} y_1 \\ \vdots \\ y_{2m}\end{pmatrix} = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix} 1 & & & & & 1\\ & \ddots & 0 & & \unicode{x22F0} & \\ & 0 & 1 & 1 & 0 & \\ & & 1 & -1 & & \\ & \unicode{x22F0} & 0 & & \ddots & \\ 1 & & & & & -1\end{pmatrix}}_{P}\begin{pmatrix} x_1 \\ \vdots \\ x_{2m}\end{pmatrix}$$

Let $\mathcal{B}$ be the basis of $\mathbb{R}^n$ such that $P = P(\mathcal{B} \to \mathcal{B}_0)$. Note th

$\mathcal{B}$ can be found explicitly as follows. Put $\mathcal{B} = (w_1, \dots, w_n)$. Then

$$P^{-1} = P(\mathcal{B}_0 \to \mathcal{B}) = \left([w_1]_{\mathcal{B}_0}\ [w_2]_{\mathcal{B}_0} \cdots [w_n]_{\mathcal{B}_0}\right).$$

Then
$$[x]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ \vdots \\ y_{2m}\end{pmatrix}.$$

Thus, in basis $\mathcal{B}$, Q has the standard form $Q(x) = y_1^2 + \cdots + y_m^2 - y_{m+1}^2 -$

Because the number of positive terms and negative terms are equa the ~~sg~~ signature of Q is 0.

n is odd, e.g. $n = 2m+1$

Then $Q(x) = 2x_1 x_{2m+1} + \cdots + 2x_m x_{m+2} + x_{m+1}^2$.

$$= \left[\left(\frac{x_1 + x_{2m+1}}{\sqrt{2}}\right)^2 - \left(\frac{x_1 - x_{2m+1}}{\sqrt{2}}\right)^2\right] + \cdots + \left[\left(\frac{x_m + x_{m+2}}{\sqrt{2}}\right)^2 - \left(\frac{x_m - x_{m+2}}{\sqrt{2}}\right)^2\right] + x_{m+1}^2$$

$$= y_1^2 + \cdots + y_m^2 + y_{m+1}^2 - y_{m+2}^2 - \cdots - y_{2m+1}^2,$$

where $y_1 = \dfrac{x_1 + x_{2m+1}}{\sqrt{2}}$, $y_2 = \dfrac{x_2 + x_{2m}}{\sqrt{2}}$, $\ldots$, $y_m = \dfrac{x_m + x_{m+2}}{\sqrt{2}}$, $y_{m+1} = x_{m+1}$,

$y_{m+2} = \dfrac{x_1 - x_{2m+1}}{\sqrt{2}}$, $\ldots$, $y_{2m+1} = \dfrac{x_m - x_{m+2}}{\sqrt{2}}$.

We have

$$\begin{pmatrix} y_1 \\ \vdots \\ y_{2m+1} \end{pmatrix} = \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} 1 & & & 0 & & & 1 \\ & \ddots & & & & \iddots & \\ 0 & & & 1 & & & 0 \\ & & 1 & 0 & -1 & & \\ & \iddots & & & & \ddots & \\ 1 & & & 0 & & & -1 \end{pmatrix}}_{P} \begin{pmatrix} x_1 \\ \vdots \\ x_{2m+1} \end{pmatrix}$$

Let $\mathcal{B}$ be the basis of $\mathbb{R}^n$ such that $\mathcal{B} = (w_1, w_2, \ldots, w_n)$ where

$$P^{-1} = P(\mathcal{B}_0 \to \mathcal{B}) = ([w_1]_{\mathcal{B}_0}\ [w_2]_{\mathcal{B}_0} \cdots [w_n]_{\mathcal{B}_0})$$

Then $[x]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ \vdots \\ y_{2m+1} \end{pmatrix}$

Thus, in basis $\mathcal{B}$, Q has the standard form $Q(x) = y_1^2 + \cdots + y_{m+1}^2 - y_{m+2}^2 - \cdots - y_{2m+1}^2$

The signature of Q is $(m+1) - m = 1$. Therefore, we conclude

$$\operatorname{sig}(B) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

(2) Problem 3, Goodman-Wallach, p. 11.

Let B be a skew-symmetric or symmetric nondegenerate bilinear form on a vector space V. Let W be a subspace of V such that $B|_{W\times W}$ is also nondegenerate. Define $W^{\perp} = \{v \in V : B(v,w) = 0 \ \forall w \in W\}$.

Because B is bilinear, $W^{\perp}$ is a subspace of V. We'll show that $V = W \oplus W^{\perp}$. Let $x \in W \cap W^{\perp}$. Because $x \in W^{\perp}$, $B(x,v) = 0$ for all $v \in W$. Since $x \in W$ and $B|_{W\times W}$ is nondegenerate, $x = 0$. Thus $W \cap W^{\perp} = \{0\}$.

Take any $y \in V$. We'll show that $y \in W + W^{\perp}$. Note that this claim is true if W is finite-dimensional. Otherwise, it may not be true. For example, $V \cancel{W} = L^2(\mathbb{R})$ is a vector space over $\mathbb{R}$ with bilinear form $B(u,v) = \int_{\mathbb{R}} uv\,dx$.

Take $W = \mathcal{D}(\mathbb{R})$. Then $W^{\perp} = \{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} u\phi\,dx = 0 \ \forall \phi \in \mathcal{D}(\mathbb{R})\} = \{0\}$ by the fundamental theorem of calculus of variations. Then $W \oplus W^{\perp} = W \neq V$.

Now we assume that W is finite-dimensional. Suppose that $\dim W = n$. We consider 2 cases as follows.

- <u>B is a symmetric form</u>

Because B is a symmetric nondegenerate bilinear form on W, by lemm 1.1.2, Goodman-Wallach p. 4, there exists a basis $(v_1, v_2, \dots, v_n)$ of W in which

$$B(v_i, v_j) = \begin{cases} 1 & \text{if } 1 \le \cancel{p\,r}\, i = j \le p, \\ -1 & \text{if } p < i = j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

To show that $y \in W + W^\perp$, it suffices to show that $y' = y - \sum_{j=1}^{n} \frac{B(y,v_j)}{B(v_j,v_j)} v_j$ belongs to $W^\perp$. To do so, we only need to show that for each $1 \le i \le n$, $B(v_i, y') = 0$. We have

$$B(v_i, y') = B(v_i, y) - \sum_{j=1}^{n} \frac{B(y,v_j)}{B(v_j,v_j)} B(v_i, v_j) = B(v_i, y) - B(y, v_j) = 0.$$

- <u>B is a skew-symmetric form</u>

Because B is a nondegenerate skew-symmetric form on W, by Lemma 1.1.5, Goodman-Wallach p. 7, $n = \dim W$ is even, i.e. $n = 2k$. Also, there exists a basis $(v_1, \ldots, v_{2k})$ of W in which

$$B(v_i, v_j) = \begin{cases} 1 & \text{if } i+j = 2k+1,\ 1 \le i \le k, \\ -1 & \text{if } i+j = 2k+1,\ k+1 \le i \le 2k \\ 0 & \text{otherwise} \end{cases}$$

To show that $y \in W + W^\perp$, it suffices to show that $y' = y + \sum_{j=1}^{2k} \frac{B(v_j, y)}{B(v_j, v_{2k+1-j})} v_j$ belongs to $W^\perp$. To do so, we only need to show that for each $1 \le i \le 2k$, $B(v_i, y') = 0$. We have

$$B(v_i, y') = B(v_i, y) + \sum_{j=1}^{2k} \frac{B(v_j, y)}{B(v_j, v_{2k+1-j})} B(v_i, v_j) = B(v_i, y) - B(v_i, y) = 0.$$

We have proved that $V = W \oplus W^\perp$. Now we'll show that B is nondegenerate as a bilinear map on $W^\perp$. Let $x \in W^\perp$ be such that $B(x, v) = 0$ for all $v \in W^\perp$. Because $x \in W^\perp$, $B(x, v) = 0$ for all $v \in W$. Since $V = W + W^\perp$, we have $B(x,v)=0$ for all $v\in V$. Since B is nondegenerate on V, we get ~~$x\in$~~ $x=0$. Thus B is nondegenerate on $W^\perp$.

Note that if W is infinite dimensional, the conclusion of the problem may be wrong. For example, $V=C([0,1])$ is a vector space over $\mathbb{R}$ and $B: V\times V\to\mathbb{R}$, $B(u,v)=u(0)v(0)-\int_0^1 uv\,dx$ is a symmetric bilinear form on V. Take $W=\{v\in\mathcal{D}((0,1)) \mid \int_0^1 v\,dx=0\}$.

If $v_0\in C([0,1])$ satisfies $B(v_0,v)=0$ for all $v\in V$ then

$$v_0(0)v(0)=\int_0^1 v(x)v_0(x)\,dx \quad \forall v\in V.$$

In particular, $\int_0^1 v(x)v_0(x)\,dx=0$ for all $v\in\mathcal{D}((0,1))$.

Thus $v_0\equiv 0$ by the fundamental lemma of calculus of variations. Thus B is nondegenerate on V.

If $v_0\in W$ satisfies $B(v_0,v)=0$ $\forall v\in W$, then $\int_0^1 v_0 v\,dx=0$ $\forall v\in W$. Take any $\eta\in\mathcal{D}((0,1))$ such that $\int_0^1\eta\,dx=1$. Then put $c=\int_0^1 v_0\eta\,dx$. We have $\int_0^1(v_0-c)v\,dx=0$ $\forall v\in W$ and $\int_0^1(v_0-c)\eta\,dx=0$.

Because $\mathcal{D}((0,1))=W+\mathbb{R}\eta$, we get $\int_0^1(v_0-c)v\,dx=0$ $\forall v\in\mathcal{D}((0,1))$. Thus $v_0\equiv c$. Since $c=\int_0^1 v_0\,dx=0$, we get $v_0\equiv 0$. Thus B is nondegenerate on W.

We have $W^\perp=\{w\in C([0,1]) : \int_0^1 vw\,dx=0 \ \forall v\in W\}$
$=\{\text{constant functions}\}$ by the previous arguments.

For any two constant functions c_1 and c_2 on $[0,1]$, we have

$$B(c_1,c_2) = c_1(0)c_2(0) - \int_0^1 c_1 c_2\,dx = c_1c_2 - c_1c_2 = 0$$

Thus B is degenerate on $W^\perp$.

(3) Problem 4, Goodman-Wallach, p. 11

Let F be either $\mathbb{R}$ or $\mathbb{C}$, and V be the vector space (over F) of symmetric 2×2 matrices. Define a map $B: V\times V \to F$, $B(x,y) = \det(x+y) - \det(x) - \det(y)$.

(a) First, we show that B is a bilinear form.

$$V = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \,\middle|\, x_1,x_2,x_3 \in F \right\} = \operatorname{span}_F \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{v_3} \right\}$$

For $x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}$ we have

$$B(x,y) = \det\begin{pmatrix} x_1+y_1 & x_2+y_2 \\ x_2+y_2 & x_3+y_3 \end{pmatrix} - \det\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} - \det\begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix}$$

$$= (x_1+y_1)(x_3+y_3) - (x_2+y_2)^2 - (x_1x_3 - x_2^2) - (y_1y_3 - y_2^2)$$

$$= x_1x_3 + y_1x_3 + x_1y_3 + y_1y_3 - x_2^2 - 2x_2y_2 - y_2^2 - x_1x_3 + x_2^2 - y_1y_3 + y_2^2$$

$$= x_1y_3 + y_1x_3 - 2x_2y_2$$

It is easy to see that B is symmetric. For $t \in F$, $z \in V$, we have

$$B(x+tz,y) = (x_1+tz_1)y_3 + y_1(x_3+tz_3) - 2(x_2+tz_2)y_2$$

$$= B(x,y) + t\,B(z,y)$$

Therefore B is a bilinear form on V.

Suppose that for some $x \in V$, $B(x,y) = 0$ for all $y \in V$. Then

$$0 = B(x, v_1) = x_3$$
$$0 = B(x, v_2) = -2x_2$$
$$0 = B(x, v_3) = x_1$$

Thus $x = 0$. Therefore B is nondegenerate.

Next, consider the case $F = \mathbb{R}$. We'll show that $\mathrm{sig}(B) = (1,2)$.

Denote $\mathcal{B}_0 = (v_1, v_2, v_3)$ be the basis that is mentioned earlier of V.

For $x \in V$ such that $[x]_{\mathcal{B}_0} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, we have $B(x,x) = 2x_1x_3 - 2x_2^2$.

Since B is symmetric, $Q(x) = B(x,x)$ is a quadratic form on V. We have

$$Q(x) = \Big(\underbrace{\frac{x_1+x_3}{\sqrt{2}}}_{y_1}\Big)^2 - \Big(\underbrace{\frac{x_1-x_3}{\sqrt{2}}}_{y_2}\Big)^2 - (\underbrace{\sqrt{2}\,x_2}_{y_3})^2,$$

where

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}}_{P} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Let $\mathcal{B} = (w_1, w_2, w_3)$ be the basis of V such that $[x]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

This is equivalent to say $P = P(\mathcal{B} \to \mathcal{B}_0)$, or equivalently,

$$P^{-1} = P(\mathcal{B}_0 \to \mathcal{B}) = \left([w_1]_{\mathcal{B}_0} \; [w_2]_{\mathcal{B}_0} \; [w_3]_{\mathcal{B}_0}\right).$$

Thus, in basis $\mathcal{B}$ we have $Q(x) = y_1^2 - y_2^2 - y_3^2$. Thus by the Law of Inertia for quadratic forms, $\mathrm{sig}(Q) = (1,2)$. Thus $\mathrm{sig}(B) = (1,2)$.

(b) Put $SO(V,B) = O(V,B) \cap SL(V)$

$$= \{\tau \in GL(V) : \det(\tau) = 1 \text{ and } B(\tau v, \tau w) = B(v,w) \ \forall v,w \in V\}$$

We'll show that the map $\varphi : SL(2,F) \to SO(V,B)$, $\varphi(g)v = g^{T}vg$ is a group morphism. First, we'll check that φ is well-defined. For $v, w \in V$ and $g \in SL(2,F)$, we have

$$\begin{aligned} B(\varphi(g)v, \varphi(g)w) &= B(gvg^T, gwg^T) \\ &= \det(gvg^T + gwg^T) - \det(gvg^T) - \det(gwg^T) \\ &= \det(g(v+w)g^T) - \underbrace{\det(g)}_{1}\det(v)\underbrace{\det(g^T)}_{1} - \det(g)\det(w)\det \ldots \\ &= \det(v+w) - \det(v) - \det(w) \\ &= B(v,w) \end{aligned}$$

Thus $\varphi(g) \in O(V,B)$. Moreover, $\det(\varphi(g)) = \det(\varphi(g)I)$

$$= \det(gg^T) = 1.$$

Thus $\varphi(g) \in SO(V,B)$. Next, for $g_1, g_2 \in SL(2,F)$, we'll show that

$$\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$$

For each $v \in V$, we have

$$\begin{aligned} \varphi(g_1 g_2)v &= g_1 g_2 v (g_1 g_2)^T \\ &= g_1 (g_2 v g_2^T) g_1^T \\ &= \varphi(g_1)(g_2 v g_2^T) = \varphi(g_1) \circ \varphi(g_2)(v), \end{aligned}$$

Thus, $\varphi(g_1 g_2)v = \varphi(g_1) \circ \varphi(g_2)(v) \quad \forall v \in V.$

Next, we'll show that $\ker\varphi = \{I_2, -I_2\}$.

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It is easy to see that $I_2, -I_2 \in \ker \varphi$. Now let $g \in \ker\varphi$. Then $\varphi(g) = Id$, i.e. $\varphi(g)v = v \;\; \forall v \in V$. Then $gvg^T = v \;\; \forall v \in V$.

This is equivalent to $gv_j g^T = v_j$ for $j = 1, 2, 3$. Write $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$.

We have

$$gv_1 g^T = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & g_3 \\ g_2 & g_4 \end{pmatrix} = \begin{pmatrix} g_1^2 & g_1 g_3 \\ g_1 g_3 & g_3^2 \end{pmatrix}.$$

Because $gv_1 g^T = v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we get $g_1 = \pm 1$ and $g_3 = 0$.

We have

$$gv_3 g^T = \begin{pmatrix} g_1 & g_2 \\ 0 & g_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ g_2 & g_4 \end{pmatrix} = \begin{pmatrix} g_2^2 & g_2 g_4 \\ g_2 g_4 & g_4^2 \end{pmatrix}$$

Because $gv_3 g^T = v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we get $g_2 = 0$ and $g_4 = \pm 1$.

Because $\det(g) = 1$, we have $g_1 g_4 - g_2 g_3 = 0$, i.e. $g_1 g_4 = 1$.

Thus $(g_1 = 1, g_4 = 1)$ or $(g_1 = -1, g_4 = -1)$. Thus $g = I_2$ or $g = I_2$.

Therefore $\ker\varphi = \{I_2, -I_2\}$.