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Math 8271: Lie algebra

(1) Problem 1, Section 1.3.8, page 34, Goodman-Wallach.

We'll show that the exponential map $\exp: M_n(\mathbb{C}) \to GL(n,\mathbb{C})$ is surjective.

For $n=1$, the exponential map $\exp: \mathbb{C} \to \mathbb{C}^\times$ is surjective because each $z = re^{i\theta} \in \mathbb{C}^\times$, $(r>0,\ \theta\in\mathbb{R})$ can be written as $z = \exp(\log(r)+i\theta)$.

Now let us consider $n \geq 2$. Since $\mathbb{C}$ is algebraically closed, any matrix $A \in M_n(\mathbb{C})$ has n eigenvalues (counted with multiplicity).

Take any $A \in GL(n,\mathbb{C})$, we are looking for $X \in M_n(\mathbb{C})$ such that $A = \exp(X)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \neq 0$ be the eigenvalues of A. ~~By~~ We know that $\exp(B_{\log 2}(0)) = \Gamma$, where Γ is an open neighborhood of I_n. Thus, there is a number $r>0$ such that $B_r(I_n) \subset \Gamma = \exp(B_{\log 2}(0))$.

Take $\alpha = \dfrac{\sqrt{n-1}}{r.\max\{|\lambda_1|,\ldots,|\lambda_n|\}} + 1 > 0$. Then $|\alpha\lambda_1|, |\alpha\lambda_2|, \ldots, |\alpha\lambda_n| > \dfrac{\sqrt{n-1}}{r}$.

Put $B = \alpha A$. Then all eigenvalues of B, which are $\alpha\lambda_1, \ldots, \alpha\lambda_n$, have modulus greater than $\sqrt{n-1}/r$. Suppose that we can find $Y \in M_n(\mathbb{C})$ such that $B = \exp(Y)$. Then

$$\begin{aligned} A = \alpha^{-1}B &= \exp(\beta I_n)\exp(Y), \quad \text{where } \beta = \log(\alpha^{-1}) \in \mathbb{C}, \\ &= \exp(\beta I_n + Y), \quad \text{since } \beta I_n \text{ and } Y \text{ commute.} \end{aligned}$$

Then $A \in \exp(M_n(\mathbb{C}))$. Therefore, the problem can be reduced to the case in which all eigenvalues of A have modulus greater than $\sqrt{n-1}/r$. Because $\mathbb{C}$ is algebraically closed, A is similar to a matrix of Jordan form. Namely, there exist Jordan blocks $J_1, J_2, \dots, J_m$ with

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix} \in M_{n_k}(\mathbb{C})$$

and a matrix $P \in GL(n, \mathbb{C})$ such that $A = P^{-1} \operatorname{diag}(J_1, \dots, J_m) P$.

If we can find a matrix $Y \in M_n(\mathbb{C})$ such that $\operatorname{diag}(J_1, \dots, J_m) = \exp(Y)$, then $A = P^{-1} \exp(Y) P = \exp(P^{-1} Y P) \in \exp(M_n(\mathbb{C}))$. Thus, it suffices to assume that A is of the form $\operatorname{diag}(J_1, \dots, J_m)$. Note that we still have $|\lambda_1|, \dots, |\lambda_m| > \sqrt{n-1}/r$.

Suppose that for each $1 \le k \le m$, we can find $Y_k \in M_{n_k}(\mathbb{C})$ such that $J_k = \exp(Y_k)$. Then

$$A = \operatorname{diag}(J_1, \dots, J_m) = \operatorname{diag}(\exp(Y_1), \dots, \exp(Y_m))$$

$$= \exp(\operatorname{diag}(Y_1, \dots, Y_m)).$$

$$\left(\text{by using the definition } \exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}\right)$$

Thus, it suffices to reduce the problem to the case

$$A = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \in GL(n, \mathbb{C}), \text{ with } |\lambda| > \frac{\sqrt{n-1}}{r}.$$

Put $B = \begin{pmatrix} 1 & \lambda^{-1} & & \\ & 1 & \ddots & \\ & & \ddots & \lambda^{-1} \\ & & & 1 \end{pmatrix} \in GL(n, \mathbb{C})$. We have

$$\|B - I_n\| = \left\| \begin{pmatrix} 0 & \lambda^{-1} & & \\ & 0 & \ddots & \\ & & \ddots & \lambda^{-1} \\ & & & 0 \end{pmatrix} \right\| = \sqrt{\underbrace{(\lambda^{-1})^2 + \dots + (\lambda^{-1})^2}_{n-1 \text{ times}}} = \lambda^{-1}\sqrt{n-1} < r$$

Thus, $B \in B_r(I_n) \subset \exp(B_{\log 2}(0))$. Thus, there exists $Y \in M_n(\mathbb{C})$ such that $B = \exp(Y)$. We have

$$A = \lambda B = \exp(\tilde{\lambda} I_n)\exp(Y), \quad \left(\text{where } \tilde{\lambda} \in \mathbb{C}, \ \exp(\tilde{\lambda}) = \lambda\right)$$
$$= \exp(\tilde{\lambda} I_n + Y) \quad (\text{since } \tilde{\lambda} I_n \text{ and } Y \text{ commute})$$

Therefore, $A \in \exp(M_n(\mathbb{C}))$.

② Problem 2, Section 2.4.5, page 106, Goodman-Wallach.

Theorem 2.4.1, page 93, gives a survey of root systems on $\mathfrak{g} = \mathrm{Lie}(G)$ in case G is an algebraic group of type A, B, C, D. We restate these results as follows.

<u>Type A</u> $G = SL(\ell+1, \mathbb{C})$

Let K be the vector subspace of $M_{\ell+1}(\mathbb{C})$ of diagonal matrices. Let $\{\varepsilon_1, \dots, \varepsilon_{\ell+1}\}$ be the dual basis of K, i.e. $\varepsilon_i \in K^*$ and $\varepsilon_i(\mathrm{diag}(a_1, \dots, a_{\ell+1})) = a_i$. Then the roots of the maximal torus of G on $\mathfrak{g}$ are

$$\pm(\varepsilon_i - \varepsilon_j), \quad \text{with } 1 \le i < j \le \ell+1$$

<u>Type B</u> $G = SO(2\ell+1, \mathbb{C})$

Let K be the vector subspace of $M_{2\ell+1}(\mathbb{C})$ of diagonal matrices. Let $\{\varepsilon_1, \dots, \varepsilon_{2\ell+1}\}$ be the dual basis of K, i.e. $\varepsilon_i \in K^*$ and $\varepsilon_i(\mathrm{diag}(a_1, \dots, a_{2\ell+1})) = a_i$. Then the roots of the maximal torus of G on $\mathfrak{g}$ are

$$\pm(\varepsilon_i-\varepsilon_j),\ \pm(\varepsilon_i+\varepsilon_j),\ \pm\varepsilon_k,\ \text{with } 1\le i<j\le l,\ 1\le k\le l$$

Type C $G = Sp(2l,\mathbb{C})$

Let K be the vector subspace of $M_{2l}(\mathbb{C})$ of diagonal matrices. Let $\{\varepsilon_1,\dots,\varepsilon_{2l}\}$ be the dual basis of K, i.e. $\varepsilon_i\in K^*$ and $\varepsilon_i(\mathrm{diag}(a_1,\dots,a_{2l})) = a_i$. Then the roots of the maximal torus of G on $\mathfrak{g}$ are

$$\pm(\varepsilon_i-\varepsilon_j),\ \pm(\varepsilon_i+\varepsilon_j),\ \pm 2\varepsilon_k,\ \text{with } 1\le i<j\le l,\ 1\le k\le l.$$

Type D $G = SO(2l,\mathbb{C})$

Let K be the vector subspace of $M_{2l}(\mathbb{C})$ of diagonal matrices. Let $\{\varepsilon_1,\dots,\varepsilon_{2l}\}$ be the dual basis of K, i.e. $\varepsilon_i\in K^*$ and $\varepsilon_i(\mathrm{diag}(a_1,\dots,a_{2l})) = a_i$. Then the roots of the maximal torus of G on $\mathfrak{g}$ are

$$\pm(\varepsilon_i-\varepsilon_j),\ \pm(\varepsilon_i+\varepsilon_j)\quad\text{with } 1\le i<j\le l.$$

Return to the problem.

(a) Type A: $G = SL(l+1,\mathbb{C})$.

Put $\alpha_r = \varepsilon_i - \varepsilon_{i+1}$ and $\Delta = \{\alpha_1,\dots,\alpha_l\}$. For $1\le i<j\le \cancel{l}$,

$$\varepsilon_i-\varepsilon_j = (\varepsilon_i-\varepsilon_{i+1})+\cdots+(\varepsilon_{j-1}-\varepsilon_j) = \alpha_i+\cdots+\alpha_{j-1},$$

$$-(\varepsilon_i-\varepsilon_j) = -\alpha_i-\cdots-\alpha_{j-1}.$$

Thus Δ is a set of simple roots of $\mathfrak{g}$ and all positive roots of its are of the form $\alpha_i+\cdots+\alpha_j$ for $1\le i\le j\le l$.

(b) Type B: $G = SO(2l+1,\mathbb{C})$

Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le l-1$, and $\alpha_l = \varepsilon_l$. Put $\Delta = \{\alpha_1, \dots, \alpha_l\}$.

For $1 \le i < j \le l$, $1 \le k \le l$,

$$\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + \dots + (\varepsilon_{j-1} - \varepsilon_j) = \alpha_i + \dots + \alpha_{j-1},$$

$$\begin{aligned}\varepsilon_i + \varepsilon_j &= (\varepsilon_i - \varepsilon_j) + 2(\varepsilon_j - \varepsilon_l) + 2\varepsilon_l = (\alpha_i + \dots + \alpha_{j-1}) + 2(\alpha_j + \dots + \alpha_{l-1}) + 2\alpha_l \\ &= (\alpha_i + \dots + \alpha_{j-1}) + 2(\alpha_j + \dots + \alpha_l),\end{aligned}$$

$$\varepsilon_k = (\varepsilon_k - \varepsilon_l) + 2\varepsilon_l = (\alpha_k + \dots + \alpha_{l-1}) + 2\alpha_l.$$

Those are the roots in Φ^+. Thus, all roots in $\Phi^+ \setminus \Delta$ are of the forms

$$\alpha_i + \dots + \alpha_j \quad \text{for } 1 \le i < j \le l,$$

$$\alpha_i + \dots + \alpha_{j-1} + 2(\alpha_j + \dots + \alpha_l) \quad \text{for } 1 \le i < j \le l.$$

(c) Type C: $G = Sp(2l, \mathbb{C})$.

Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le l-1$ and $\alpha_l = 2\varepsilon_l$. Take $\Delta = \{\alpha_1, \dots, \alpha_l\}$.

For $1 \le i < j \le l$,

$$\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + \dots + (\varepsilon_{j-1} - \varepsilon_j) = \alpha_i + \dots + \alpha_{j-1},$$

$$\varepsilon_i + \varepsilon_j = (\varepsilon_i - \varepsilon_j) + 2(\varepsilon_j - \varepsilon_l) + 2\varepsilon_l = (\alpha_i + \dots + \alpha_{j-1}) + 2(\alpha_j + \dots + \alpha_{l-1}) + \alpha_l,$$

$$2\varepsilon_k = 2(\varepsilon_k - \varepsilon_l) + 2\varepsilon_l = 2(\alpha_k + \dots + \alpha_{l-1}) + \alpha_l.$$

Thus, Δ is a set of ~~sy~~ simple roots of $\mathfrak{g}$, and the above roots are the positive roots. Thus $\Phi^+ \setminus \Delta$ consists of the roots

$$\alpha_i + \dots + \alpha_j, \quad 1 \le i < j \le l,$$

$$(\alpha_i + \dots + \alpha_{j-1}) + 2(\alpha_j + \dots + \alpha_{l-1}) + \alpha_l, \quad 1 \le i < j < l,$$

$$2(\alpha_k + \dots + \alpha_{l-1}) + \alpha_l, \quad 1 \le k < l.$$

(d) Type D: $G = SO(2l, \mathbb{C})$.

Put $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le l-1$ and $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$.

For $1 \le i < j \le l$, $\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + \cdots + (\varepsilon_{j-1} - \varepsilon_j) = \alpha_i + \cdots + \alpha_{j-1}$.

$$\cancel{\text{For } 1 \le i < j = l,\ \varepsilon_i \mp \varepsilon_j = (\varepsilon_i - \varepsilon_l) = (\varepsilon_i - \varepsilon_{i+1}) + \cdots + (\varepsilon_{l-2} - \varepsilon_{l-1}) + (\varepsilon_{l-1} - \varepsilon_l)}$$
$$\cancel{= \alpha_i + \cdots + \alpha}$$

For $1 \le i < j \le l-1$,
$$\varepsilon_i + \varepsilon_j = (\varepsilon_i - \varepsilon_j) + 2(\varepsilon_j - \varepsilon_{l-1}) + (\varepsilon_{l-1} - \varepsilon_l) + (\varepsilon_{l-1} + \varepsilon_l)$$
$$= (\alpha_i + \cdots + \alpha_{j-1}) + 2(\alpha_j + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l.$$

For $1 \le i < j = l$,
$$\varepsilon_i + \varepsilon_l = (\varepsilon_i - \varepsilon_{l-1}) + (\varepsilon_{l-1} + \varepsilon_l)$$
$$= (\varepsilon_i - \varepsilon_{i+1}) + \cdots + (\varepsilon_{l-2} - \varepsilon_{l-1}) + (\varepsilon_{l-1} + \varepsilon_l)$$
$$= \alpha_i + \cdots + \alpha_{l-2} + \alpha_l$$

Thus, Δ is a set of simple roots. The positive roots are of the forms

$$\alpha_i + \cdots + \alpha_j, \quad 1 \le i \le j < l,$$
$$\alpha_i + \cdots + \alpha_{l-2} + \alpha_l, \quad 1 \le i < l-1,$$
$$\alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l, \quad 1 \le i < j \le l-1.$$

Therefore, $\Phi^+ \setminus \Delta$ consists of the roots

$$\alpha_i + \cdots + \alpha_j, \quad 1 \le i < j < l,$$
$$\alpha_i + \cdots + \alpha_l, \quad 1 \le i < l-1,$$
$$\alpha_i + \cdots + \alpha_{l-2} + \alpha_l, \quad 1 \le i < l-1,$$
$$\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l, \quad 1 \le i < j < l-1.$$

Now we'll use the results of (a)-(d) to prove the assertions (2)-(5) in Lemma 2.4.10, page 103, Goodman-Wallach.

~~Assertion (2) Let $\beta \in \Phi^+ \setminus \Delta$. We'll show that there exist $\gamma, \delta \in \Phi^+$ such that $\beta = \gamma + \delta$.~~

~~We see from (a)–(d) that any non-simple root β of the maximal torus of G on $\mathfrak{g}$ is a sum of at least two simple roots. Thus, we can group those summands into two groups, the sum of each is still a positive root. Thus $\beta = \gamma + \delta$ where both γ and δ are positive roots.~~

Assertion (3) Let $\tilde{\alpha}$ be the highest root relative to $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. We'll show that $\tilde{\alpha} = n_1\alpha_1 + \dots + n_\ell\alpha_\ell$ with $n_i \geq 1$ for $i = 1, 2, \dots, \ell$.

From (a), the highest root of type A is $\tilde{\alpha} = \alpha_1 + \dots + \alpha_\ell$, with $ht(\tilde{\alpha}) = \ell$.

From (b), the highest root of type B is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_\ell$, with $ht(\tilde{\alpha}) = 2\ell - 1$.

From (c), the highest root of type C is $\tilde{\alpha} = 2\alpha_1 + \dots + 2\alpha_{\ell-1} + \alpha_\ell$, with $ht(\tilde{\alpha}) = 2\ell - 1$

From (d), the highest root of type D is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$,

with $ht(\tilde{\alpha}) = 1 + 2(\ell-3) + 1 + 1 = 2\ell - 3$.

Assertion (4) Take any $\beta \in \Phi^+$, $\beta \neq \tilde{\alpha}$. We'll show that there exists $\alpha \in \Phi^+$ such that $\beta + \alpha \in \Phi^+$.

* Type A: $G = SL(\ell+1, \mathbb{C})$

As given in Assertion (3), $\tilde{\alpha} = \varepsilon_1 - \varepsilon_{\ell+1}$. Recall that all positive roots on $\mathfrak{g}$ are of the form $\varepsilon_i - \varepsilon_j$ for $1 \leq i < j \leq \ell+1$. Thus $\beta = \varepsilon_i - \varepsilon_j$. Since $\beta \neq \tilde{\alpha}$, either $i > 1$ or $j < \ell+1$. If $i > 1$ then we choose $\alpha = \varepsilon_1 - \varepsilon_i$. In this case $\beta + \alpha = (\varepsilon_i - \varepsilon_j) + (\varepsilon_1 - \varepsilon_i) = \varepsilon_1 - \varepsilon_j \in \Phi^+$. If $j < \ell+1$ then we choose $\alpha = \varepsilon_j - \varepsilon_{\ell+1}$. Then $\alpha + \beta = (\varepsilon_i - \varepsilon_j) + (\varepsilon_j - \varepsilon_{\ell+1}) = \varepsilon_i - \varepsilon_{\ell+1} \in \Phi^+$.

* Type B: $G = SO(2\ell+1, \mathbb{C}), \ell \geq 2$

As given in Assertion (3), $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$. Recall that all positive roots on $\mathfrak{g}$ relative to Δ are of the forms $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, ε_k for $1 \leq i < j \leq \ell$, $1 \leq k \leq \ell$.

If $\beta = \varepsilon_i - \varepsilon_j$ then we can choose $\alpha = \varepsilon_j$. In this case, $\alpha + \beta = \varepsilon_i \in \Phi^+$.

If $\beta = \varepsilon_k$ then we can choose $\alpha = \varepsilon_s$ for any $1 \leq s \leq \ell$, $\ell \neq k$. Then $\alpha + \beta =$
$= \varepsilon_k + \varepsilon_s \in \Phi^+$.

If $\beta = \varepsilon_i + \varepsilon_j$ then ~~either $i > 1$ or~~ $j > 2$. Take any $1 \leq s < j$ and $s \neq i$. Put $\alpha = \varepsilon_s - \varepsilon_j \in \Phi^+$. Then $\alpha + \beta = (\varepsilon_s - \varepsilon_j) + (\varepsilon_i + \varepsilon_j) = \varepsilon_s + \varepsilon_i \in \Phi^+$.

* Type C: $G = Sp(2\ell, \mathbb{C})$, $\ell \geq 2$

As given in Assertion (3), $\tilde{\alpha} = 2\varepsilon_1$. Recall that all positive roots on $\mathfrak{g}$ relative to Δ are of the form $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, $2\varepsilon_k$ for $1 \leq i < j \leq \ell$, $1 \leq k \leq \ell$.

If $\beta = \varepsilon_i - \varepsilon_j$ then we choose $\alpha = \varepsilon_i + \varepsilon_j \in \Phi^+$. Then $\alpha + \beta = 2\varepsilon_i \in \Phi^+$.

If $\beta = \varepsilon_i + \varepsilon_j$ then we choose $\alpha = \varepsilon_i - \varepsilon_j \in \Phi^+$. Then $\alpha + \beta = 2\varepsilon_i \in \Phi^+$.

If $\beta = 2\varepsilon_k$ for $1 < k \leq \ell$ then we choose $\alpha = \varepsilon_1 - \varepsilon_k \in \Phi^+$. Then
$\alpha + \beta = \varepsilon_1 + \varepsilon_k \in \Phi^+$.

* Type D: $G = SO(2\ell, \mathbb{C})$, $\ell \geq 3$

As given in Assertion (3), $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$. Recall that all positive roots on $\mathfrak{g}$ relative to Δ are of the form $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq \ell$.

If $\beta = \varepsilon_i - \varepsilon_j$: we choose $\alpha = \varepsilon_2 + \varepsilon_\ell$ if $j = 2$. In this case $\beta + \alpha = \varepsilon_i + \varepsilon_\ell \in \Phi^+$.

In case $j > 2$, we take any $1 \leq s < j$, $s \neq i$ and put $\alpha = \varepsilon_s + \varepsilon_j \in \Phi^+$.

Then $\alpha+\beta = \varepsilon_i+\varepsilon_s \in \Phi^+$.

If $\beta = \varepsilon_i+\varepsilon_j$ then $j>2$ because $\beta \neq \tilde{\alpha}$. Take any $1\leq s<j$, $s\neq i$ and put $\alpha = \varepsilon_s-\varepsilon_j \in \Phi^+$. Then $\alpha+\beta = \varepsilon_i+\varepsilon_s \in \Phi^+$.

Assertion (2) Let $\beta \in \Phi^+\setminus\Delta$. We'll show that there exist $\gamma, \delta \in \Phi^+$ such that $\beta = \gamma+\delta$.

- For types A, B, C, D, if $\beta = \alpha_i+\dots+\alpha_j$ for $1\leq i<j\leq l$, then we choose $\gamma = \alpha_i$, $\delta = \alpha_{i+1}+\dots+\alpha_j$. This choice is valid except for one case, namely when $\beta = \alpha_{l-2}+\alpha_{l-1}+\alpha_l$ in type D. In this case, we choose $\gamma = \alpha_{l-1}$ and $\delta = \alpha_{l-2}+\alpha_l$.
- For type B, if $\beta = \alpha_i+\dots+\alpha_{j-1}+2\alpha_j+\dots+2\alpha_l$ for $1\leq i<j\leq l$ then we can choose $\gamma = \alpha_j$, $\delta = \alpha_i+\dots+\alpha_j+2\alpha_{j+1}+\dots+2\alpha_l$,
- For type C: if $\beta = \alpha_i+\dots+\alpha_{j-1}+2\alpha_j+\dots+2\alpha_{l-1}+\alpha_l$ for $1\leq i<j<l$ then we choose $\gamma = \alpha_i$, $\delta = \underbrace{\alpha_{i+1}+\dots+\alpha_{j-1}}_{=0 \text{ if } i=j-1}+2\alpha_j+\dots+2\alpha_{l-1}+\alpha_l$.

If $\beta = 2\alpha_i+\dots+2\alpha_{l-1}+\alpha_l$, for $1\leq i<l$, then we choose $\gamma = \alpha_i$, and $\delta = \alpha_i+\underbrace{2\alpha_{i+1}+\dots+2\alpha_{l-1}}_{=0 \text{ if } i=l-1}+\alpha_l$.

- For type D: if $\beta = \alpha_i+\dots+\alpha_{l-2}+\alpha_l$ for $1\leq i<l-1$ then we choose $\gamma = \alpha_i+\dots+\alpha_{l-2}$ and $\delta = \alpha_l$. If $\beta = \alpha_i+\dots+\alpha_{j-1}+2\alpha_j+\dots+2\alpha_{l-2}+\alpha_{l-1}+\alpha_l$ for $1\leq i<j<l-1$ then we choose $\gamma = \alpha_i+\dots+\alpha_l$ and $\delta = \alpha_j+\dots+\alpha_{l-2}$.

Assertion 5 Let $\alpha \in \Phi^+$, $\alpha \neq \tilde{\alpha}$. We'll find $1\leq k_1,\dots,k_r\leq l$ such that

$\alpha = \tilde{\alpha} - \alpha_{k_1} - \cdots - \alpha_{k_r}$ and $\tilde{\alpha} - \alpha_{k_1} - \cdots - \alpha_{k_j} \in \Phi^+$ for all $1 \leq j \leq r$.

Type A: $\tilde{\alpha} = \alpha_1 + \cdots + \alpha_\ell$

We have $\alpha = \alpha_i + \cdots + \alpha_j$ for $1 \leq i \leq j \leq \ell$. We then choose

$\alpha_{k_1} = \alpha_1, \ldots, \alpha_{k_{i-1}} = \alpha_{i-1}$,

$\alpha_{k_i} = \alpha_\ell, \ldots, \alpha_{k_{\ell+i-j-1}} = \alpha_{j+1}$,

$r = \ell + i - j - 1$.

Type B: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell$

~~We have α~~ Consider the following cases of α.

- $\alpha = \alpha_i + \cdots + \alpha_j$, $1 \leq i \leq j \leq \ell$. We choose

  $\alpha_{k_1} = \alpha_2, \ldots, \alpha_{k_{\ell-1}} = \alpha_\ell$,

  $\alpha_{k_\ell} = \alpha_1, \ldots, \alpha_{k_{\ell+i-2}} = \alpha_{i-1}$,

  $\alpha_{k_{\ell+i-1}} = \alpha_\ell, \ldots, \alpha_{k_{2\ell+i-j-2}} = \alpha_{j+1}$,

  $r = 2\ell + i - j - 2$.

- $\alpha = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_\ell$, $1 \leq i < j \leq \ell$. We choose

  $\alpha_{k_1} = \alpha_1, \ldots, \alpha_{k_{j-1}} = \alpha_{j-1}$,

  $\alpha_{k_j} = \alpha_1, \ldots, \alpha_{k_{i+j-2}} = \alpha_{i-1}$,

  $r = i + j - 2$.

~~$\alpha =$~~ Type C: $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$

Consider the following cases of α.

- $\alpha = \alpha_i + \cdots + \alpha_j$, $1 \leq i \leq j \leq \ell$. We choose

  $\alpha_{k_1} = \alpha_1, \ldots, \alpha_{k_{\ell-1}} = \alpha_{\ell-1}$,

  $\alpha_{k_\ell} = \alpha_1, \ldots, \alpha_{k_{\ell+i-2}} = \alpha_{i-1}$,

  $\alpha_{k_{\ell+i-1}} = \alpha_\ell, \ldots, \alpha_{k_{2\ell+i-j-2}} = \alpha_{j+1}$, and $r = 2\ell + i - j - 2$.

- $\alpha = \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{\ell-1} + \alpha_\ell$, $1 \le i < j < \ell$. We choose

  $\alpha_{k_1} = \alpha_1, \dots, \alpha_{k_{j-1}} = \alpha_{j-1}$,

  $\alpha_{k_j} = \alpha_1, \dots, \alpha_{k_{i+j-2}} = \alpha_{i-1}$,

  $r = i+j-2$.

- $\alpha = 2\alpha_i + \dots + 2\alpha_{\ell-1} + \alpha_\ell$, $1 \le i < \ell$. We choose

  $\alpha_{k_1} = \alpha_1, \dots, \alpha_{k_{i-1}} = \alpha_{i-1}$,

  $\alpha_{k_i} = \alpha_1, \dots, \alpha_{k_{2i-2}} = \alpha_{i-1}$,

  $r = 2i-2$.

<u>Type D</u>: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$.

Consider the following cases of α.

- $\alpha = \alpha_i + \dots + \alpha_j$ for $1 \le i \le j < \ell$. We choose

  $\alpha_{k_1} = \alpha_2, \dots, \alpha_{k_{\ell-3}} = \alpha_{\ell-2}$,

  $\alpha_{k_{\ell-2}} = \alpha_1, \dots, \alpha_{k_{i+\ell-4}} = \alpha_{i-1}$,

  $\alpha_{k_{i+\ell-3}} = \alpha_\ell, \dots, \alpha_{k_{2\ell+i-j-4}} = \alpha_{j+1}$,

  $r = 2\ell + i - j - 4$.

- $\alpha = \alpha_\ell$. We choose

  $\alpha_{k_1} = \alpha_1, \dots, \alpha_{k_{\ell-2}} = \alpha_{\ell-2}, \alpha_{k_{\ell-1}} = \alpha_{\ell-1}$,

  $\alpha_{k_\ell} = \alpha_1, \dots, \alpha_{k_{2\ell-3}} = \alpha_{\ell-2}$,

  $r = 2\ell-3$.

- $\alpha = \alpha_i + \dots + \alpha_\ell$ for $1 \le i < \ell-1$. We choose

  $\alpha_{k_1} = \alpha_2, \dots, \alpha_{k_{\ell-3}} = \alpha_{\ell-2}$,

  $\alpha_{k_{\ell-2}} = \alpha_1, \dots, \alpha_{k_{i+\ell-4}} = \alpha_{i-1}$,

  $r = i+\ell-4$.

- $\alpha = \alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell$ for $1 \le i < \ell-1$. We choose

$\alpha_{k_1} = \alpha_2, \ldots, \alpha_{k_{\ell-3}} = \alpha_{\ell-2}$,

$\alpha_{k_{\ell-2}} = \alpha_1, \ldots, \alpha_{k_{i+\ell-4}} = \alpha_{i-1}$,

$\alpha_{k_{i+\ell-3}} = \alpha_{\ell-1}$,

$r = i+\ell-3$.

- $\alpha = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$ for $1 \le i < j < \ell-1$. We choose

$\alpha_{k_1} = \alpha_2, \ldots, \alpha_{k_{j-2}} = \alpha_{j-1}$,

$\alpha_{k_{j-1}} = \alpha_1, \ldots, \alpha_{k_{i+j-3}} = \alpha_{i-1}$,

$r = i+j-3$.

(3) Problem 1, Section 3.1.5, page 145, Goodman-Wallach.

Let $G \subset GL(n,\mathbb{C})$ be a classical group of type A, B, C or D. Set $V = \sum_{i=1}^{n} \mathbb{R}\varepsilon_i$ ~~giv~~ and give V the inner product $(\cdot,\cdot)$ such that $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

(a) Show that $(\alpha,\alpha) \in \{1,2,4\}$ for all $\alpha \in \Phi$ and that at most two distinct lengths occur.

Since $(-\alpha,-\alpha) = (\alpha,\alpha)$, we can assume $\alpha \in \Phi^+$. The survey of root systems at the beginning of Problem 2 gives us all possibilities for α. Let us consider each type of G.

Type A: $G = SL(\ell+1, \mathbb{C})$.

Then $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \le i < j \le \ell+1$. We have

$$(\alpha,\alpha) = (\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = \underbrace{(\varepsilon_i,\varepsilon_i)}_{=1} - \underbrace{(\varepsilon_i,\varepsilon_j)}_{=0} - \underbrace{(\varepsilon_j,\varepsilon_i)}_{=0} + \underbrace{(\varepsilon_j,\varepsilon_j)}_{=1} = 2.$$

Thus, 2 is the only length occurring in Φ.

<u>Type B</u> : $G = SO(2l+1, \mathbb{C})$.

Then
$$\alpha = \begin{cases} \varepsilon_i - \varepsilon_j & \text{for } 1 \le i < j \le l, \\ \varepsilon_i + \varepsilon_j & \text{for } 1 \le i < j \le l, \\ \varepsilon_k & \text{for } 1 \le k \le l \end{cases}$$

In the first two cases, $(\alpha, \alpha) = (\varepsilon_i, \varepsilon_i) + (\varepsilon_j, \varepsilon_j) = 2$. In the last case, $(\alpha, \alpha) = (\varepsilon_k, \varepsilon_k) = 1$. Thus, 1 and 2 are the only lengths occurring in Φ.

<u>Type C</u> : $G = Sp(2l, \mathbb{C})$.

Then
$$\alpha = \begin{cases} \varepsilon_i - \varepsilon_j & \text{for } 1 \le i < j \le l, \\ \varepsilon_i + \varepsilon_j & \text{for } 1 \le i < j \le l, \\ 2\varepsilon_k & \text{for } 1 \le k \le l. \end{cases}$$

In the first two cases, $(\alpha, \alpha) = (\varepsilon_i, \varepsilon_i) + (\varepsilon_j, \varepsilon_j) = 2$. In the last case, $(\alpha, \alpha) = (2\varepsilon_k, 2\varepsilon_k) = 4(\varepsilon_k, \varepsilon_k) = 4$. Thus, 2 and 4 are the only lengths occurring in Φ.

<u>Type D</u> : $G = SO(2l, \mathbb{C})$.

Then
$$\alpha = \begin{cases} \varepsilon_i - \varepsilon_j & \text{for } 1 \le i < j \le l, \\ \varepsilon_i + \varepsilon_j & \text{for } 1 \le i < j \le l. \end{cases}$$

This both cases, $(\alpha, \alpha) = (\varepsilon_i, \varepsilon_i) + (\varepsilon_j, \varepsilon_j) = 2$. Thus, 2 is the only length occurring in Φ.

(b) Let $\alpha, \beta \in \Phi$ with $(\alpha, \alpha) = (\beta, \beta)$. We'll show that there exists $w \in W_G$ such that $w.\alpha = \beta$.

Let H be the maximal torus of G and $\mathfrak{h}$ be its Lie algebra. By definition, the Weyl group $W_G = Norm_G(H)/H$ acts on $\mathfrak{h}^*$ via the group morphism $W_G \to GL(\mathfrak{h}^*)$ defined by $\langle w.\alpha, x\rangle = \langle \alpha, w^{-1}xw\rangle \ \forall \alpha \in \mathfrak{h}^*, \forall x \in H$.

This definition does not depend on the specific choice of representative in to coset wH.

For each $\alpha \in \Phi$, the root reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is defined by

$$s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha .$$

By Lemma 3.1.6, p. 133, Goodman-Wallach, W_G is generated by $\{s_\alpha : \alpha \in \Delta\}$.

For $i \neq j$, we have

$$s_{\varepsilon_i+\varepsilon_j}(\varepsilon_k) = \varepsilon_k - \frac{2(\varepsilon_k, \varepsilon_i+\varepsilon_j)}{(\varepsilon_i+\varepsilon_j, \varepsilon_i+\varepsilon_j)}(\varepsilon_i+\varepsilon_j) = \varepsilon_k - (\varepsilon_k, \varepsilon_i+\varepsilon_j)(\varepsilon_i+\varepsilon_j)$$

$$= \varepsilon_k - [(\varepsilon_k,\varepsilon_i)+(\varepsilon_k,\varepsilon_j)](\varepsilon_i+\varepsilon_j)$$

Thus,

$$s_{\varepsilon_i+\varepsilon_j}(\varepsilon_k) = \begin{cases} -\varepsilon_i & \text{if } k=j, \\ -\varepsilon_j & \text{if } k=i, \\ \varepsilon_k & \text{if } k\neq i,j. \end{cases} \qquad (*)$$

Consider G in each type A, B, C, D.

<u>Type A</u>: $G = SL(\ell+1, \mathbb{C})$

By Lemma 3.1.6, p. 133, Goodman-Wallach, $W_G \simeq \mathfrak{S}_{\ell+1}$ acts on $\mathfrak{h}^*$ by permutation of $\varepsilon_1, \dots, \varepsilon_{\ell+1}$. Any roots $\alpha, \beta \in \Phi$ with $(\alpha,\alpha) = (\beta,\beta)$ are of the form $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_k - \varepsilon_s$ where $i\neq j$ and $k \neq s$. There exists $w \in W_G$ such that $w.\varepsilon_i = \varepsilon_k$ and $w.\varepsilon_j = \varepsilon_s$. Then

$$w.\alpha = w.(\varepsilon_i - \varepsilon_j) = w.\varepsilon_i - w.\varepsilon_j = \varepsilon_k - \varepsilon_s = \beta.$$

By a Remark in 3.1.6, page 133, Goodman-Wallach, if G is of type B, C, D then W_G contains $\mathfrak{S}_\ell$, an element of which permutes $\varepsilon_1, \dots, \varepsilon_\ell$.

Type B: $G = SO(2\ell+1, \mathbb{C})$

Consider two roots α, β such that $(\alpha, \alpha) = (\beta, \beta)$. If $w.\alpha = \beta$ then $w^{-1}.\beta = \alpha$ and $w.(-\alpha) = -\beta$. Thus, it is sufficient to consider the following cases.

$$(\mathrm{I}) \begin{cases} \alpha = \varepsilon_i - \varepsilon_j \\ \beta = \varepsilon_k - \varepsilon_s \end{cases}, \quad (\mathrm{II}) \begin{cases} \alpha = \varepsilon_i + \varepsilon_j \\ \beta = \varepsilon_k - \varepsilon_s \end{cases}, \quad (\mathrm{III}) \begin{cases} \alpha = \varepsilon_i + \varepsilon_j, \\ \beta = \varepsilon_k + \varepsilon_s \end{cases}, \quad (\mathrm{IV}) \begin{cases} \alpha = \varepsilon_i, \\ \beta = \varepsilon_k, \end{cases} \quad (\mathrm{V}) \begin{cases} \alpha = -\varepsilon_i, \\ \beta = \varepsilon_k. \end{cases}$$

Let $w_0 \in W_G \simeq \mathfrak{S}_\ell$ be an element such that $w_0 \cdot \varepsilon_i = \varepsilon_k$ and $w_0 \cdot \varepsilon_j = \varepsilon_s$.

Then $w = w_0$ satisfies $w.\alpha = \beta$ in cases (I), (III), (IV).

For case (II), we put $w = \begin{cases} s_{\varepsilon_j + \varepsilon_s} \cdot w_0 & \text{if } j \neq k, \\ s_{\varepsilon_j + \varepsilon_s} & \text{if } j = k. \end{cases}$

If $j \neq k$, $\varepsilon_i \xmapsto{w_0} \varepsilon_k \xmapsto{s_{\varepsilon_j + \varepsilon_s}} \varepsilon_k$, Thus $w.\alpha = \beta$

$\varepsilon_j \xmapsto{w_0} \varepsilon_s \xmapsto{s_{\varepsilon_j + \varepsilon_s}} -\varepsilon_j$.

~~Thus~~, ~~ca~~ If $j = k$, ~~$w\,\varepsilon_i$~~ $k \neq i, s$. Then $w.\alpha = s_{\varepsilon_j + \varepsilon_s} \cdot (\varepsilon_i + \varepsilon_j)$

$$= s_{\varepsilon_j + \varepsilon_s} \cdot (\varepsilon_i) + s_{\varepsilon_j + \varepsilon_s} \cdot \varepsilon_j$$

$$= \cancel{x}\, \varepsilon_i - \varepsilon_s = \beta.$$

For case (V), if $i \neq k$ then we simply take $w = s_{\varepsilon_i + \varepsilon_k}$. If $i = k$ then we take $1 \leq j \leq \ell$, $j \neq i$. Let $w_1 \in W_G$ be such that $w_1 \cdot \varepsilon_i = \varepsilon_j$.

Then we take $w = s_{\varepsilon_i+\varepsilon_j}\cdot w_1$. Then $w\cdot\alpha = w\cdot\varepsilon_i = s_{\varepsilon_i+\varepsilon_j}\cdot(w_1\cdot\varepsilon_i)$

$$= s_{\varepsilon_i+\varepsilon_j}\cdot\varepsilon_j = -\varepsilon_i = \beta.$$

Type C : $G = Sp(2l,\mathbb{C})$

Consider two roots α,β such that $(\alpha,\alpha)=(\beta,\beta)$. Then we still have case (I), (II), (III) as in Type B, in which w is chosen by the same way. However, we have two more cases, namely

$$\begin{cases}\alpha = 2\varepsilon_k,\\ \beta = 2\varepsilon_s\end{cases} \quad \text{and} \quad \begin{cases}\alpha = 2\varepsilon_k,\\ \beta = -2\varepsilon_s.\end{cases}$$

If $w\cdot\alpha=\beta$ then $w\cdot(2\alpha) = 2\beta$. Thus, these cases reduce to the cases (IV) and (V) as in ~~Case~~ Type B, which are solved.

Type D : $G = SO(2l,\mathbb{C})$

Consider two roots α,β such that $(\alpha,\alpha)=(\beta,\beta)$. Then we still have Cases (I), (II), (III) as in Type B. Then we choose w by the same way as in Type B.