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Math 8272: Lie Algebra

① Problem 1, p. 234 Goodman-Wallach.

Put $G = SL(2,\mathbb{C})$ and $F^{(2)} = \{ f(x,y) = a_0x^2 + 2a_1xy + a_2y^2 \mid a_0, a_1, a_2 \in \mathbb{C} \}$.

The unique irreducible representation of G on $F^{(2)}$ is given in the proof of Proposition 2.3.5, p. 86, namely

$$(g\cdot f)(x,y) = f(ax+cy,\ bx+dy)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $f \in F^{(2)}$. We identify $P(F^{(2)})$ with $\mathbb{C}[a_0, a_1, a_2]$.

(a) Consider $D \in P(F^{(2)})$, $D(f) = a_1^2 - a_0a_2$. We'll show that D is G-invariant, i.e. $D(g\cdot f) = D(f)$ for all $g \in G$. For $f(x,y) = a_0x^2 + 2a_1xy + a_2y^2$, we can write $f(x,y) = \cancel{(x\ y)\begin{pmatrix} a_1 & a_2 \\ a_0 & a_1 \end{pmatrix}}\ (x\ y)\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Denote $\mu(f) = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have

$$(g\cdot f)(x,y) = f(ax+cy,\ bx+dy) = (ax+cy\quad bx+dy)\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}\begin{pmatrix} ax+cy \\ bx+dy \end{pmatrix}$$

$$= (x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (x\ y)\, g\, \mu(f)\, g^t \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, $\mu(g\cdot f) = g\,\mu(f)\,g^t$. We have $D(f) = a_1^2 - a_0a_2 = -\det(\mu(f))$.

Therefore, $D(g\cdot f) = -\deg(\mu(g\cdot f)) = -\det(g\,\mu(f)\,g^t)$

$$= -\underbrace{\det(g)}_{1}\ \det(\mu(f))\ \underbrace{\det(g^t)}_{1}$$

$$= -\det(\mu(g)) = D(g).$$

(b) Denote by $P^k(F^{(2)})$ the space of polynomials that are homogeneous with order k on $F^{(2)}$. First, we show that the representation $P^k(F^{(2)})$ has the decomposition

$$P^k(F^{(2)}) \simeq F^{(2k)} \oplus F^{(2k-4)} \oplus F^{(2k-8)} \oplus \cdots$$

To do so, we will compute and compare the characters of both sides. The space $P^k(F^{(2)})$ is isomorphic to $S^k(F^{(2)})$ as representations of G according to the proof of Theorem 4.1.20, page 189. Thus,

$$\mathrm{ch}\, P^k(F^{(2)})(d(q)) = \begin{bmatrix} k+2 \\ k \end{bmatrix}_q \quad \forall q \in \mathbb{C}\setminus\{0\},$$

where $d(q) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \in G$. We have

$$\begin{bmatrix} k+2 \\ k \end{bmatrix}_q = \frac{[k+2]_q!}{[2]_q!\,[k]_q!} = \frac{[k+1]_q\,[k+2]_q}{[2]_q} = \frac{\dfrac{q^{k+1}-q^{-k-1}}{q-q^{-1}}\;\dfrac{q^{k+2}-q^{-k-2}}{q-q^{-1}}}{\dfrac{q^2-q^{-2}}{q-q^{-1}}}$$

$$= \frac{(q^{k+1}-q^{-k-1})(q^{k+2}-q^{-k-2})}{(q^2-q^{-2})(q-q^{-1})}.$$

Therefore, $$\mathrm{ch}\, P^k(F^{(2)})(d(q)) = \frac{(q^{k+1}-q^{-k-1})(q^{k+2}-q^{-k-2})}{(q^2-q^{-2})(q-q^{-1})} \qquad (1).$$

By Proposition 4.1.17,

$$\mathrm{ch}(F^{(2k)} \oplus F^{(2k-4)} \oplus F^{(2k-8)} \oplus \cdots) = \mathrm{ch}(F^{(2k)}) + \mathrm{ch}(F^{(2k-4)}) + \mathrm{ch}(F^{(2k-8)}) + \cdots$$

The formula at the bottom of page 188 reads

$$\mathrm{ch}\, F^{(m)}(d(q)) = q^m + q^{m-2} + \cdots + q^{-m+2} + q^{-m}$$

$$= \frac{(q^2)^{m+1} - 1}{q^2 - 1} q^{-m} = \frac{q^{m+2} - q^{-m}}{q^2 - 1} = \frac{q^{m+1} - q^{-m-1}}{q - q^{-1}}$$

$$= \frac{(q^{m+1} - q^{-m-1})(q^2 - q^{-2})}{(q^2 - q^{-2})(q - q^{-1})} = \frac{(q^{m+3} + q^{-m-3}) - (q^{m-1} + q^{-m+1})}{(q^2 - q^{-2})(q - q^{-1})}.$$

Thus,

$$ch\, F^{(2k)}(d(q)) + ch\, F^{(2k-4)}(d(q)) + ch\, F^{(2k-8)}(d(q)) + \cdots$$

$$= \frac{(q^{2k+3} + q^{-2k-3}) - (q + q^{-1})}{(q^2 - q^{-2})(q - q^{-1})} = \frac{(q^{k+2} - q^{-k-2})(q^{k+1} - q^{-k-1})}{(q^2 - q^{-2})(q - q^{-1})} \qquad (2)$$

From (1) and (2) we have

$$ch\, P^k(F^{(2)})(d(q)) = ch\, F^{(2k)}(d(q)) + ch\, F^{(2k-4)}(d(q)) + ch\, F^{(2k-8)}(d(q)) + \cdots$$

therefore, $P^k(F^{(2)}) \simeq F^{(2k)} \oplus F^{(2k-4)} \oplus F^{(2k-8)} \oplus \cdots$ as representations of G.

Next, we'll show that $\dim P^k(F^{(2)})^G = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ 1 & \text{if } k \text{ is even}. \end{cases}$

Because $P^k(F^{(2)}) \simeq F^{(2k)} \oplus F^{(2k-4)} \oplus F^{(2k-8)} \oplus \cdots$, we have a decomposition

$$P^k(F^{(2)}) = A^{(2k)} \oplus A^{(2k-4)} \oplus A^{(2k-8)} \oplus \cdots$$

where $A^{(m)} \simeq F^{(m)}$ as representations of G. For each $v \in P^k(F^{(2)})$, we write $v = v^{(2k)} + v^{(2k-4)} + v^{(2k-8)} + \cdots$ with $v^{(m)} \in A^{(m)}$. We notice that

$$g.v = v \quad \forall g \in G$$

$$\Leftrightarrow \underbrace{g.v^{(2k)}}_{\in A^{(2k)}} + \underbrace{g.v^{(2k-4)}}_{\in A^{(2k-4)}} + \underbrace{g.v^{(2k-8)}}_{\in A^{(2k-8)}} + \cdots = \underbrace{v^{(2k)}}_{\in A^{(2k)}} + \underbrace{v^{(2k-4)}}_{\in A^{(2k-4)}} + \underbrace{v^{(2k-8)}}_{\in A^{(2k-8)}} + \cdots$$

$$\Leftrightarrow \begin{cases} g.v^{(2k)} = v^{(2k)}, \\ g.v^{(2k-4)} = v^{(2k-4)}, \quad \forall g \in G \\ \cdots \end{cases}$$

$$\Leftrightarrow \begin{cases} v^{(2k)} \in \left(A^{(2k)}\right)^G, \\ v^{(2k-4)} \in \left(A^{(2k-4)}\right)^G, \\ \cdots \end{cases}$$

Therefore, $\mathcal{P}^k(F^{(2)})^G \simeq (F^{(2k)})^G \oplus (F^{(2k-4)})^G \oplus \cdots$

Thus, $\dim \mathcal{P}^k(F^{(2)})^G = \dim (F^{(2k)})^G + \dim (F^{(2k-4)})^G + \cdots$

Hence, it suffices to show that

$$\dim (F^{(m)})^G = \begin{cases} 1 & \text{if } m=0, \\ 0 & \text{if } m \geqslant 1. \end{cases}$$

For $m=0$, $F^{(0)} \cong \mathbb{C}$. We know that the trivial representation of $G = SL(2,\mathbb{C})$ on $\mathbb{C}$ is irreducible. Thus, $(F^{(0)})^G = F^{(0)} \cong \mathbb{C}$. Then $\dim (F^{(0)})^G = 1$.

Consider $m \geqslant 1$. Then $F^{(m)}$ can be taken as the space of binary forms of degree m, i.e. $F^{(m)} = \{ f(x,y) = a_0 x^m + a_1 x^{m-1} y + \cdots + a_{m-1} x y^{m-1} + a_m y^m \mid a_0, a_1, \ldots a_m \in \mathbb{C} \}$.
The irreducible representation of G on $F^{(k)}$ is given in the proof of Proposition 2.3.5, page 86, Goodman – Wallach, namely

$$(g.f)(x,y) = f(ax+cy,\ bx+dy), \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \text{ and } f \in F^{(m)}.$$

Take $f \in (F^{(m)})^G$. Then $f(ax+cy, bx+dy) \equiv f(x,y)$ $\forall\, a,b,c,d \in \mathbb{C}$, $ad-bc=1$.

Write $f(x,y) = \sum_{j=0}^{m} a_j x^j y^{k-j}$.

Take $b=c=0$ and $d=a^{-1}$, we have $f(ax, a^{-1}y) \equiv f(x,y)$ $\forall a \in \mathbb{C}\setminus\{0\}$

Thus $\sum_{j=0}^{m} a_j a^{2j-m} x^j y^{m-j} = \sum_{j=0}^{m} a_j x^j y^{m-j}$.

This happens only if $a_j = a_j a^{2j-m}$ $\forall\, 0 \leq j \leq m$.

Thus, $a_j(a^{2j-m}-1)=0 \quad \forall 0\leq j\leq m \quad \forall a\in\mathbb{C}\setminus\{0\}$.

Thus, $a_j=0$ if $j\neq \frac{m}{2}$. If m is odd then $f\equiv 0$, i.e. $(F^{(m)})^G=\{0\}$.

If $m=2k$ then $f(x,y) = a_k x^k y^k \quad (k\geq 1)$. Now choose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{C}).$$

Then $f(ax+cy,\, bx+dy) = f(x, x+y) = a_k x^k (x+y)^k$. We must have

$$a_k x^k y^k \equiv a_k x^k (x+y)^k$$

This is possible only if $a_k=0$. Therefore $f(x,y)\equiv 0$ and thus $(F^{(m)})^G=\{0\}$.

Hence, $\dim (F^{(m)})^G = 0$ for all $m\geq 1$.

(c) We show that $\mathcal{P}(F^{(2)})^G = \mathbb{C}[D]$. We have the decomposition of $\mathcal{P}(F^{(2)})$ into subrepresentations $\mathcal{P}(F^{(2)}) = \bigoplus_{k=0}^{\infty} \mathcal{P}^k(F^{(2)})$. Thus,

$$\mathcal{P}(F^{(2)})^G = \bigoplus_{k=0}^{\infty} \mathcal{P}^k(F^{(2)})^G.$$

Because $\dim \mathcal{P}^k(F^{(2)})^G = 0$ if k is odd, we have

$$\mathcal{P}(F^{(2)})^G = \bigoplus_{m=0}^{\infty} \mathcal{P}^{2m}(F^{(2)})^G \qquad (*)$$

For each $m\geq 0$, we put $\varphi_m \in \mathcal{P}^{2m}(F^{(2)})$, $\varphi_m(f) = (D(f))^m = (a_1^2 - a_0a_2)^m$ for all $f\in F^{(2)}$. Then $\varphi_m(g\cdot f) = (D(g\cdot f))^m = (D(f))^m = \varphi_m(f)$ for all $g\in G$. Thus, $\varphi_m \in \mathcal{P}^{2m}(F^{(2)})^G$. Because $\dim \mathcal{P}^{2m}(F^{(2)})^G = 1$, we have

$$\mathcal{P}^{2m}(F^{(2)})^G = \langle \varphi_m\rangle = \langle D^m\rangle$$

From $(*)$, we get $\mathcal{P}(F^{(2)})^G = \bigoplus_{m=0}^{\infty} \langle D^m\rangle = \mathbb{C}[D]$.

$\boxed{6}$

② Problem 1, page 254, Goodman - Wallach.

Put $V = \mathbb{C}^n$, $G = GL(n, \mathbb{C})$. Let G act on V by ~~usually~~ usual multiplication, i.e. $g.v = gv$ $\forall g \in G$, $\forall v \in \mathbb{C}^n$. Let G act on M_n by conjugation, i.e. $g.X = gXg^{-1}$ $\forall g \in G$, $\forall X \in M_n$. Consider a linear map $T: V \otimes V^* \to M_n$ defined by $T(v \otimes v^*) = A \in M_n$ where $v^*(w)\, v = Aw$ $\forall w \in V$.

We admit that T is a linear isomorphism. Now we'll show that T intertwines with the action of G on $V \otimes V^*$ and M_n. Consider $g \in G$, $v \in V$, $v^* \in V^*$. We have $g.(v \otimes v^*) = (g.v) \otimes (g.v^*)$. ~~For $w \in V$~~,

$$T(g.(v \otimes v^*)) = A_1, \text{ where } A_1 w = (g.v^*)(w)\,(g.v) \quad \forall w \in V.$$

By the definition of the action of G on V^*, we have

$$A_1 w = v^*(g^{-1}w)(g.v) = g.(v^*(g^{-1}w)\, v) = g.(Ag^{-1}w)) = (gAg^{-1})w, \quad (1)$$

where $A = T(v \otimes v^*)$. We have

$$g.\, T(v \otimes v^*) = g.A = gAg^{-1} \overset{(1)}{=\!=} A_1 = T(g(v \otimes v^*)).$$

Thus, T is an intertwining map, and $V \otimes V^*$ and M_n are equivalent representation of G.

(a) The map $T^{\otimes k}: (V \otimes V^*)^{\otimes k} \to M_n^{\otimes k}$, $T^{\otimes k}(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = T(v_1 \otimes v_1^*) \otimes \cdots \otimes T(v_k \otimes v_k^*)$ is also a linear isomorphism. For $g \in G$, we have

$$g.(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = g.v_1 \otimes g.v_1^* \otimes \cdots \otimes g.v_k \otimes g.v_k^*.$$

Thus, $T^{\otimes k}(g.(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*)) = T^{\otimes k}(g.v_1 \otimes g.v_1^* \otimes \cdots \otimes g.v_k \otimes g.v_k^*)$

$= T(g\cdot v_1 \otimes g\cdot v_1^*) \otimes \cdots \otimes T(g\cdot v_k \otimes g\cdot v_k^*)$

$= T(g\cdot(v_1 \otimes v_1^*)) \otimes \cdots \otimes T(g\cdot(v_k \otimes v_k^*))$

$= g\cdot T(v_1 \otimes v_1^*) \otimes \cdots \otimes g\cdot T(v_k \otimes v_k^*)$ (because T is intertwining)

$= g\cdot \left(T(v_1 \otimes v_1^*) \otimes \cdots \otimes T(v_k \otimes v_k^*)\right)$

$= g\cdot T^{\otimes k}(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*)$.

Therefore, $T^{\otimes k}$ is intertwining. Hence, $(V \otimes V^*)^{\otimes k}$ and $M_n^{\otimes k}$ are equivalent representations of G.

(b) Consider a cyclic permutation $c \in \mathfrak{S}_k$ of $\{1, 2, \dots, k\}$. Write

$$c = (m_1 \to m_2 \to \cdots \to m_k \to m_1).$$

Define $\lambda_c \in \left[(V \otimes V^*)^{\otimes k}\right]^*$ as follows.

$$\lambda_c(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{j=1}^{k} \langle v_{m_j}^*, v_{m_{j+1}} \rangle.$$

Put $X_j = T(v_j \otimes v_j^*)$ $\forall 1 \le j \le k$. We want to show that

$$\lambda_c(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = tr(X_{m_1} X_{m_2} \cdots X_{m_k}).$$

This is equivalent to showing that

$$\prod_{j=1}^{k} \langle v_{m_j}^*, v_{m_{j+1}} \rangle = tr(X_{m_1} X_{m_2} \cdots X_{m_k}). \qquad (2)$$

Because both sides of (2) are multilinear maps in $(v_1, v_1^*, \dots, v_k, v_k^*)$, it suffices to show (2) for $v_1, \dots, v_k \in \{e_1, \dots, e_n\}$ and $v_1^*, \dots, v_k^* \in \{e_1^*, \dots, e_k^*\}$. Write $v_j^* = e_{s_j}^*$ and $v_j = e_{r_j}$. Then

$$LHS(2) = \cancel{\prod_{j=1}^{k} \langle e_{m_{s_j}}^*, e} \prod_{j=1}^{k} \langle e_{s_{m_j}}^*, e_{r_{m_{j+1}}} \rangle$$

Thus, $$\mathrm{LHS}(2) = \begin{cases} 1 & \text{if } s_{m_j} = r_{m_{j+1}} \quad \forall 1\le j\le k, \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

We have $X_{m_j} = T(v_{m_j} \otimes v^*_{m_j}) = T(e_{r_{m_j}} \otimes e^*_{s_{m_j}})$. For every $w \in V$,

$$X_{m_j} w = e^*_{s_{m_j}}(w)\, e_{r_{m_j}} = w_{s_{m_j}}\, e_{r_{m_j}} = \begin{array}{cc} & \\ r_{m_j} & \left[\begin{array}{ccc} & & \\ & & 1 \\ & & \end{array}\right] \\ & \quad\quad s_{m_j} \end{array} w.$$

Thus, $$X_{m_j} = \begin{array}{cc} \left[\begin{array}{ccc} & & \\ & 1 & \\ & & \end{array}\right] & r_{m_j} \\ \quad s_{m_j} & \end{array}.$$

then $$X_{m_1} X_{m_2} \cdots X_{m_k} = \begin{cases} \left[\begin{array}{ccc} \ddots & & \\ & 1 & \\ & & \ddots \end{array}\right] \text{ only one entry on the diagonal} = 1, \text{ others are } 0 & \text{if } s_{m_j} = r_{m_{j+1}} \quad \forall 1\le j\le k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $$\mathrm{RHS}(2) = \mathrm{tr}(X_{m_1} X_{m_2} \cdots X_{m_k}) = \begin{cases} 1 & \text{if } s_{m_j} = r_{m_{j+1}} \quad \forall 1\le j\le k \\ 0 & \text{otherwise.} \end{cases} \qquad (4).$$

From (3) and (4), we have (2).

(c) Let $s \in \mathfrak{S}_k$. Write $s = c_1 c_2 \cdots c_r$ where ~~ea~~ $c_1, c_2, \ldots, c_r$ are disjoint cyclic permutations. Write $c_i = (m_{1,i} \to m_{2,i} \to \cdots \to m_{k_i,i} \to m_{1,i})$. Define $\lambda_s \in [(V\otimes V^*)^{\otimes k}]^*$ as follows.

$$\lambda_s(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{i=1}^{r} \prod_{j=1}^{k_i} \langle v^*_{m_{j,i}}, v_{m_{j+1,i}} \rangle. \qquad (5)$$

By Equation (2) in part (b), we have $\prod_{j=1}^{k_i} \langle v^*_{m_{j,i}}, v_{m_{j+1,i}} \rangle = \mathrm{tr}(X_{m_{1,i}} X_{m_{2,i}} \cdots X_{m_{k_i,i}})$.

Therefore, (5) becomes $\lambda_s(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) = \prod_{i=1}^{r} tr(X_{m_{1,i}} X_{m_{2,i}} \cdots X_{m_{k_i,i}})$.

(3) Problem 2, page 255, Goodman-Wallach.

Put $G = GL(n,\mathbb{C})$. Let G act on M_n by conjugation, i.e. $g.X = gXg^{-1}$ for all $g \in G$, $X \in M_n$. For $1 \leq i \leq n$, we define $u_i \in \mathcal{P}(M_n)$ as $u_i(X) = tr(X^i)$.

We'll show that $\mathcal{P}(M_n)^G = \mathbb{C}[u_1, u_2, \ldots, u_n]$.

First, we'll show that $\mathbb{C}[u_1, u_2, \ldots, u_n] \subset \mathcal{P}(M_n)^G$. Because $\mathcal{P}(M_n)^G$ is an algebra over $\mathbb{C}$, it suffices to show that $u_i \in \mathcal{P}(M_n)^G$ for every $1 \leq i \leq n$. For any $g \in G$, we have

$$u_i(g.X) = u_i(gXg^{-1}) = tr((gXg^{-1})^i) = tr(gX^ig^{-1}) = tr(X^i) = u_i(X).$$

Therefore, $u_i \in \mathcal{P}(M_n)^G$.

Next, we'll show that $\mathcal{P}(M_n)^G \subset \mathbb{C}[u_1, u_2, \ldots, u_n]$. We have the decomposition

$$\mathcal{P}(M_n) = \bigoplus_{k=0}^{\infty} \mathcal{P}^k(M_n),$$

where $\mathcal{P}^k(M_n)$ is the set all homogeneous polynomials on M_n of degree k.

Thus, $\mathcal{P}(M_n)^G = \bigoplus_{k=0}^{\infty} \mathcal{P}^k(M_n)^G$. Hence, it suffices to show that for every $k \geq 0$, $\mathcal{P}^k(M_n)^G \subset \mathbb{C}[u_1, u_2, \ldots, u_n]$. Because $\mathcal{P}^0(M_n) = \mathbb{C}$, we only need to consider the case $k \geq 1$. We'll follow 9 steps.

<u>Step 1</u> Consider a map

$$\phi : \mathcal{P}^k(M_n) \longrightarrow (S^k(M_n))^*$$
$$f \longmapsto (X^{\otimes k} \mapsto f(X)).$$

Recall that $S^k(M_n)$ is linearly spanned by the elements $X^{\otimes k}$, for $X \in M_n$ (see Lemma B.2.3, page 651). We admit that ϕ is well-defined and is a linear isomorphism. We'll show that ϕ ~~is intertwining~~ intertwines with the action of G on $\mathcal{P}^k(M_n)$ and $(S^k(M_n))^*$. Consider $g \in G$, $f \in \mathcal{P}^k(M_n)$.

$$g \cdot f \in \mathcal{P}^k(M_n), \quad (g \cdot f)(X) = f(g^{-1} \cdot X) = f(g^{-1} X g).$$

Then $\phi(g \cdot f) = (Z^{\otimes k} \mapsto (g \cdot f)(Z)) = (Z^{\otimes k} \mapsto f(g^{-1} Z g))$. (1)

On the other hand,

$$\begin{aligned} g \cdot (\phi(f)) &= (Z \mapsto \phi(f)(g^{-1} \cdot Z)) \\ &= (Z^{\otimes k} \mapsto \phi(f)(g^{-1} \cdot Z^{\otimes k})) \\ &= (Z^{\otimes k} \mapsto \phi(f)((g^{-1} \cdot Z)^{\otimes k})) \\ &= (Z^{\otimes k} \mapsto f(g^{-1} \cdot Z)) \\ &= (Z^{\otimes k} \mapsto f(g^{-1} Z g)) \qquad (2) \end{aligned}$$

By (1) and (2), $\phi(g \cdot f) = g \cdot (\phi(f))$. Thus, ϕ is intertwining. Therefore, $\mathcal{P}^k(M_n)$ and $(S^k(M_n))^*$ are equivalent representations of G.

Step 2 We have a natural injection map $S^k(M_n) \hookrightarrow M_n^{\otimes k}$. Thus, we have a natural dual map, which is surjective, $(M_n^{\otimes k})^* \longrightarrow (S^k(M_n))^*$. Hence,

$$[S^k(M_n)^*]^G = \{ f|_{S^k(M_n)} : f \in (M_n^{\otimes k})^G \}.$$

Step 3 Consider the map $B: V^{\otimes k}\otimes V^{*\otimes k} \longrightarrow [(V\otimes V^*)^{\otimes k}]^*$,

$$B(v_1\otimes\cdots\otimes v_k\otimes v_1^*\otimes\cdots\otimes v_k^*) = \Big(w_1\otimes w_1^*\otimes\cdots\otimes w_k\otimes w_k^* \mapsto \prod_{i=1}^k \langle v_i^*, w_i\rangle\langle w_i^*, v_i\rangle\Big).$$

We admit that B is a linear isomorphism. We'll show that B is intertwining.

For any $g\in G$, we have

$$B(g.(v_1\otimes\cdots\otimes v_k\otimes v_1^*\otimes\cdots\otimes v_k^*)) = B(g.v_1\otimes\cdots\otimes g.v_k\otimes g.v_1^*\otimes\cdots\otimes g.v_k^*)$$

$$= \Big(w_1\otimes w_1^*\otimes\cdots\otimes w_k\otimes w_k^* \mapsto \prod_{i=1}^k \langle g.v_i^*, w_i\rangle\langle w_i^*, g v_i\rangle\Big)$$

$$= \Big(w_1\otimes w_1^*\otimes\cdots\otimes w_k\otimes w_k^* \mapsto \prod_{i=1}^k \langle v_i^*, g^{-1}w_i\rangle\langle w_i^*, g v_i\rangle\Big) \quad (3)$$

$$g.B(v_1\otimes\cdots\otimes v_k\otimes v_1^*\otimes\cdots\otimes v_k^*) = g.\Big(w_1\otimes w_1^*\otimes\cdots\otimes w_k\otimes w_k^* \mapsto \prod_{i=1}^k \langle v_i^*, w_i\rangle\langle w_i^*, v_i\rangle\Big)$$

$$= \Big(w_1\otimes w_1^*\otimes\cdots\otimes w_k\otimes w_k^* \mapsto \prod_{i=1}^k \langle v_i^*, g^{-1}w_i\rangle\langle g^{-1}w_i^*, v_i\rangle\Big)$$

$$= \Big(w_1\otimes w_1^*\otimes\cdots\otimes w_k\otimes w_k^* \mapsto \prod_{i=1}^k \langle v_i^*, g^{-1}w_i\rangle\langle w_i^*, g v_i\rangle\Big) \quad (4)$$

From (3) and (4), we conclude that B is an intertwining map. Therefore, $V^{\otimes k}\otimes V^{*\otimes k}$ and $[(V\otimes V^*)^{\otimes k}]^*$ are equivalent representions of G.

Step 4 By Theorem 5.3.1, page 247, Goodman-Wallach,

$$(V^{\otimes k}\otimes V^{*\otimes k})^G = \operatorname{span}\{C_s : s\in\mathfrak{S}_k\}, \text{ where } C_s = \sum_{|I|=k} e_{s.I}\otimes e_I^*,$$

$s.I = s.(i_1,\dots,i_k) = (i_{s^{-1}(1)},\dots,i_{s^{-1}(k)})$, $e_I = e_{i_1}\otimes\cdots\otimes e_{i_k}$.

Now by Step 3, we get $\{[(V\otimes V^*)^{\otimes k}]^*\}^G = \operatorname{span}\{B(C_s) : s\in\mathfrak{S}_k\}$.

Step 5 We will compute $B(C_s)$.

$$B(C_s) = B\Big(\sum_{|I|=k} e_{s.I}\otimes e_I^*\Big) = \sum_{|I|=k} B\left(e_{s.I}\otimes e_I^*\right). \qquad (5)$$

$$B(e_{s.I}\otimes e_I^*) = B\left(e_{i_{s^{-1}(1)}}\otimes\cdots\otimes e_{i_{s^{-1}(k)}}\otimes e_{i_1}^*\otimes\cdots\otimes e_{i_k}^*\right)$$

$$= \Big(v_1\otimes v_1^*\otimes\cdots\otimes v_k\otimes v_k^* \longmapsto \prod_{j=1}^{k}\langle e_{i_j}^*, v_j\rangle \prod_{j=1}^{k}\langle v_j^*, e_{i_{s^{-1}(j)}}\rangle\Big)$$

replace j by s(j)

$$= \Big(v_1\otimes v_1^*\otimes\cdots\otimes v_k\otimes v_k^* \longmapsto \prod_{j=1}^{k}\langle e_{i_j}^*, v_j\rangle \prod_{j=1}^{k}\langle v_{s(j)}^*, e_{i_j}\rangle\Big)$$

$$= \Big(v_1\otimes v_1^*\otimes\cdots\otimes v_k\otimes v_k^* \longmapsto \prod_{j=1}^{k}(v_j)_{i_j}(v_{s(j)}^*)_{i_j}\Big)$$

Then (5) becomes

$$B(C_s) = \sum_{|I|=k}\Big(v_1\otimes v_1^*\otimes\cdots\otimes v_k\otimes v_k^* \longmapsto \prod_{j=1}^{k}(v_j)_{i_j}(v_{s(j)}^*)_{i_j}\Big)$$

$$= \Big(v_1\otimes v_1^*\otimes\cdots\otimes v_k\otimes v_k^* \longmapsto \sum_{|I|=k}\prod_{j=1}^{k}(v_j)_{i_j}(v_{s(j)}^*)_{i_j}\Big)$$

$$= \Big(v_1\otimes v_1^*\otimes\cdots\otimes v_k\otimes v_k^* \longmapsto \langle v_{s(1)}^*, v_1\rangle\cdots\langle v_{s(k)}^*, v_k\rangle\Big) \qquad (6)$$

Define $\lambda_s \in \left[(V\otimes V^*)^{\otimes k}\right]^*$ as follows.

$$\lambda_s\left(v_1\otimes v_1^*\otimes\cdots\otimes v_k\otimes v_k^*\right) := \langle v_1^*, v_{s(1)}\rangle\cdots\langle v_k^*, v_{s(k)}\rangle. \qquad (7)$$

Then (6) implies $B(C_s) = \lambda_{s^{-1}}$.

Step 6 With the result $B(C_s) = \lambda_{s^{-1}}$, Step 4 gives us

$$\left\{\left[(V\otimes V^*)^{\otimes k}\right]^*\right\}^G = \mathrm{span}\{\lambda_{s^{-1}} : s\in\mathfrak{S}_k\} = \mathrm{span}\{\lambda_s : s\in\mathfrak{S}_k\}$$

For $s \in \mathfrak{S}_k$, we write $s = c_1 c_2 \cdots c_r$ where $c_1, c_2, \ldots, c_r$ are disjoint cyclic ~~decomposition~~ permutations. Write $c_i = (m_{1,i} \to m_{2,i} \to \cdots \to m_{k_i,i} \to m_{1,i})$. Then

$$\lambda_s(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^*) \overset{(7)}{=} \langle v_1^*, v_{s(1)} \rangle \cdots \langle v_k^*, v_{s(k)} \rangle$$

$$= \prod_{i=1}^{r} \prod_{j=1}^{k_i} \langle v^*_{m_{j,i}}, v_{m_{j+1,i}} \rangle$$

$$= \prod_{i=1}^{r} tr(X_{m_{1,i}} X_{m_{2,i}} \cdots X_{m_{k_i,i}}) \quad \text{(by Part (c), Prob 2)} \quad (8)$$

where $X_j = T(v_j \otimes v_j^*)$.

<u>Step 7</u> By Part (a), Problem 2, we have an isomorphism of representations

$$T^{\otimes k} : (V \otimes V^*)^k \longrightarrow M_n^{\otimes k}.$$

Thus, we have an isomorphism $(T^{\otimes k})^* : (M_n^{\otimes k})^* \longrightarrow [(V \otimes V^*)^k]^*$,

$$f \longmapsto f \circ T^{\otimes k}.$$

A map $f \in (M_n^{\otimes k})^*$ can be considered as a multi-linear map on M_n^k.

Under $T^{\otimes k}$, $\underbrace{f(X_1, \ldots, X_k)}_{\text{multi-linear}} \longmapsto \left(v_1 \otimes v_1^* \otimes \cdots \otimes v_k \otimes v_k^* \mapsto f(\tilde{X}_1, \ldots, \tilde{X}_k) \right)$

where $\tilde{X}_j = T(v_j \otimes v_j^*)$.

We have

$$f_s := [(T^{\otimes k})^*]^{-1}(\lambda_s) \overset{(8)}{=} \left(X_1 \otimes \cdots \otimes X_k \longmapsto \prod_{i=1}^{r} tr(X_{m_{1,i}} X_{m_{2,i}} \cdots X_{m_{k_i,i}}) \right) \quad (9)$$

Because $\{[(V \otimes V^*)^{\otimes k}]^*\}^G = span\{\lambda_s : s \in \mathfrak{S}_k\}$, we have

$$[(M_n^{\otimes k})^*]^G = span\{f_s : s \in \mathfrak{S}_k\}.$$

Step 8 According to Step 2, we have

$$[S^k(M_n)^*]^G = \text{span}\{ f_s|_{S^k(M_n)} : s \in \mathfrak{S}_k \}.$$

Put $g_s = f_s|_{S^k(M_n)}$. We have

$$g_s(X^{\otimes k}) = f_s(X^{\otimes k}) \overset{(9)}{=} \prod_{i=1}^{r} tr(X^{k_i}) = u_{k_1}(X)\, u_{k_2}(X) \dots u_{k_r}(X).$$

Step 9 According to Step 1, we have

$$\phi^{-1}(g_s) = (X \mapsto u_{k_1}(X) \dots u_{k_r}(X)) \in \mathbb{C}[u_1, \dots, u_n].$$

Thus $\mathcal{P}^k(M_n)^G = \text{span}\{\phi^{-1}(g_s) : s \in \mathfrak{S}_k\} \subset \mathbb{C}[u_1, \dots, u_n]$.

(4) Problem #1, page 277, Goodman-Wallach.

Let Ω be the following bilinear form on $\mathbb{C}^4$, $\Omega(x,y) = x^t J_2 y \;\; \forall x,y \in \mathbb{C}^4$, where $J_2 = \begin{pmatrix} 0 & s_2 \\ -s_2 & 0 \end{pmatrix}$ and $s_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Put $G = Sp(\mathbb{C}^4, \Omega)$, which is a classical group of Type C_2.

Put $V = \mathbb{C}^4$. Let (π, V) be the representation of ~~G~~ in usual sense, i.e. $\pi(g)v = \underset{\text{usual multiplication}}{gv}$, $\forall g \in G, v \in V$. This representation induces a representation on $(\pi \otimes \pi, V \otimes V)$ of G. Put $\tilde{V} = \Lambda^2(V) \subset V \otimes V$. Denote by $(\rho, \tilde{V})$ the subrepresentation of $(\pi \otimes \pi, V \otimes V)$.

(a) We will find the weights of ρ. We understand these weights as the weights of the representation $(d\rho, \tilde{V})$ of $\mathfrak{g} = \mathfrak{sp}(\mathbb{C}^4, \Omega)$, which are defined ~~on~~ in

Theorem 3.1.16, page 138. Let (e_1, e_2, e_3, e_4) be the standard basis of $V = \mathbb{C}^4$.

Then an ordered basis of the vector space $\tilde{V} = \Lambda^2(V)$ is

$$(e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4).$$

By a result on page 73, Goodman-Wallach, a Cartan subalgebra of $\underbrace{\mathfrak{g}}_{\mathfrak{g}=Lie(G)}$ is

$$\mathfrak{h} = \{ diag(a_1, a_2, -a_2, -a_1) : a_1, a_2 \in \mathbb{C} \}$$

Thus, a basis of the dual vector space $\mathfrak{h}^*$ is $\{\varepsilon_1, \varepsilon_2\}$, where

$$\varepsilon_1(diag(a_1, a_2, -a_2, -a_1)) = a_1,$$

$$\varepsilon_2(diag(a_1, a_2, -a_2, -a_1)) = a_2.$$

For each $\lambda \in \mathfrak{h}^*$, put

$$\tilde{V}(\lambda) = \{ w \in \tilde{V} : d\rho(h) w = \langle \lambda, h \rangle w \quad \forall h \in \mathfrak{h} \}.$$

We write $w = c_{12} e_1 \wedge e_2 + c_{13} e_1 \wedge e_3 + c_{14} e_1 \wedge e_4 + c_{23} e_2 \wedge e_3 + c_{24} e_2 \wedge e_4 + c_{34} e_3 \wedge e_4,$

$$h = diag(a_1, a_2, -a_2, -a_1),$$

$$\lambda = \beta_1 \varepsilon_1 + \beta_2 \varepsilon_2.$$

Then $\langle \lambda, h \rangle = \beta_1 \langle \varepsilon_1, h \rangle + \beta_2 \langle \varepsilon_2, h \rangle = \beta_1 a_1 + \beta_2 a_2$. Thus,

$$\langle \lambda, h \rangle w = \sum_{1 \le i < j \le 4} (\beta_1 a_1 + \beta_2 a_2) c_{ij} \, e_i \wedge e_j \qquad (1)$$

We have $d\rho(h) w = \displaystyle\sum_{1 \le i < j \le 4} c_{ij} \, d\rho(h)(e_i \wedge e_j) = \sum_{1 \le i < j \le 4} c_{ij} \frac{d}{dt}\Big|_{t=0} \rho(\underbrace{\exp(th)}_{A_t})(e_i \wedge e_j)$

$$= \sum_{1 \le i < j \le 4} c_{ij} \frac{d}{dt}\Big|_{t=0} (A_t e_i \wedge A_t e_j) =$$

16

$$= \sum_{1\le i<j\le 4} c_{ij}\left[\left(\frac{d}{dt}\Big|_{t=0} A_t e_i\right)\wedge e_j + e_i\wedge\left(\frac{d}{dt}\Big|_{t=0} A_t e_j\right)\right] \qquad (*)$$

We have $\frac{d}{dt}\Big|_{t=0} A_t = \frac{d}{dt}\Big|_{t=0} \exp(th) = h$. Thus, $(*)$ implies

$$d\rho(h)w = \sum_{1\le i<j\le 4} c_{ij}\left[(he_i)\wedge e_j + e_i\wedge(he_j)\right]. \qquad (2)$$

We have

$$he_1 = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & -a_2 & \\ & & & -a_1 \end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}a_1\\0\\0\\0\end{pmatrix} = a_1 e_1.$$

Similarly, $he_2 = a_2 e_2$, $he_3 = -a_2 e_3$, $he_4 = -a_1 e_4$.

Then $(**)$ can be written as

$$\begin{aligned} d\rho(h)w = {} & c_{12}(a_1 e_1\wedge e_2 + a_2 e_1\wedge e_2) + c_{13}(a_1 e_1\wedge e_3 - a_2 e_1\wedge e_3) + \\ & + c_{14}(\underbrace{a_1 e_1\wedge e_4 - a_1 e_1\wedge e_4}_{=0}) + c_{23}(\underbrace{a_2 e_2\wedge e_3 - a_2 e_2\wedge e_3}_{=0}) + \\ & + c_{24}(a_2 e_2\wedge e_4 - a_1 e_2\wedge e_4) + c_{34}(-a_2 e_3\wedge e_4 - a_1 e_3\wedge e_4) \\ = {} & c_{12}(a_1+a_2)\, e_1\wedge e_2 + c_{13}(a_1-a_2)\, e_1\wedge e_3 + c_{24}(a_2-a_1)\, e_2\wedge e_4 + c_{34}(-a_1-a_2)\, e_3\wedge e_4 \end{aligned} \qquad (3)$$

We now compare (1) ~~and~~ to (3). The equation $\langle\lambda,h\rangle w = d\rho(h)w$ $\forall h\in\mathfrak{h}$ is equivalent to

$$\begin{cases} (\beta_1 a_1 + \beta_2 a_2)\, c_{12} = c_{12}(a_1+a_2), & (4) \\ (\beta_1 a_1 + \beta_2 a_2)\, c_{13} = c_{13}(a_1-a_2), & (5) \\ (\beta_1 a_1 + \beta_2 a_2)\, c_{14} = 0, & (6) \\ (\beta_1 a_1 + \beta_2 a_2)\, c_{23} = 0, & (7) \\ (\beta_1 a_1 + \beta_2 a_2)\, c_{24} = c_{24}(a_2-a_1), & (8) \\ (\beta_1 a_1 + \beta_2 a_2)\, c_{34} = c_{34}(-a_1-a_2) & (9) \end{cases} \qquad \forall a_1,a_2\in\mathbb{C}$$

By (8) and (7), $c_{14} = c_{23} = 0$. If $(\beta_1, \beta_2) \notin \{(\pm 1, \pm 1)\}$ then the system (4)-(9) only has trivial solution $c_{ij} \equiv 0$.

- $\beta_1 = \beta_2 = 1$: then $\lambda = \varepsilon_1 + \varepsilon_2$. The system (4)-(9) has solutions

$$\begin{cases} c_{12} \in \mathbb{C}, \\ c_{13} = c_{24} = c_{34} = 0. \end{cases}$$

Thus, $\tilde{V}(\varepsilon_1 + \varepsilon_2) = \mathrm{span}\{e_1 \wedge e_2\}$.

- $\beta_1 = \beta_2 = -1$: then $\lambda = -\varepsilon_1 - \varepsilon_2$. The system (4)-(9) has solutions

$$\begin{cases} c_{34} \in \mathbb{C}, \\ c_{12} = c_{13} = c_{24} = 0. \end{cases}$$

Thus, $\tilde{V}(-\varepsilon_1 - \varepsilon_2) = \mathrm{span}\{e_3 \wedge e_4\}$.

- $\beta_1 = 1, \beta_2 = -1$: then $\lambda = \varepsilon_1 - \varepsilon_2$. The system (4)-(9) has solutions

$$\begin{cases} c_{13} \in \mathbb{C}, \\ c_{12} = c_{24} = c_{34} = 0 \end{cases}$$

Thus, $\tilde{V}(\varepsilon_1 - \varepsilon_2) = \mathrm{span}\{e_1 \wedge e_3\}$.

- $\beta_1 = -1, \beta_2 = 1$: then $\lambda = -\varepsilon_1 + \varepsilon_2$. The system (4)-(9) has solutions

$$\begin{cases} c_{24} \in \mathbb{C} \\ c_{12} = c_{13} = c_{34} = 0 \end{cases}$$

Thus, $\tilde{V}(-\varepsilon_1 + \varepsilon_2) = \mathrm{span}\{e_2 \wedge e_4\}$.

We conclude that the set of weights of $(\rho, \tilde{V})$ is $\mathcal{X}(\tilde{V}) = \{\pm \varepsilon_1 \pm \varepsilon_2\}$.

Next, we'll show that $\mathcal{X}(\tilde{V})$ is invariant under the action of Weyl group.

18

Let W be the Weyl group of $G = Sp(\mathbb{C}^4, \Omega)$. By Lemma 3.1.6, p. 133, W is generated by the reflections $\{s_\alpha : \alpha \in \Delta\}$. Recall that

$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2 \, , \, \alpha_2 = 2\varepsilon_2\}$$

is the set of simple roots of $G = Sp(\mathbb{C}^4, \Omega)$ (see page 101, Goodman-Wallach). By Equation (3.6), page 132, the reflection $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is given by

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

Thus,
$$s_{\alpha_1}(\beta) = \beta - \frac{2(\beta, \alpha_1)}{(\alpha_1, \alpha_1)}\alpha_1 = \beta - \frac{2(\beta, \varepsilon_1 - \varepsilon_2)}{2}(\varepsilon_1 - \varepsilon_2)$$
$$= \beta - (\beta, \varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_2) \qquad (10),$$

$$s_{\alpha_2}(\beta) = \beta - \frac{2(\beta, \alpha_2)}{(\alpha_2, \alpha_2)}\alpha_2 = \beta - \frac{2(\beta, 2\varepsilon_2)}{4}(2\varepsilon_2) = \beta - 2(\beta, \varepsilon_2)\varepsilon_2 \qquad (11).$$

Because the group W is generated by $\{s_{\alpha_1}, s_{\alpha_2}\}$, to show that $\mathcal{X}(\tilde{V})$ is invariant under the action of W, it suffices to show that $\mathcal{X}(\tilde{V})$ is invariant under s_{α_1} and s_{α_2}. Moreover, because $s_\alpha(-\beta) = -s_\alpha(\beta)$, it suffices to show that

$$s_{\alpha_1}(\varepsilon_1 + \varepsilon_2), \; s_{\alpha_1}(\varepsilon_1 - \varepsilon_2), \; s_{\alpha_2}(\varepsilon_1 + \varepsilon_2), \; s_{\alpha_2}(\varepsilon_1 - \varepsilon_2) \in \mathcal{X}(\tilde{V}).$$

Thanks to (10) and (11), we have

$$s_{\alpha_1}(\varepsilon_1 + \varepsilon_2) = (\varepsilon_1 + \varepsilon_2) - \underbrace{(\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2)}_{=0}(\varepsilon_1 - \varepsilon_2) = \varepsilon_1 + \varepsilon_2 \in \mathcal{X}(\tilde{V}).$$

$$s_{\alpha_1}(\varepsilon_1 - \varepsilon_2) = (\varepsilon_1 - \varepsilon_2) - \underbrace{(\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2)}_{=2}(\varepsilon_1 - \varepsilon_2) = -(\varepsilon_1 - \varepsilon_2) \in \mathcal{X}(\tilde{V}).$$

$$s_{\alpha_2}(\varepsilon_1+\varepsilon_2)=(\varepsilon_1+\varepsilon_2)-2(\varepsilon_1+\varepsilon_2,\varepsilon_2)\varepsilon_2=\varepsilon_1+\varepsilon_2-2\varepsilon_2=\varepsilon_1-\varepsilon_2\in\mathfrak{X}(\tilde{V}).$$

$$s_{\alpha_2}(\varepsilon_1-\varepsilon_2)=(\varepsilon_1-\varepsilon_2)-2(\varepsilon_1-\varepsilon_2,\varepsilon_2)\varepsilon_2=\varepsilon_1-\varepsilon_2+2\varepsilon_2=\varepsilon_1+\varepsilon_2\in\mathfrak{X}(\tilde{V}).$$

(5) Problem 1, Section 7.1.4, page 339, Goodman-Wallach.

We will verify the Weyl Denominator Formula for a classical group G of types A_ℓ, B_ℓ, C_ℓ, D_ℓ: $\Delta_G=\sum_{s\in W}\mathrm{sgn}(s)\,e^{s.\rho}$.

By the definition of Δ_G, we are supposed to show that

$$e^\rho\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})=\sum_{s\in W}\mathrm{sgn}(s)\,e^{s.\rho}, \qquad (1)$$

where $\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and W is the Weyl group of G. A survey of Φ^+ is given on pages 100-102, Goodman-Wallach.

Type A_ℓ $(G=SL(\ell+1,\mathbb{C}))$: $\Phi^+=\{\varepsilon_i-\varepsilon_j : 1\le i<j\le \ell+1\}$,

Type B_ℓ $(G=SO(2\ell+1,\mathbb{C}))$: $\Phi^+=\{\varepsilon_i-\varepsilon_j,\ \varepsilon_i+\varepsilon_j : 1\le i<j\le\ell\}\cup\{\varepsilon_i : 1\le i\le\ell\}$,

Type C_ℓ $(G=Sp(\ell,\mathbb{C}))$: $\Phi^+=\{\varepsilon_i-\varepsilon_j,\ \varepsilon_i+\varepsilon_j : 1\le i<j\le\ell\}\cup\{2\varepsilon_i : 1\le i\le\ell\}$,

Type D_ℓ $(G=SO(2\ell,\mathbb{C}),\ \ell\ge 3)$: $\Phi^+=\{\varepsilon_i-\varepsilon_j,\ \varepsilon_i+\varepsilon_j : 1\le i<j\le\ell\}$.

We will very (1) in each type of G.

<u>Type A_ℓ: $G=SL(\ell+1,\mathbb{C})$</u>

$$\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha=\frac{1}{2}\sum_{1\le i<j\le\ell+1}(\varepsilon_i-\varepsilon_j)=\frac{\ell}{2}\varepsilon_1+\frac{\ell-2}{2}\varepsilon_2+\dots+\frac{2-\ell}{2}\varepsilon_\ell+\frac{-\ell}{2}\varepsilon_{\ell+1}. \qquad (2)$$

Recall that $\mathfrak{h}=\{\mathrm{diag}(a_1,\dots,a_{\ell+1}) : \sum_{i=1}^{\ell+1}a_i=0\}$ and $\varepsilon_i(\mathrm{diag}(a_1,\dots,a_{\ell+1}))=a_i$.

Thus, $\varepsilon_1+\varepsilon_2+\dots+\varepsilon_{\ell+1}=0$. Then (2) becomes

$$\rho=\frac{\ell}{2}\varepsilon_1+\frac{\ell-2}{2}\varepsilon_2+\dots+\frac{2-\ell}{2}\varepsilon_\ell+\frac{\ell}{2}(\varepsilon_1+\dots+\varepsilon_\ell)=\ell\varepsilon_1+(\ell-1)\varepsilon_2+\dots+\varepsilon_\ell. \qquad (3)$$

Thus, as a character on $H=\{h=\mathrm{diag}(x_1,\dots,x_{\ell+1}) \mid x_1\cdots x_\ell x_{\ell+1}=1\}$,

$$e^\rho = x_1^\ell x_2^{\ell-1}\cdots x_\ell. \qquad (4)$$

For $\alpha=\varepsilon_i-\varepsilon_j$, we have $e^{-\alpha}=e^{+\varepsilon_j-\varepsilon_i}=x_i^{-1}x_j$. Thus,

$$LHS(1)=x_1^\ell x_2^{\ell-1}\cdots x_\ell\prod_{1\le i<j\le \ell+1}(1-x_i^{-1}x_j)=\prod_{1\le i<j\le \ell+1}x_i(1-x_i^{-1}x_j)=\prod_{1\le i<j\le \ell+1}(x_i-x_j). \qquad (5)$$

Now we will compute RHS(1). For $\lambda=\lambda_1\varepsilon_1+\dots+\lambda_\ell\varepsilon_\ell+\lambda_{\ell+1}\varepsilon_{\ell+1}$, with $\lambda_i\in\mathbb{Z}$,

we have $\lambda=\lambda_1\varepsilon_1+\dots+\lambda_\ell\varepsilon_\ell+\lambda_{\ell+1}(-\varepsilon_1-\dots-\varepsilon_\ell)=(\lambda_1-\lambda_{\ell+1})\varepsilon_1+\dots+(\lambda_\ell-\lambda_{\ell+1})\varepsilon_\ell$.

Thus, $e^\lambda=x_1^{\lambda_1-\lambda_{\ell+1}}\cdots x_\ell^{\lambda_\ell-\lambda_{\ell+1}}=x_1^{\lambda_1}\cdots x_\ell^{\lambda_\ell}(x_1\cdots x_\ell)^{-\lambda_{\ell+1}}=x_1^{\lambda_1}\cdots x_\ell^{\lambda_\ell}x_{\ell+1}^{\lambda_{\ell+1}}$. Thus,

$$e^{\lambda_1\varepsilon_1+\dots+\lambda_{\ell+1}\varepsilon_{\ell+1}}=x_1^{\lambda_1}\cdots x_{\ell+1}^{\lambda_{\ell+1}}. \qquad (6)$$

In the proof of Lemma 3.16, page 133, the Weyl group W acts on $\mathfrak{h}^*$ by permuting $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{\ell+1}$. By (3) we get

$$s\cdot\rho=\ell\varepsilon_{s(1)}+(\ell-1)\varepsilon_{s(2)}+\dots+\varepsilon_{s(\ell)}.$$

Then by (6) we get $e^{s\rho}=x_{s(1)}^\ell x_{s(2)}^{\ell-1}\cdots x_{s(\ell)}$. Thus,

(7) $$RHS(1)=\sum_{s\in\mathfrak{S}_{\ell+1}}\mathrm{sgn}(s)\,x_{s(1)}^\ell x_{s(2)}^{\ell-1}\cdots x_{s(\ell)}=\det\begin{pmatrix}x_1^\ell & x_1^{\ell-1} & \cdots & 1\\ x_2^\ell & x_2^{\ell-1} & \cdots & 1\\ \vdots & \vdots & & \vdots\\ x_{\ell+1}^\ell & x_{\ell+1}^{\ell-1} & \cdots & 1\end{pmatrix}=\prod_{1\le i<j\le\ell+1}(x_i-x_j).$$

By (5) and (7), we obtain the identity (1).

Type B_ℓ: $G=SO(2\ell+1,\mathbb{C})$.

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \frac{1}{2}\sum_{1\le i<j\le \ell}[(\varepsilon_i-\varepsilon_j)+(\varepsilon_i+\varepsilon_j)] + \frac{1}{2}\sum_{i=1}^{\ell}\varepsilon_i = \sum_{1\le i<j\le\ell}\varepsilon_i + \frac{1}{2}\sum_{i=1}^{\ell}\varepsilon_i$$

$$= \left(\ell-\frac{1}{2}\right)\varepsilon_1 + \left(\ell-\frac{3}{2}\right)\varepsilon_2 + \cdots + \frac{1}{2}\varepsilon_\ell. \qquad (8)$$

Because the coefficients of $\varepsilon_1,\dots,\varepsilon_\ell$ are not integers, we don't know how to define e^ρ. The textbook (near the bottom of page 330) suggested that we need some knowledge about the spin group $\mathrm{Spin}(2\ell+1,\mathbb{C})$ to make sense of e^ρ. However, we will continue to work <u>formally</u> on this type.

$$e^\rho \text{ "=" } x_1^{\ell-\frac{1}{2}}x_2^{\ell-\frac{3}{2}}\cdots x_\ell^{\frac{1}{2}} = x_1^{\ell-1}x_2^{\ell-2}\cdots x_{\ell-1}(x_1\cdots x_\ell)^{1/2}. \qquad (9)$$

For $\alpha = \varepsilon_i - \varepsilon_j$, $e^{-\alpha} = e^{\varepsilon_j-\varepsilon_i} = x_i^{-1}x_j$.

For $\alpha = \varepsilon_i + \varepsilon_j$, $e^{-\alpha} = e^{-\varepsilon_i-\varepsilon_j} = x_i^{-1}x_j^{-1}$.

For $\alpha = \varepsilon_i$, $e^{-\alpha} = e^{-\varepsilon_i} = x_i^{-1}$.

Thus, $\mathrm{LHS}(1) = x_1^{\ell-1}x_2^{\ell-2}\cdots x_{\ell-1}(x_1\cdots x_\ell)^{1/2}\left(\prod_{1\le i<j\le\ell}(1-x_i^{-1}x_j)(1-x_i^{-1}x_j^{-1})\right)\left(\prod_{k=1}^{\ell}(1-x_k^{-1})\right)$

$$= \left(\prod_{1\le i<j\le\ell} x_i(1-x_i^{-1}x_j)(1-x_i^{-1}x_j^{-1})\right)\left(\prod_{k=1}^{\ell} x_k^{1/2}(1-x_k^{-1})\right)$$

$$= \prod_{1\le i<j\le\ell}(x_i+x_i^{-1}-x_j-x_j^{-1})\prod_{k=1}^{\ell}(x_k^{1/2}-x_k^{-1/2}). \qquad (10)$$

On page 128, Goodman-Wallach, an action of W on $\mathcal{X}(H)$ is defined by

$$s.e^\lambda := e^{s.\lambda}.$$

By Lemma 3.1.3, page 131, Goodman-Wallach, the Weyl group W acts on the coordinate functions in $\mathcal{X}(H)$ by $x_i \mapsto (x_{\sigma(i)})^{\pm1}$ $(i=1,2,\dots,\ell)$ for every $\sigma\in\mathfrak{S}_\ell$ and choice ±1 of exponents. Then,

$$s\cdot e^{\rho} \overset{(9)}{=\!=} s\cdot\left(x_1^{l-\frac{1}{2}} x_2^{l-\frac{3}{2}} \cdots x_l^{\frac{1}{2}}\right) = (s.x_1)^{l-\frac{1}{2}} (s.x_2)^{l-\frac{3}{2}} \cdots (sx_l)^{\frac{1}{2}}$$
$$= (x_{\sigma(1)})^{\pm(l-\frac{1}{2})} (x_{\sigma(2)})^{\pm(l-\frac{3}{2})} \cdots (x_{\sigma(l)})^{\pm\frac{1}{2}} \qquad (11)$$

When viewed as an element in $GL(\mathfrak{h}^*)$, we have $s(\varepsilon_i) = \pm\varepsilon_{\sigma(i)}$ (the choice of ± 1 is according to that of $(x_{\sigma(i)})^{\pm 1}$). By the definition of sgn(s) on the top of page 331,

$$\mathrm{sgn}(s) = \det\left((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l) \mapsto (\pm\varepsilon_{\sigma(1)}, \ldots, \pm\varepsilon_{\sigma(l)})\right) = \mathrm{sgn}(\sigma)(-1)^{\#\text{ of minus signs}} \qquad (12)$$

By (11) and (12), we have

$$RHS(1) = \sum_{\substack{\sigma\in\mathfrak{S}_l \\ \text{choices } \pm 1}} \mathrm{sgn}(\sigma)(-1)^{\#\text{ of "}-\text{"}} (x_{\sigma(1)})^{\pm(l-\frac{1}{2})} (x_{\sigma(2)})^{\pm(l-\frac{3}{2})} \cdots (x_{\sigma(l)})^{\pm\frac{1}{2}} \qquad (13)$$

When (10) is expanded, it will be a sum of $4^{\binom{l}{2}} 2^l = 4^{\frac{l(l-1)}{2}} 2^l = 2^{l^2}$ terms, each of which is of the form $\pm x_1^{r_1} x_2^{r_2} \cdots x_l^{r_l}$, where each r_i is an integer added by $\frac{1}{2}$. We'll show that after the sum of these 2^{l^2} terms is reduced, only the terms with $|r_1|, |r_2|, \ldots, |r_l|$ pairwise distinct remain. To do so, let us ~~on~~ consider two typical terms: $x_1^{r} x_2^{r} x_3^{r_3} \cdots x_l^{r_l}$ and $x_1^{r} x_2^{-r} x_3^{r_3} \cdots x_l^{r_l}$. Write

$$\prod_{1\le i<j\le l} (x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{l} (x_k^{1/2} - x_k^{-1/2}) = \sum_r A_r x_1^r x_2^r + B, \qquad (14)$$

where A_r is independent of x_1 and x_2, and B is a sum of terms $\pm x_1^{r_1} x_2^{r_2} \cdots x_l^{r_l}$ with $r_1 \ne r_2$. Switching x_1 and x_2, we get

$$-\prod_{1\le i<j\le l} (x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{l} (x_k^{1/2} - x_k^{-1/2}) = \sum_r A_r x_2^r x_1^r + B', \qquad (15)$$

where B' is a sum of terms $\pm x_1^{r_1} x_2^{r_2} \cdots x_\ell^{r_\ell}$ with $r_1 \neq r_2$ (but the choices of ± 1 may be different from those in B). Adding (14) and (15), we get

$$\sum_r A_r (x_1 x_2)^r = -\frac{B+B'}{2}. \qquad (16)$$

LHS(16) is a polynomial of x_1x_2 while RHS(16) is not unless it is identically zero. Thus $\sum_r A_r (x_1x_2)^r \equiv 0$. Therefore, each $A_r \equiv 0$. Similarly, we write

$$\prod_{1\le i<j\le \ell} (x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{\ell} (x_k^{1/2} - x_k^{-1/2}) = \sum_r C_r x_1^r x_2^{-r} + D, \qquad (17)$$

where C_r is independent of x_1 and x_2, and D is a sum of terms $\pm x_1^{r_1} x_2^{r_2} \cdots x_\ell^{r_\ell}$ with $r_1 \neq -r_2$. Switching x_1 and x_2^{-1}, we get

$$-\prod_{1\le i<j\le \ell} (x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{\ell} (x_k^{1/2} - x_k^{-1/2}) = \sum_r C_r x_2^{-r} x_1^{r} + D', \qquad (18)$$

where D' is a sum of terms $\pm x_1^{r_1} x_2^{r_2} \cdots x_\ell^{r_\ell}$ with $r_1 \neq -r_2$ (but the choices of ± 1 may be different from those in D). Adding (17) and (18), we get

$$\sum_r C_r x_1^r x_2^{-r} = \frac{D+D'}{2}. \qquad (19)$$

LHS(19) is a polynomial of $x_1 x_2^{-1}$ while RHS(19) is not unless it is identically zero. Thus, $\sum_r C_r (x_1 x_2^{-1})^r \equiv 0$. Therefore, each $C_r \equiv 0$.

We have showed that after (10) is expanded and reduced, only the terms $\pm x_1^{r_1} x_2^{r_2} \cdots x_\ell^{r_\ell}$ with $|r_1|, |r_2|, \ldots, |r_\ell|$ pairwise distinct remain. Put $a_i = x_i + x_i^{-1}$. Then

$$\prod_{1\le i<j\le \ell} (x_i + x_i^{-1} - x_j - x_j^{-1}) = \prod_{1\le i<j\le \ell} (a_i - a_j) = \sum_{\tau \in S_\ell} \operatorname{sgn}(\tau) a_{\tau(1)}^{\ell-1} a_{\tau(2)}^{\ell-2} \cdots a_{\tau(\ell-1)}.$$

Thus, (10) becomes

$$\mathrm{LHS}(1) = \sum_{\tau\in\mathfrak{S}_\ell} \mathrm{sgn}(\tau)\,(x_{\tau(1)}+x_{\tau(1)}^{-1})^{\ell-1}(x_{\tau(2)}+x_{\tau(2)}^{-1})^{\ell-2}\cdots(x_{\tau(\ell-1)}+x_{\tau(\ell-1)}^{-1})\prod_{k=1}^{\ell}(x_k^{1/2}-x_k^{-1/2})$$

$$= \sum_{\tau\in\mathfrak{S}_\ell} \alpha\,\mathrm{sgn}(\tau)\,(x_{\tau(1)})^{e_1}(x_{\tau(2)})^{e_2}\cdots(x_{\tau(\ell-1)})^{e_{\ell-1}}\prod_{k=1}^{\ell}(x_k^{1/2}-x_k^{-1/2}) \qquad (20)$$

(where α's are numbers depending on each summand)

$$= \sum \alpha\,\mathrm{sgn}(\tau)\,(x_{\tau(1)})^{e_1\pm\frac{1}{2}}(x_{\tau(2)})^{e_2\pm\frac{1}{2}}\cdots x_{\tau(\ell-1)}^{e_{\ell-1}\pm\frac{1}{2}}\,x_{\tau(\ell)}^{\pm\frac{1}{2}}. \qquad (21)$$

Note that $e_1, e_2, \ldots, e_{\ell-1}$ are integers satisfying $|e_k|\le \ell-k$. As we showed earlier, in the reduced form of (21), the values $|e_1\pm\frac{1}{2}|, |e_2\pm\frac{1}{2}|, \ldots, |e_{\ell-1}\pm\frac{1}{2}|, |\pm\frac{1}{2}|$ are pairwise distinct. Suppose by contradiction that there is $1\le k\le \ell-1$ such that $|e_k|\le \ell-k-1$. Then $|e_k|, |e_{k+1}|, \ldots, |e_{\ell-1}|\le \ell-k-1$. Then

$$|e_k\pm\tfrac{1}{2}|, |e_{k+1}\pm\tfrac{1}{2}|, \ldots, |e_{\ell-1}\pm\tfrac{1}{2}|, |\pm\tfrac{1}{2}| \le \ell-k-\tfrac{1}{2}.$$

Thus,

$$\underbrace{|e_k\pm\tfrac{1}{2}|, |e_{k+1}\pm\tfrac{1}{2}|, \ldots, |e_{\ell-1}\pm\tfrac{1}{2}|, |\pm\tfrac{1}{2}|}_{\ell-k \text{ distinct numbers}} \in \underbrace{\left\{\tfrac{1}{2}, \tfrac{3}{2}, \ldots, \ell-k-\tfrac{1}{2}\right\}}_{\ell-k-1 \text{ elements}}.$$

This is a contradiction. Thus, $|e_k| = \ell-k$ for all $1\le k\le \ell-1$. Thus, $e_k = \pm(\ell-k)$.

Consequently, the factors α in (20) and (21) are ±1. We have

$$|e_k\pm\tfrac{1}{2}| \le |e_k| + \tfrac{1}{2} = \ell-k+\tfrac{1}{2}.$$

Thus, $|e_1\pm\frac{1}{2}|, |e_2\pm\frac{1}{2}|, \ldots, |e_{\ell-1}\pm\frac{1}{2}|, |\pm\frac{1}{2}| \in \{\frac{1}{2}, \frac{3}{2}, \ldots, \ell-\frac{1}{2}\}$. Hence

$$|e_k\pm\tfrac{1}{2}| = \ell-k+\tfrac{1}{2} \quad \forall\, 1\le k\le \ell-1.$$

Hence, if we chose $e_k = l-k$ in (21) for some k then we must have

$$e_k \pm \frac{1}{2} = e_k + \frac{1}{2} = l-k+\frac{1}{2},$$

i.e. the factor $x_k^{1/2}$ in (20) is chosen to multiply with $(x_{\tau(1)})^{e_1}(x_{\tau(2)})^{e_2}\cdots(x_{\tau(l-1)})^{e_{l-1}}$.

If $e_k = -(l-k)$ for some k then we must have

$$e_k \pm \frac{1}{2} = e_k - \frac{1}{2} = -\left(l-k+\frac{1}{2}\right),$$

i.e. the factor $(-x_k^{-1/2})$ in (20) is chosen to multiply with $(x_{\tau(1)})^{e_1}(x_{\tau(2)})^{e_2}\cdots(x_{\tau(l-1)})^{e_{l-1}}$.

Therefore,

$$\alpha = \begin{cases} -1 & \text{if the number of "$-$" chosen for } \pm\left(l-k+\frac{1}{2}\right),\ 1\le k\le l, \text{ is odd.} \\ +1 & \text{otherwise.} \end{cases}$$

Then (21) can be written as

$$LHS(1) = \sum_{\tau\in\mathfrak{S}_l} \operatorname{sgn}(\tau)\,(-1)^{\#\text{ of "}-\text{"}} (x_{\tau(1)})^{\pm(l-\frac{1}{2})}(x_{\tau(2)})^{\pm(l-\frac{3}{2})}\cdots(x_{\tau(l)})^{\pm\frac{1}{2}}.$$

Comparing with (13), we get the identity (1).

<u>Type C_l: $G = Sp(l,\mathbb{C})$</u>

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \frac{1}{2}\sum_{1\le i<j\le l}\left[(\varepsilon_i-\varepsilon_j)+(\varepsilon_i+\varepsilon_j)\right] + \frac{1}{2}\sum_{i=1}^{l} 2\varepsilon_i = \sum_{1\le i<j\le l}\varepsilon_i + \sum_{i=1}^{l}\varepsilon_i$$

$$= l\varepsilon_1 + (l-1)\varepsilon_2 + \cdots + \varepsilon_l. \qquad (22)$$

Thus, $e^\rho = x_1^l x_2^{l-1}\cdots x_l$. $\qquad$ (23)

For $\alpha = \varepsilon_i - \varepsilon_j$, $e^{-\alpha} = e^{\varepsilon_j-\varepsilon_i} = x_i^{-1}x_j$.

For $\alpha = \varepsilon_i + \varepsilon_j$, $e^{-\alpha} = e^{-\varepsilon_i-\varepsilon_j} = x_i^{-1}x_j^{-1}$.

For $\alpha = 2\varepsilon_k$, $e^{-\alpha} = e^{-2\varepsilon_k} = x_k^{-2}$.

Thus, $\mathrm{LHS}(1) = x_1^{\ell} x_2^{\ell-1} \cdots x_\ell \prod_{1\le i<j\le \ell} (1-x_i^{-1}x_j)(1-x_i^{-1}x_j^{-1}) \prod_{k=1}^{\ell} (1-x_k^{-2})$

$$= \prod_{1\le i<j\le \ell} x_i(1-x_i^{-1}x_j)(1-x_i^{-1}x_j^{-1}) \prod_{k=1}^{\ell} x_k(1-x_k^{-2})$$

$$= \prod_{1\le i<j\le \ell} (x_i+x_i^{-1}-x_j-x_j^{-1}) \prod_{k=1}^{\ell} (x_k-x_k^{-1}). \qquad (24)$$

By Lemma 3.1.2, page 331, Goodman-Wallach, the Weyl group W acts on the coordinate functions in $\mathfrak{X}(H)$ by $x_i \mapsto (x_{\sigma(i)})^{\pm 1}$ $(1\le i\le \ell)$ for every $\sigma \in \mathfrak{S}_\ell$ and choice ± 1 of exponents. Then

$$s.e^{\rho} \overset{(23)}{=\!=} s.(x_1^{\ell} x_2^{\ell-1}\cdots x_\ell) = (s.x_1)^{\ell}(s.x_2)^{\ell-1}\cdots(s.x_\ell)$$

$$= (x_{\sigma(1)})^{\pm\ell}(x_{\sigma(2)})^{\pm(\ell-1)}\cdots\cdots(x_{\sigma(\ell)})^{\pm 1}. \qquad (25)$$

From now, the arguments will follow very closely to those in Type B_ℓ. When viewed as an element in $GL(\mathfrak{h}^*)$, ~~we~~ s satisfies $s(\varepsilon_i) = \pm\varepsilon_{\sigma(i)}$ (the choice of ± 1 is according to that of $(x_{\sigma(i)})^{\pm 1}$). By the definition of $\mathrm{sgn}(s)$ on top of page 331,

$$\mathrm{sgn}(s) = \det\left((\varepsilon_1,\ldots,\varepsilon_\ell) \mapsto (\pm\varepsilon_{\sigma(1)},\ldots,\pm\varepsilon_{\sigma(\ell)})\right) = \mathrm{sgn}(\sigma)(-1)^{\#\text{ of minus signs}} \qquad (26).$$

By (25) and (26), we have

$$\mathrm{RHS}(1) = \sum_{\substack{\sigma\in\mathfrak{S}_\ell \\ \text{choices of } \pm 1}} \mathrm{sgn}(\sigma)(-1)^{\#\text{ of "}-\text{"}}(x_{\sigma(1)})^{\pm\ell}(x_{\sigma(2)})^{\pm(\ell-1)}\cdots(x_{\sigma(\ell)})^{\pm 1}. \qquad (27)$$

When (24) is expanded, it will be a sum of $4^{\binom{\ell}{2}} 2^{\ell} = 2^{\ell^2}$ terms, each of which is of the form $\pm x_1^{r_1}x_2^{r_2}\cdots x_\ell^{r_\ell}$, where each r_i is an integer. ~~add~~ We'll

Show that after the sum of these 2^{ℓ^2} terms is reduced, only the terms with $|r_1|, |r_2|, \ldots, |r_\ell|$ pairwise distinct remain. The proof is exactly the same as in Type B_ℓ, so we will not rewrite it here. Among these remaining terms, we'll show that $|r_k| \geq 1$ for all $1 \leq k \leq \ell$. Suppose by contradiction that there exists some $1 \leq k \leq \ell$ such that $r_k = 0$. We can assume $k=1$. Write

$$\prod_{1\leq i<j\leq \ell}(x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{\ell}(x_k - x_k^{-1}) = A + B, \qquad (28)$$

where A is independent of x_1, and B is a sum of terms $\pm x_1^{r_1} x_2^{r_2} \ldots x_\ell^{r_\ell}$ with $r_1 \neq 0$.

Switching x_1 and x_1^{-1}, we get

$$-\prod_{1\leq i<j\leq \ell}(x_i + x_i^{-1} - x_j - x_j^{-1}) \prod_{k=1}^{\ell}(x_k - x_k^{-1}) = A + B', \qquad (29)$$

where B' is a sum of terms $\pm x_1^{r_1} x_2^{r_2} \ldots x_\ell^{r_\ell}$ with $r_1 \neq 0$ (but the choices of ± 1 may be different from those in B). Adding (28) and (29), we get

$$A = -\frac{B + B'}{2}. \qquad (30)$$

LHS(30) is indepedent of x_1 while RHS(30) is not unless it is identically zero. thus $A \equiv 0$. This is a contradiction.

We have showed that after (24) is expanded and reduced, only the terms $\pm x_1^{r_1} x_2^{r_2} \ldots x_\ell^{r_\ell}$ with $|r_1|, |r_2|, \ldots, |r_\ell| \geq 1$ and pairwise distinct remain. By the virtue of (20) and (21) we have:

$$LHS(1) = \sum_{\tau \in \mathfrak{S}_l} \alpha \, \mathrm{sgn}(\tau) (x_{\tau(1)})^{e_1} (x_{\tau(2)})^{e_2} \cdots (x_{\tau(l-1)})^{e_{l-1}} \prod_{k=1}^{l} (x_k - x_k^{-1}) \qquad (31)$$

$$= \sum \alpha \, \mathrm{sgn}(\tau) (x_{\tau(1)})^{e_1 \pm 1} (x_{\tau(2)})^{e_2 \pm 1} \cdots x_{\tau(l-1)}^{e_{l-1} \pm 1} x_{\tau(l)}^{\pm 1} . \qquad (32)$$

Note that $|e_k| \leq l-k$. Suppose by contradiction that there is $1 \leq k \leq l-1$ such that $|e_k| \leq l-k-1$. Then $|e_k|, |e_{k+1}|, \dots, |e_{l-1}| \leq l-k-1$. Then

$$\underbrace{|e_k \pm 1|, |e_{k+1} \pm 1|, \dots, |e_{l-1} \pm 1|, |\pm 1|}_{l-k \text{ integers in } \{1,2,\dots\}} \leq l-k-1 .$$

This is a contradiction. Thus, $|e_k| = l-k$ for all $1 \leq k \leq l-1$. Thus, $e_k = \pm(l-k)$.

Consequently, the factors α in (31) and (32) are ± 1. We have

$$|e_k \pm 1| \leq |e_k| + 1 = l-k+1$$

Thus, $|e_1 \pm 1|, |e_2 \pm 1|, \dots, |e_{l-1} \pm 1|, |\pm 1| \in \{1, 2, \dots, l\}$. Hence, $|e_k \pm 1| = l-k+1$.

Hence, if we chose $e_k = l-k$ in (31) for some k then we must have

$$e_k \pm 1 = l-k+1,$$

i.e. the factor x_k in (31) is chosen to multiply with $(x_{\tau(1)})^{e_1}(x_{\tau(2)})^{e_2} \cdots (x_{\tau(l-1)})^{e_{l-1}}$.

If $e_k = -(l-k)$ for some k then we must have

$$e_k \pm 1 = -(l-k) - 1,$$

i.e. the factor $(-x_k^{-1})$ in (31) is chosen to multiply with $(x_{\tau(1)})^{e_1}(x_{\tau(2)})^{e_2} \cdots (x_{\tau(l-1)})^{e_{l-1}}$.

Therefore,

$$\alpha = \begin{cases} -1 & \text{if the number of "$-$" chosen for } \pm(l-k+1), 1 \leq k \leq l, \text{ is odd}, \\ +1 & \text{otherwise.} \end{cases}$$

Then (32) can be written as

$$\mathrm{LHS}(1) = \sum_{\substack{\tau\in\mathfrak{S}_\ell \\ \text{Choices of } \pm 1}} \operatorname{sgn}(\tau)(-1)^{\#\text{ of } "-"} (x_{\tau(1)})^{\pm\ell}(x_{\tau(2)})^{\pm(\ell-1)}\cdots(x_{\tau(\ell)})^{\pm 1}.$$

Comparing with (27), we get the identity (1).

<u>Type D_ℓ: $G = SO(2\ell,\mathbb{C})$, $\ell\geq 3$</u>

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \frac{1}{2}\sum_{1\leq i<j\leq\ell}[(\varepsilon_i-\varepsilon_j)+(\varepsilon_i+\varepsilon_j)] = \sum_{1\leq i<j\leq\ell}\varepsilon_i = (\ell-1)\varepsilon_1+(\ell-2)\varepsilon_2+\dots+\varepsilon_{\ell-1}.$$

Thus, $e^\rho = x_1^{\ell-1}x_2^{\ell-2}\cdots x_{\ell-1}$. (33)

For $\alpha=\varepsilon_i-\varepsilon_j$, $e^{-\alpha} = e^{\varepsilon_j-\varepsilon_i} = x_i^{-1}x_j$.

For $\alpha=\varepsilon_i+\varepsilon_j$, $e^{-\alpha} = e^{-\varepsilon_i-\varepsilon_j} = x_i^{-1}x_j^{-1}$.

$$\text{Thus, } \mathrm{LHS}(1) = x_1^{\ell-1}x_2^{\ell-2}\cdots x_{\ell-1}\prod_{1\leq i<j\leq\ell}(1-x_i^{-1}x_j)(1-x_i^{-1}x_j^{-1})$$

$$= \prod_{1\leq i<j\leq\ell} x_i(1-x_i^{-1}x_j)(1-x_i^{-1}x_j^{-1})$$

$$= \prod_{1\leq i<j\leq\ell}(x_i+x_i^{-1}-x_j-x_j^{-1}). \qquad (34)$$

By Lemma 3.1.4, page 132, Goodman-Wallach, the Weyl group W acts on the coordinate functions in $\mathfrak{X}(H)$ by $x_i\mapsto(x_{\sigma(i)})^{\pm 1}$ $(1\leq i\leq\ell)$ for every $\sigma\in\mathfrak{S}_\ell$ and choice ± 1 of exponent with an even number of negative exponents. Then

$$s.e^\rho \overset{(33)}{=\!=} s.(x_1^{\ell-1}x_2^{\ell-2}\cdots x_{\ell-1}) = (s.x_1)^{\ell-1}(s.x_2)^{\ell-2}\cdots(s.x_{\ell-1})$$

$$= (x_{\sigma(1)})^{\pm(\ell-1)}(x_{\sigma(2)})^{\pm(\ell-2)}\cdots(x_{\sigma(\ell-1)})^{\pm 1}. \qquad (35)$$

When viewed as an element in $\mathfrak{h}^*$, s satisfies $s(\varepsilon_i)=\pm\varepsilon_{\sigma(i)}$ (the choice of sign is according to that of $(x_{\sigma(i)})^{\pm 1}$). By the definition of $\operatorname{sgn}(s)$ on the top of page 331,

$$sgn(s) = det\left((\varepsilon_1,\dots,\varepsilon_\ell) \mapsto (\pm\varepsilon_{\sigma(1)},\dots,\pm\varepsilon_{\sigma(\ell)})\right) = sgn(\sigma)(-1)^{\#\text{ of } \text{"}-\text{"}} = sgn(\sigma). \quad (36)$$

By (35) and (36), we have

$$RHS(1) = \sum_{\sigma\in\mathfrak{S}_\ell} sgn(\sigma)(x_{\sigma(1)})^{\pm(\ell-1)}(x_{\sigma(2)})^{\pm(\ell-2)}\cdots(x_{\sigma(\ell-1)})^{\pm 1}. \quad (37)$$

When the product at (34) is expanded, (34) can be written as

$$LHS(1) = \prod_{1\le i<j\le\ell}(x_i + x_i^{-1} - x_j - x_j^{-1})$$

$$= \prod_{\tau\in\mathfrak{S}_\ell}(x_{\tau(1)} + x_{\tau(1)}^{-1})^{\ell-1}(x_{\tau(2)} + x_{\tau(2)}^{-1})^{\ell-2}\cdots(x_{\tau(\ell-1)} + x_{\tau(\ell-1)}^{-1}). \quad (38)$$

$$= \sum \alpha (x_{\tau(1)})^{e_1}(x_{\tau(2)})^{e_2}\cdots(x_{\tau(\ell-1)})^{e_{\ell-1}}, \quad (39)$$

where α is a number depending on each summand and $e_k\in\mathbb{Z}$, $|e_k|\le\ell-k$ for all $1\le k\le\ell-1$. We'll show that in the reduced form, the sum at (39) contains only terms $(x_{\tau(1)})^{e_1}(x_{\tau(2)})^{e_2}\cdots(x_{\tau(\ell-1)})^{e_{\ell-1}}$ with $|e_1|,\dots,|e_{\ell-1}|$ pairwise distinct. The proof is exactly the same as in Type B_ℓ, so we will not rewrite it here. Suppose by contradiction that among those ~~m~~ remaining terms, there are terms with $e_k = 0$ for some $1\le k\le\ell-1$. Such a term is independent of at least two of the factors $x_1, x_2, \dots, x_\ell$. Write

$$\prod_{1\le i<j\le\ell}(x_i + x_i^{-1} - x_j - x_j^{-1}) = A + B, \quad (40)$$

where A is independent of x_1 and x_2, and B is a sum of terms

$\pm \alpha x_1^{r_1} x_2^{r_2} \dots x_\ell^{r_\ell}$ with $(r_1, r_2) \neq (0,0)$. Switching x_1 and x_2, we get

$$-\prod_{1 \le i < j \le \ell} (x_i + x_i^{-1} - x_j - x_j^{-1}) = A + B', \qquad (41)$$

where B' is another sum of terms $\pm \alpha x_1^{r_1} x_2^{r_2} \dots x_\ell^{r_\ell}$ with $(r_1, r_2) \neq (0,0)$. Adding (40) and (41) together, we get
$$A = -\frac{B + B'}{2}. \qquad (42)$$

LHS(42) is independent of x_1 and x_2, while RHS(42) is not unless it is identically zero. Thus, $A \equiv 0$. ~~The~~ We have showed that after (39) is reduced, only the terms with $|e_1|, |e_2|, \dots, |e_{\ell-1}|$ pairwise distinct and $|e_k| \geq 1$ for all $1 \le k \le \ell - 1$ remain. By the same argument as in Type C_ℓ, we get $|e_k| = \ell - k$ for all $1 \le k \le \ell - 1$. Thus, the coefficients α's in (39) are equal to 1. Thus, (39) can be written as

$$LHS(1) = \sum_{\substack{\tau \in \mathfrak{S}_\ell \\ \text{choices of signs}}} (x_{\tau(1)})^{\pm(\ell-1)} (x_{\tau(2)})^{\pm(\ell-2)} \cdots (x_{\tau(\ell-1)})^{\pm 1}.$$

Comparing with (37), we get the identity (1).

(6) Problem 1, Section 9.1.4, Goodman-Wallach, p. 396.

We'll show that $I^{[k-2,2]} \simeq G^{[k-2,2]} \oplus G^{[k-1,1]} \oplus G^{[k]}$. By Proposition 9.1.3, so called the reciprocity law for multiplicities of the irreducible representation of $\mathfrak{S}_n$ in I^μ, we have

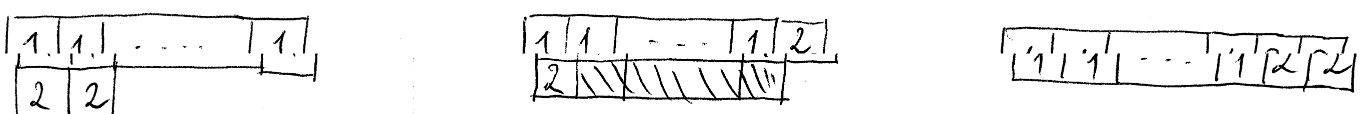
$$I^{[k-2,2]} \simeq \bigoplus_{\lambda \in Par(k,n)} \dim F_n^\lambda([k-2,2]) G^\lambda.$$

We need to show that

$$\dim F_n^\lambda([k-2,2]) = \begin{cases} 1 & \text{if } \lambda \in \{[k],[k-1,1],[k-2,2]\} \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 8.1.5, so called the Gelfand–Cetlin Basis, $\dim F_n^\lambda([k-2,2])$ is equal to the number of n-fold branching patterns of shape λ and weight $[k-2,2]$. Write such an n-fold branching pattern as $\gamma = \{\mu^{(n)}, \mu^{(n-1)}, \dots, \mu^{(1)}\}$. Because γ has weight $[k-2,2]$, its Young diagram consists of k boxes, $k-2$ of which are filled with 1 and two are filled with 2. By the rule of writing a Young diagram for an n-fold branching pattern in page 366, Goodman-Wallach, there are only 3 possibilities for γ:

$\gamma = \{[k-2,2],[k-2]\}$ $\qquad$ $\gamma = \{[k-1,1],[k-2]\}$ $\qquad$ $\gamma = \{[k],[k-2]\}$



(a) $\qquad$ (b) $\qquad$ (c)

Recall that λ is the shape of γ. For case (a), $\lambda = [k-2,2]$. For case (b), $\lambda = [k-1,1]$. For case (c), $\lambda = [k]$. Therefore,

$$\dim F_n^\lambda([k-2,2]) = \begin{cases} 1 & \text{if } \lambda \in \{[k-2,2],[k-1,1],[k]\}, \\ 0 & \text{otherwise.} \end{cases}$$

(7) Problem 1, Section 11.1.5, p.490, Goodman-Wallach.

Put $N = \left\{ u(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$ and $T = \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}$.

We will show that Γ is Zarski dense in N.

By Lemma A.1.7, p.615, Goodman-Wallach, the Zarski topology on N consists of ~~finitely many~~ finite unions of principal open sets

$$N^f = \{u(z) \in N : f(u(z)) \neq 0\} \quad \forall f \in \mathcal{O}[N] \setminus \{0\}.$$

For each $f \in \mathcal{O}[GL(2,\mathbb{C})]$, $f\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right)$ is a polynomial of variables $x_{11}, x_{12}, x_{21}, x_{22}, (x_{11}x_{22} - x_{12}x_{21})^{-1}$. Thus, $f(u(z))$ is simply a polynomial of z. We can, therefore, identify $\mathcal{O}[N]$ with $\mathbb{C}[z]$. Then

$$N^f = \{u(z) \in N : f(z) \neq 0\} \quad \forall f \in \mathbb{C}[z].$$

Take $u(a) \in N$ and let V be a Zarski open nbhd of $u(a)$ in N. We'll show that $V \cap \Gamma \neq \phi$. Because V is a finite ~~sum~~ union of principle open sets, we can assume that V is a principle open set, i.e. $V = N^f$ for some $f \in \mathbb{C}[z]$. Because $f(a) \neq 0$, f is not the zero polynomial. Thus, f has finitely many roots in $\mathbb{C}$. Thus, there exists $k \in \mathbb{Z}$ such that $f(k) \neq 0$. Then $u(k) \in N^f = V$. Hence, $u(k) \in V \cap \Gamma$.

9 Problem 1, Section 12.1.4, Goodman-Wallach, page 550.

Let G be a reductive group acting linearly on a vector space V. In other words, V is a representation of the group G. This representation induces a representation of G on $\mathcal{P}(V)$, the space of polynomial functions on V, via

$$(g.f)(v) := f(g^{-1}v) \quad \forall g \in G, f \in \mathcal{P}(V), v \in V.$$

Let $\mathbb{C}^\times$ act linearly on V by scalar multiplication. This action induces a representation of $\mathbb{C}^\times$ on $\mathcal{P}(V)$ via

$$(\lambda.f)(v) := f(\lambda^{-1}v) \quad \forall \lambda \in \mathbb{C}^\times, f \in \mathcal{P}(V), v \in V.$$

The actions of G and of $\mathbb{C}^\times$ on $\mathcal{P}(V)$ commute because

$$(\lambda.(g.f))(v) = (g.f)(\lambda^{-1}v) = f(g^{-1}\lambda^{-1}v) = f(\lambda^{-1}g^{-1}v) = (\lambda.f)(g^{-1}v) = (g.(\lambda.f))(v)$$
$$\forall \lambda \in \mathbb{C}^\times, g \in G, f \in \mathcal{P}(V).$$

Then we can define an action of $G \times \mathbb{C}^\times$ on $\mathcal{P}(V)$ via

$$((g,\lambda).f)(v) := (g.(\lambda.f))(v) = f(\lambda^{-1}g^{-1}v) \quad \forall \lambda \in \mathbb{C}^\times, g \in G, f \in \mathcal{P}(V)$$

For each integer $k \geq 0$, denote by $\mathcal{P}^k(V)$ the space of homogeneous polynomial functions of degree k. We have a decomposition of vector spaces

$$\mathcal{P}(V) = \bigoplus_{k \geq 0} \mathcal{P}^k(V).$$

Denote by $\widehat{G}$ the set of all equivalent classes of finite-dimensional irreducible G-modules. The primary decomposition of $\mathcal{P}^k(V)$ as a G-module is

$$\mathcal{P}^k(V) = \bigoplus_{\omega \in \widehat{G}} \mathcal{P}^k(V)_{(\omega)},$$

where

$$\mathcal{P}^k(V)_{(\omega)} = \sum_{\substack{U \subset \mathcal{P}^k(V) \\ [U] = \omega}} U. \qquad (1)$$

Hence, we get a decomposition of vector spaces

$$\mathcal{P}(V) = \bigoplus_{k \geq 0} \bigoplus_{\omega \in \widehat{G}} \mathcal{P}^k(V)_{(\omega)}. \qquad (2)$$

We want to show that (2), after eliminating trivial summands on the right hand side, is the primary decomposition of $\mathcal{P}(V)$ as a $G\times\mathbb{C}^\times$-module. First, we show that each $\mathcal{P}^k(V)_{(\omega)}$ is a $G\times\mathbb{C}^\times$-module. For each $(g,\lambda)\in G\times\mathbb{C}^\times$ and $f\in\mathcal{P}^k(V)_{(\omega)}$ we have

$$(g,\lambda).f = (v\mapsto f(\lambda^{-1}g^{-1}v)) = (v\mapsto \lambda^{-k}f(g^{-1}v)) = \lambda^{-k}(v\mapsto f(g^{-1}v)) = \lambda^{-k}(g.f).$$

Since $\mathcal{P}^k(V)_{(\omega)}$ is a G-module, $g.f\in\mathcal{P}^k(V)_{(\omega)}$ and thus $\lambda^{-k}(g.f)\in\mathcal{P}^k(V)_{(\omega)}$. Hence, $(g,\lambda).f\in\mathcal{P}^k(V)_{(\omega)}$. We have showed that $\mathcal{P}^k(V)_{(\omega)}$ is a $G\times\mathbb{C}^\times$-module. Thus, (2) is a decomposition of $G\times\mathbb{C}^\times$-modules.

Denote by $\widehat{G\times\mathbb{C}^\times}$ the set of all equivalent classes of finite dimensional irreducible $G\times\mathbb{C}^\times$-modules. Then we have the primary decomposition

$$\mathcal{P}(V) = \bigoplus_{\sigma\in\widehat{G\times\mathbb{C}^\times}} \mathcal{P}(V)_{(\sigma)}. \qquad (3)$$

For each $\sigma\in\widehat{G\times\mathbb{C}^\times}$, we show that there exist $k_\sigma\geq 0$ and $\omega_\sigma\in\widehat{G}$ such that

$$\mathcal{P}(V)_{(\sigma)}\subset \mathcal{P}^{k_\sigma}(V)_{(\omega_\sigma)}. \qquad (4)$$

If this can be done then

$$\bigoplus_{k\geq 0}\bigoplus_{\omega\in\widehat{G}}\mathcal{P}^k(V)_{(\omega)} \overset{(2)}{=\!=} \mathcal{P}(V) \overset{(3)}{=\!=} \bigoplus_{\sigma\in\widehat{G\times\mathbb{C}^\times}}\mathcal{P}(V)_{(\sigma)} \overset{(4)}{\subset} \overset{(*)}{\sum_{\sigma\in\widehat{G\times\mathbb{C}^\times}}} \mathcal{P}^{k_\sigma}(V)_{(\omega_\sigma)};$$

then the equality must hold; thus (1) is the primary decomposition of $\mathcal{P}(V)$ as a $G\times\mathbb{C}^\times$-module.

By the definition of isotypic components, (4) is equivalent to

$$\sum_{\substack{W \subset \mathcal{P}(V) \\ W \in \sigma}} W \subset \mathcal{P}^{k_\sigma}(V)_{(\omega_\sigma)}. \qquad (5)$$

For each $W \subset \mathcal{P}(V)$, $W \in \sigma$, we have

$$W = W \cap \mathcal{P}(V) \overset{(2)}{=\!=} \bigoplus_{k \geq 0} \bigoplus_{\omega \in \widehat{G}} \underbrace{\left(W \cap \mathcal{P}^k(V)_{(\omega)}\right)}_{G \times \mathbb{C}^\times\text{-module.}}$$

Since W is an irreducible $G \times \mathbb{C}^\times$-module, there exist an integer $k_{\sigma,W} \geq 0$ and $\omega_{\sigma,W} \in \widehat{G}$ such that $W = W \cap \mathcal{P}^{k_{\sigma,W}}(V)_{(\omega_{\sigma,W})}$. Thus, $W \subset \mathcal{P}^{k_{\sigma,W}}(V)_{(\omega_{\sigma,W})}$.

For $W_1, W_2 \in \sigma$ and $W_1 \subset \mathcal{P}^{i_1}(V)_{(\omega_1)}$, $W_2 \subset \mathcal{P}^{i_2}(V)_{(\omega_2)}$, we show that $i_1 = i_2$ and $\omega_1 = \omega_2$.

Because $W_1 \simeq W_2$ as $G \times \mathbb{C}^\times$-modules, there exists a linear isomorphism $\varphi : W_1 \to W_2$ which intertwines with the actions of $G \times \mathbb{C}^\times$ on W_1 and W_2. For each $\lambda \in \mathbb{C}^\times$ and $f \in W_1$, we have $\varphi(\lambda . f) = \lambda . \varphi(f)$. On the other hand,

$$\lambda . f = (v \mapsto f(\lambda^{-1} v)) = (v \mapsto \lambda^{-i_1} f(v)) = \lambda^{-i_1} f,$$

$$\varphi(\lambda . f) = \varphi(\lambda^{-i_1} f) = \lambda^{-i_1} \varphi(f), \qquad (6)$$

$$\lambda . \varphi(f) = (v \mapsto \varphi(f)(\lambda^{-1} v)) = (v \mapsto \lambda^{-i_2} \varphi(f)(v)) = \lambda^{-i_2} \varphi(f). \qquad (7)$$

By (6) and (7), $\lambda^{-i_1} \varphi(f) = \lambda^{-i_2} \varphi(f)$ for all $\lambda \in \mathbb{C}^\times$ and $f \in W_1$. Thus, $i_1 = i_2$ $(= i)$. Hence, $W_1, W_2 \subset \mathcal{P}^i(V)$.

By identifying G with $G \times \{1\} \subset G \times \mathbb{C}^\times$, we have $W_1 \simeq W_2$ as

G-modules via the intertwining map φ. Now suppose by contradiction that W_1 is a reducible G-module. Write

$$W_1 = U_1 \oplus U_2, \qquad (8)$$

where $U_1, U_2 \neq \{0\}$ are G-modules. For $\lambda \in \mathbb{C}^\times$, $g \in G$, $f \in U_1$,

$$(g,\lambda)\cdot f = (v \mapsto f(\lambda^{-1} g^{-1} v)) = (v \mapsto \lambda^{-i} f(g^{-1}v)) = \lambda^{-i}(v \mapsto f(g^{-1}v)) = \lambda^{-i}(g\cdot f).$$

Since U_1 is a G-module, $g\cdot f \in U_1$ and thus $\lambda^{-i}(g\cdot f) \in U_1$. Hence, $(g,\lambda)\cdot f \in U_1$. Thus, U_1 is a $G\times\mathbb{C}^\times$-module. Similarly, U_2 is also a $G\times\mathbb{C}^\times$-module. Then (8) contradicts the fact that W_1 is an irreducible $G\times\mathbb{C}^\times$-module.

Therefore, W_1 is an irreducible G-module. Similarly, W_2 is also an irreducible G-module. ~~Moreover, W_1 and~~ So far, we know that W_1 and W_2 are isomorphic irreducible G-submodules of $\mathcal{P}^i(V)$. Moreover, W_1 and W_2 are finite-dimensional. Thus, they are contained in the same isotypic component of $\mathcal{P}^i(V)$ as a G-module. This means $\omega_1 = \omega_2$. Therefore,

$$\begin{cases} k_{\sigma,W} = k_\sigma \\ \omega_{\sigma,W} = \omega_\sigma \end{cases} \quad \forall\, W \subset \mathcal{P}(V),\ W \in \sigma.$$

We obtain (5).