Verify Some Points in Chapter 3, Lieb-Seiringer

1 Verify equation (3.1.22)

By (3.1.21), $\Gamma = \sum_{j=1}^{\infty} \lambda_j T_{\varphi_j}$ and then

$$\Gamma\psi = \sum_{j=1}^{\infty} \lambda_j T_{\varphi_j}(\psi) = \sum_{j=1}^{\infty} \lambda_j \langle \psi, \varphi_j \rangle \varphi_j$$

$$(\Gamma\psi)(\underline{z}) = \sum_{j=1}^{\infty} \lambda_j \langle \psi, \varphi_j \rangle \varphi_j(\underline{z}) = \sum_{j=1}^{\infty} \lambda_j \int \psi(\underline{z}') \overline{\varphi}_j(\underline{z}') d\underline{z}' \varphi_j(\underline{z})$$

$$= \sum_{j=1}^{\infty} \int \lambda_j \psi(\underline{z}') \overline{\varphi}_j(\underline{z}') \varphi_j(\underline{z}) d\underline{z}' \qquad (1)$$

We have to show that the infinite sum can be placed inside the integration. Put

$$T_n(\underline{z}') = \sum_{j=1}^{n} \lambda_j \psi(\underline{z}') \overline{\varphi}_j(\underline{z}') \varphi_j(\underline{z}) \cancel{dz}$$

~~we'll s~~ By Dominated Convergence Theorem, we show that $|T_n(\underline{z}')|$ is bounded by an integrable function. Since $\psi \in L^2$, by Holder

$$\cancel{|T_n(\underline{z}')| \leq \sum_{j=1}^{\infty} \lambda_j |\psi(\underline{z}')| |\overline{\varphi}_j(\underline{z}')| |\varphi_j(\underline{z})| = G(\underline{z}')}$$

Inequality, we only have to show that $|\tilde{T}_n(\underline{z}')|$ ~~is~~ is bounded by a function in L^2, where

$$\tilde{T}_n(\underline{z}') = \sum_{j=1}^{n} \lambda_j \overline{\varphi}_j(\underline{z}') \varphi_j(\underline{z})$$

$$|\tilde{T}_n(\underline{z}')| \leq \sum_{j=1}^{\infty} \lambda_j |\overline{\varphi_j}(\underline{z}')| |\varphi_j(\underline{z})| = G(\underline{z}')$$

$$\| G \|_2 \leq \sum_{j=1}^{\infty} \lambda_j \|\overline{\varphi_j}\|_2 |\varphi_j(\underline{z})| = \sum_{j=1}^{\infty} \lambda_j |\varphi_j(\underline{z})|$$

Moreover,

$$\| \sum_{j=1}^{\infty} \lambda_j |\varphi_j| \|_2 \leq \sum_{j=1}^{\infty} \lambda_j \|\varphi_j\|_2 = \sum_{j=1}^{\infty} \lambda_j < \infty$$

Thus, the set $\{ \underline{z} \in \mathbb{R}^{3N} : \sum_{j=1}^{\infty} \lambda_j |\varphi_j(\underline{z})| = \infty \}$ is of measure zero.

thus, $\|G\|_2 < \infty$ for almost every $\underline{z}$. That means we can put the infinite sum inside the integration in (1) for a.e $\underline{z}$.

Then $$(\Gamma\psi)(\underline{z}) = \int \Gamma(\underline{z}, \underline{z}') \psi(\underline{z}') d\underline{z}' \quad \text{for a.e } \underline{z}$$

and

$$\cancel{\Gamma(\cdot, \underline{z}') = \sum_{j=1}^{\infty} \lambda_j \varphi_j(\cdot) \overline{\varphi_j}(\underline{z}')}$$

$$\Gamma(\underline{z}, \cdot) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(\underline{z}) \overline{\varphi_j}(\cdot) \in L^2 \quad \text{for a.e } \underline{z}.$$

$\boxed{2}$ Verify equation 3.1.28

The N-particle density matrix's kernel is

$$\Gamma(z, z') = \psi(\underline{z}) \overline{\psi(\underline{z}')}$$

where ψ is the totally antisymmetric function given at (3.1.16)

$$\Psi(z_1,\dots,z_N)=\frac{1}{\sqrt{N!}}\det\{u_i(z_j)\}_{i,j=1}^N=\frac{1}{\sqrt{N!}}\sum_{\sigma\in S_N}(-1)^\sigma u_{\sigma(1)}(z_1)\dots u_{\sigma(N)}(z_N)$$

The kernal of the k-particle reduced density matrix by definition is

$$\gamma^{(k)}(z_1,\dots,z_k;z_1',\dots,z_k')=\frac{N!}{(N-k)!}\int\Gamma(z_1,\dots,z_k,z_{k+1},\dots,z_N;z_1',\dots,z_k',z_{k+1},\dots,z_N)\,dz_{k+1}\dots dz_N$$

$$=\frac{N!}{(N-k)!}\int\Psi(z_1,\dots,z_k,z_{k+1},\dots,z_N)\overline{\Psi(z_1',\dots,z_k',z_{k+1},\dots,z_N)}\,dz_{k+1}\dots dz_N$$

$$=\frac{N!}{(N-k)!}\int\frac{1}{\sqrt{N!}}\sum_{\sigma\in S_N}(-1)^\sigma u_{\sigma(1)}(z_1)\dots u_{\sigma(k)}(z_k)\,u_{\sigma(k+1)}(z_{k+1})\dots u_{\sigma(N)}(z_N)\times$$

$$\times\frac{1}{\sqrt{N!}}\sum_{\pi\in S_N}(-1)^\pi\overline{u_{\pi(1)}(z_1')}\dots\overline{u_{\pi(k)}(z_k')}\,\overline{u_{\pi(k+1)}(z_{k+1})}\dots\overline{u_{\pi(N)}(z_N)}\,dz_{k+1}\dots dz_N$$

$$=\frac{1}{(N-k)!}\sum_{\sigma,\pi\in S_N}(-1)^\sigma(-1)^\pi u_{\sigma(1)}(z_1)\dots u_{\sigma(k)}(z_k)\,\overline{u_{\pi(1)}(z_1')}\dots\overline{u_{\pi(k)}(z_k')}\times$$

$$\times\int u_{\sigma(k+1)}(z_{k+1})\overline{u_{\pi(k+1)}(z_{k+1})}\,dz_{k+1}\dots\int u_{\sigma(N)}(z_N)\overline{u_{\pi(N)}(z_N)}\,dz_N$$

$$=\frac{1}{(N-k)!}\sum_{\sigma,\pi\in S_N}(-1)^\sigma(-1)^\pi u_{\sigma(1)}(z_1)\dots u_{\sigma(k)}(z_k)\,\overline{u_{\pi(1)}(z_1')}\dots\overline{u_{\pi(k)}(z_k')}\,\delta_{\sigma(k+1),\pi(k+1)}\dots\delta_{\sigma(N),\pi(N)}\tag{2}$$

Thus we only count $\sigma,\pi\in S_N$ such that $\sigma(i)=\pi(i)\ \forall i=k+1,\dots,N$.

To form such a pair (σ,π), we do the following steps:

1) Choose a k-combination τ from $\{1,\dots,N\}$

2) $(\sigma(1),\dots,\sigma(k))$ is a permutation of τ

3) $(\pi(1),\dots,\pi(k))$ is a permutation of τ

4) $(\sigma(k+1),\dots,\sigma(N)) = (\pi(k+1),\dots,\pi(N))$ is a permutation of the rest $(N-k)$ elements.

There are $\binom{N}{k} = \dfrac{N!}{(N-k)!\,k!}$ ways to do the first step, $k!$ ways to do the second, $k!$ ways to do the third, and $(N-k)!$ ways to do the last step. Thus there are

$$\frac{N!}{(N-k)!\,k!}\,k!\,k!\,(N-k)! = N!\,k!$$

ways to construct a pair (σ,π). Then

$$(2) = \frac{1}{(N-k)!}\sum_{\tau}\sum_{\substack{\sigma\in S_N\\ \{\sigma(1),\dots,\sigma(k)\}=\tau}}\ \sum_{\substack{\pi\in S_N\\ \{\pi(1),\dots,\pi(k)\}=\tau\\ \pi(i)=\sigma(i)\ \forall i>k}} (-1)^{\sigma} u_{\sigma(1)}(z_1)\dots u_{\sigma(k)}(z_k)\,\overline{u_{\pi(1)}(z'_1)}\dots\overline{u_{\pi(k)}(z'_k)} \qquad (3)$$

Each such σ corresponds to a permutation of τ, called σ', such that

$$\sigma'(i) = \sigma(i) \quad \forall i=1,\dots,k$$

Similarly, π corresponds to $\pi' \in S_\tau$ such that

$$\pi'(i) = \pi(i) \quad \forall i = 1,\dots,k$$

Moreover,

$$(-1)^\sigma = (-1)^{\sigma'} \varepsilon_{\tau,(\sigma(k+1),\dots,\sigma(N))},$$

where $\varepsilon = \pm 1$ and depends only on τ and $(\sigma(k+1),\dots,\sigma(N))$.

$$(-1)^\pi = (-1)^{\pi'} \varepsilon_{\tau,(\pi(k+1),\dots,\pi(N))}.$$

Because $\pi(i)=\sigma(i)$ $\forall i = k+1,\dots,N$, we have $(-1)^\sigma(-1)^\pi = (-1)^{\sigma'}(-1)^{\pi'}$. Thus

$$(3) = \frac{1}{(N-k)!} \sum_\tau \sum_{\sigma' \in S_\tau} \sum_{\pi' \in S_\tau} (-1)^{\sigma'}(-1)^{\pi'} u_{\sigma'(1)}(z_1)\dots u_{\sigma'(k)}(z_k)\, \overline{u_{\pi'(1)}(z_1')}\dots\overline{u_{\pi'(k)}(z_k')}\,(N-k)!$$

$$= \sum_\tau \Big(\sum_{\sigma'\in S_\tau} (-1)^{\sigma'} u_{\sigma'(1)}(z_1)\dots u_{\sigma'(k)}(z_k)\Big)\Big(\sum_{\pi'\in S_\tau}(-1)^{\pi'}\overline{u_{\pi'(1)}(z_1')\dots u_{\pi'(k)}(z_k')}\Big)$$

$$= \sum_\tau \det\{u_{\tau_i}(z_j)\}_{i,j=1}^{k}\ \overline{\det\{u_{\tau_i}(z_j')\}_{i,j=1}^{k}}$$

where $(\tau_1,\dots,\tau_k)$ is a list (in an arbitrary order) of the elements of τ.

$\boxed{3}$ Verify equation (3.1.29)

By Eq. (3.1.28), we have

$$\gamma^{(2)}(z_1,z_2;z_1',z_2') = \sum_{1\le i<j\le N} \det\begin{pmatrix} u_i(z_1) & u_i(z_2)\\ u_j(z_1) & u_j(z_2)\end{pmatrix} \det\begin{pmatrix} \overline{u_i(z_1')} & \overline{u_i(z_2')}\\ \overline{u_j(z_1')} & \overline{u_j(z_2')}\end{pmatrix}$$

Thus,

$$\gamma^{(2)}(z_1,z_2;z_1,z_2) = \sum_{i<j} \det\begin{pmatrix} u_i(z_1) & u_i(z_2)\\ u_j(z_1) & u_j(z_2)\end{pmatrix} \det\begin{pmatrix} \overline{u_i(z_1)} & \overline{u_i(z_2)}\\ \overline{u_j(z_1)} & \overline{u_j(z_2)}\end{pmatrix}$$

$$= \sum_{i<j} \big(u_i(z_1)u_j(z_2) - u_j(z_1)u_i(z_2)\big)\big(\overline{u_i(z_1)}\,\overline{u_j(z_2)} - \overline{u_j(z_1)}\,\overline{u_i(z_2)}\big)$$

(6)

$$= \sum_{1\le i<j\le N} u_i(z_1)\overline{u_i(z_1)}\,u_j(z_2)\overline{u_j(z_2)} + \sum_{1\le i<j\le N} u_j(z_1)\overline{u_j(z_1)}\,u_i(z_2)\overline{u_i(z_2)}$$

$$-\sum_{1\le i<j\le N} u_i(z_1)\overline{u_i(z_2)}\,\overline{u_j(z_1)}\,u_j(z_2) - \sum_{1\le i<j\le N} \overline{u_i(z_1)}\,u_i(z_2)\,u_j(z_1)\overline{u_j(z_2)}$$

$$= \sum_{i,j} u_i(z_1)\overline{u_i(z_1)}\,u_j(z_2)\overline{u_j(z_2)} - \sum_{i,j} u_i(z_1)\overline{u_i(z_2)}\,\overline{u_j(z_1)}\,u_j(z_2)$$

$$= \left(\sum_{i=1}^{N} u_i(z_1)\overline{u_i(z_1)}\right)\left(\sum_{j=1}^{N} u_j(z_2)\overline{u_j(z_2)}\right) - \left(\sum_i u_i(z_1)\overline{u_i(z_2)}\right)\left(\sum_j \overline{u_j(z_1)}\,u_j(z_2)\right)$$

$$= \gamma^{(1)}(z_1,z_1)\,\gamma^{(1)}(z_2,z_2) - \gamma^{(1)}(z_1,z_2)\,\gamma^{(1)}(z_2,z_1)$$

$$= \gamma^{(1)}(z_1,z_1)\,\gamma^{(1)}(z_2,z_2) - |\gamma^{(1)}(z_1,z_2)|^2.$$

$\boxed{4}$ <u>Verify that $\gamma^{(k)}$ is positive semidefinite and self-adjoint (Eq. (3.1.30))</u>

By Point 6, Number 2, report on "Compact operator on Hilbert space", it is sufficient to show that for any $\phi \in L^2(\mathbb{R}^{3k}; \mathbb{C}^{q^k})$,

$$(\phi, \gamma^{(k)}\phi) \geq 0$$

We have

$$(\phi, \gamma^{(k)}\phi) = \int \gamma^{(k)}\phi(z_1,\dots,z_k)\,\overline{\phi}(z_1,\dots,z_k)\,dz_1\dots dz_k$$

$$= \iint \gamma^{(k)}(z_1,\dots,z_k; z_1',\dots,z_k')\,\phi(z_1',\dots,z_k')\,\overline{\phi}(z_1,\dots,z_k)\,dz_1'\dots dz_k'\,dz_1\dots dz_k$$

$$= \iint \gamma^{(k)}(\underline{z}^{(k)}; \underline{z}^{(k)\prime})\,\overline{\Phi}(\underline{z}^{(k)})\,\phi(\underline{z}^{(k)\prime})\,d\underline{z}^{(k)}\,d\underline{z}^{(k)\prime}$$

$$= C \iiint \Gamma(\underline{z}^{(k)}, \underline{z}^{(N-k)}; \underline{z}^{(k)'}, \underline{z}^{(N-k)})\, \bar{\phi}(\underline{z}^{(k)})\, \phi(\underline{z}^{(k)'})\, d\underline{z}^{(N-k)} d\underline{z}^{(k)} d\underline{z}^{(k)'} \quad (4)$$

where $C = \dfrac{N!}{(N-k)!}$.

By Eq. (3.1.23),

$$(4) = C \iiint \sum_{j=1}^{\infty} \lambda_j \underbrace{\psi_j(\underline{z}^{(k)}, \underline{z}^{(N-k)})\, \bar{\psi}_j(\underline{z}^{(k)'}, \underline{z}^{(N-k)})\, \bar{\phi}(\underline{z}^{(k)})\, \phi(\underline{z}^{(k)'})}_{f_j(\underline{z}^{(k)}, \underline{z}^{(k)'}, \underline{z}^{(N-k)})} d\underline{z}^{(N-k)} d\underline{z}^{(k)} d\underline{z}^{(k)'} \quad (5)$$

We wish to take the infinite sum out of the integration. To do so, we need to show that

$$\sum_{j=1}^{\infty} \lambda_j \iiint |f_j(\underline{z}^{(k)}, \underline{z}^{(k)'}, \underline{z}^{(N-k)})|\, d\underline{z}^{(N-k)} d\underline{z}^{(k)} d\underline{z}^{(k)'} < \infty \quad (6)$$

We have

$$\iiint |f_j(\underline{z}^{(k)}, \underline{z}^{(k)'}, \underline{z}^{(N-k)})|\, d\underline{z}^{(N-k)} d\underline{z}^{(k)} d\underline{z}^{(k)'}$$

$$= \int \left| \left(\int \psi_j(\underline{z}^{(k)}, \underline{z}^{(N-k)})\, \bar{\phi}(\underline{z}^{(k)})\, d\underline{z}^{(k)} \right) \left(\int \bar{\psi}_j(\underline{z}^{(k)'}, \underline{z}^{(N-k)})\, \phi(\underline{z}^{(k)'})\, d\underline{z}^{(k)'} \right) \right| d\underline{z}^{(N-k)}$$

$$= \int \left| \int \psi_j(\underline{z}^{(k)}, \underline{z}^{(N-k)})\, \bar{\phi}(\underline{z}^{(k)})\, d\underline{z}^{(k)} \right|^2 d\underline{z}^{(N-k)}$$

$$\overset{\text{Schwarz}}{\leq} \int \int |\psi_j(\underline{z}^{(k)}, \underline{z}^{(N-k)})|^2 d\underline{z}^{(k)} \int |\bar{\phi}(\underline{z}^{(k)})|^2 d\underline{z}^{(k)}\, d\underline{z}^{(N-k)}$$

$$= \|\phi\|^2 \iint |\psi_j(\underline{z}^{(k)}, \underline{z}^{(N-k)})\, d\underline{z}^{(k)} d\underline{z}^{(N-k)}$$

$$= \|\phi\|^2 \int |\psi_j(\underline{z})|^2 d\underline{z} = \|\phi\|^2 \|\psi_j\|^2 = \|\phi\|^2$$

Then
$$\text{LHS}(6) \leq \sum_{j=1}^{\infty} \lambda_j \|\phi\|^2 = \|\phi\|^2 \sum_{j=1}^{\infty} \lambda_j = \|\phi\|^2 \mathrm{Tr}(\Gamma) < \infty$$

Hence we can take out the infinite sum at (5):

$$(5) = C \sum_{j=1}^{\infty} \lambda_j \iiint f_j(\underline{z}^{(k)}, \underline{z}^{(k)'}, \underline{z}^{(N-k)}) \, d\underline{z}^{(k)} d\underline{z}^{(k)'} d\underline{z}^{(N-k)}$$

$$= C \sum_{j=1}^{\infty} \lambda_j \int \left| \int \psi_j(\underline{z}^{(k)}, \underline{z}^{(N-k)}) \overline{\Phi}(\underline{z}^{(k)}) \, d\underline{z}^{(k)} \right|^2 d\underline{z}^{(N-k)} \geq 0 \quad (7)$$

And of course $(5) \leq \text{LHS}(6) < \infty$.

$\boxed{5}$ Verify that $\mathrm{Tr}\,\gamma^{(k)} = N!/(N-k)!$

Since $\gamma^{(k)}$ is a ~~semi~~ positive semidefinite compact operator, we only need to find an orthonormal basis $\{\phi_i\}_{i \in \mathbb{N}}$ of $L^2(\mathbb{R}^{3N}; \mathbb{C}^{q^N})$ such that

$$\sum_{i=1}^{\infty} (\phi_i, \gamma^{(k)} \phi_i) = \frac{N!}{(N-k)!} = C$$

By Point 2, Part IV, report on "Hilbert Schmidt operator and ...", we have

$$L^2(\mathbb{R}^{3N}; \mathbb{C}^{q^N}) = L^2(\mathbb{R}^{3k}; \mathbb{C}^{q^k}) \hat{\otimes} L^2(\mathbb{R}^{3(N-k)}; \mathbb{C}^{q^{N-k}})$$

Let $\{\phi_i^{(k)}\}_{i \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^{3k}; \mathbb{C}^{q^k})$,

$\{\phi_i^{(N-k)}\}_{i \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^{3(N-k)}; \mathbb{C}^{q^{N-k}})$.

Then accordingly an orthonormal basis of $L^2(\mathbb{R}^{3N}; \mathbb{C}^{q^N})$ is $\{\phi_i\}_{i\in\mathbb{N}}$ where

$$\phi_i(\underline{z}) = \phi_i(\underline{z}^{(k)}, \underline{z}^{(N-k)}) = \phi_i^{(k)}(\underline{z}^{(k)})\, \phi_i^{(N-k)}(\underline{z}^{(N-k)})$$

By (7) we have (now ψ_j is replaced by ϕ_i)

$$(\phi_i^{(k)}, \gamma^{(k)}\phi_i^{(k)}) = C\sum_{j=1}^{\infty} \lambda_j \int \left| \int \phi_j(\underline{z}^{(k)}, \underline{z}^{(N-k)})\, \bar{\phi}_i^{(k)}(\underline{z}^{(k)})\, d\underline{z}^{(k)} \right|^2 d\underline{z}^{(N-k)}$$

$$= C\sum_{j=1}^{\infty} \lambda_j \int \left| \int \phi_j^{(k)}(\underline{z}^{(k)})\, \phi_j^{(N-k)}(\underline{z}^{(N-k)})\, \bar{\phi}_i^{(k)}(\underline{z}^{(k)})\, d\underline{z}^{(k)} \right|^2 d\underline{z}^{(N-k)}$$

$$= C\sum_{j=1}^{\infty} \lambda_j \int \left| \int \phi_j^{(k)}(\underline{z}^{(k)})\, \bar{\phi}_i^{(k)}(\underline{z}^{(k)})\, d\underline{z}^{(k)} \right|^2 \left|\phi_j^{(N-k)}(\underline{z}^{(N-k)})\right|^2 d\underline{z}^{(N-k)}$$

$$= C\sum_{j=1}^{\infty} \lambda_j \left(\phi_i^{(k)}, \phi_j^{(k)}\right)_{L^2(\mathbb{R}^{3k}; \mathbb{C}^{q^k})} \int \left|\phi_j^{(N-k)}(\underline{z}^{(N-k)})\right|^2 d\underline{z}^{(N-k)}$$

$$= C\sum_{j=1}^{\infty} \lambda_j\, \delta_{ij}\, \|\phi_j^{(N-k)}\|_{L^2(\mathbb{R}^{3(N-k)}; \mathbb{C}^{q^{N-k}})}$$

$$= C\sum_{j=1}^{\infty} \lambda_j\, \delta_{ij}$$

$$= C\lambda_i$$

Thus, $\displaystyle\sum_{i=1}^{\infty} (\phi_i^{(k)}, \gamma^{(k)}\phi_i^{(k)}) = C\sum_{i=1}^{\infty}\lambda_i = C\,\mathrm{Tr}\,\Gamma = C.$

$\boxed{6}$ Verify that the one-particle density $\rho_\psi(x)$ is the diagonal part of $\gamma^{(1)}$

Let ψ be the N-particle wavefunction. Put $\Gamma = \Gamma_\psi$. We need

To show that $\rho_\Psi(\underline{z}^{(1)}) = \gamma^{(1)}(\underline{z}^{(1)}, \underline{z}^{(1)})$. We have

$$\Gamma(\underline{z}; \underline{z}') = \Psi(\underline{z})\,\bar{\Psi}(\underline{z}')$$

and

$$\gamma^{(1)}(\underline{z}^{(1)}, \underline{z}^{(1)}) = \frac{N!}{(N-1)!}\int \Gamma(\underline{z}^{(1)}, \underline{z}^{(N-1)}; \underline{z}^{(1)}, \underline{z}^{(N-1)})\, d\underline{z}^{(N-1)}$$

$$= N\int \Psi(\underline{z}^{(1)}, \underline{z}^{(N-1)})\,\bar{\Psi}(\underline{z}^{(1)}, \underline{z}^{(N-1)})\, d\underline{z}^{(N-1)}$$

$$= N\int |\Psi(\underline{z}^{(1)}, \underline{z}^{(N-1)})|^2\, d\underline{z}^{(N-1)}$$

$$= N\rho^1_\Psi(\underline{z}^{(1)})$$

Because Ψ is totally antisymmetric, we have

$$\cancel{\rho^i_\Psi(z^1, z^2, \dots, z^N)}\ \rho^i_\Psi(z) = \int |\Psi(z_1, \dots, z, \dots, z_N)|^2\, dz_1 \dots \widehat{dz_i} \dots dz_N$$

$$= \int |\Psi(z, z_2, \dots, z_N)|^2\, dz_2 \dots dz_N$$

$$= \rho^i_\Psi(z) \qquad \forall z$$

Thus $\rho_\Psi(z) = \sum_{i=1}^{N} \rho^i_\Psi(z) = N\rho^1_\Psi(z)$ and hence

$$\gamma^{(1)}(\underline{z}^{(1)}, \underline{z}^{(1)}) = \rho_\Psi(\underline{z}^{(1)}).$$

That means $\gamma^{(1)}(\underline{z}^{(1)}, \underline{z}^{(1)})$ is the (average) number of particles at $\underline{z}^{(1)}$.

Analogously, $\gamma^{(k)}(\underline{z}^{(k)}, \underline{z}^{(k)})$ is the (average) number of k-~~tuples~~ combinations of particles at $\underline{z}^{(k)} = (z_1, z_2, \dots, z_k)$.

$\boxed{7}$ Verify inequality (3.1.33)

First, we show that $\mathring{\gamma}^{(1)}$ is positively semidefinite. For each $\phi \in L^2(\mathbb{R}^3)$, we have

$$(\phi, \mathring{\gamma}^{(1)}\phi) = \int \mathring{\gamma}^{(1)}\phi(x)\,\bar{\phi}(x)dx = \iint \mathring{\gamma}^{(1)}(x,x')\phi(x')\bar{\phi}(x)\,dx'dx$$

$$= \sum_{\sigma=1}^{q} \iint \gamma^{(1)}(x,\sigma;x',\sigma)\,\phi(x')\,\bar{\phi}(x)\,dx'dx$$

$$\overset{(3.1.31)}{=\!=} \sum_{\sigma=1}^{q} \iint \sum_{j=1}^{\infty} \lambda_j^{(1)} f_j(x,\sigma)\bar{f}_j(x',\sigma)\,\phi(x')\,\bar{\phi}(x)\,dx'dx \qquad (8)$$

where $\{f_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^3;\mathbb{C}^q)$ consisting of eigenfunctions of $\gamma^{(1)}$. To take the infinite sum out of the integral, we have to show that

$$\sum_{j=1}^{\infty} \lambda_j^{(1)} \iint \left| f_j(x,\sigma)\,\bar{f}_j(x',\sigma)\,\phi(x')\,\bar{\phi}(x)\right| dx'dx < \infty \qquad (9)$$

$$\mathrm{LHS}(9) = \sum_{j=1}^{\infty} \lambda_j^{(1)} \iint |f_j(x,\sigma)|\,|\phi(x)|\,|f_j(x',\sigma)|\,|\phi(x')|\,dx'dx$$

$$= \sum_{j=1}^{\infty} \lambda_j^{(1)} \left(\int |f_j(x,\sigma)|\,|\phi(x)|\,dx\right)^2$$

$$\overset{\text{Schwarz}}{\leq} \sum_{j=1}^{\infty} \lambda_j^{(1)} \int |f_j(x,\sigma)|^2 dx \int |\phi(x)|^2 dx$$

$$= \|\phi\|^2 \sum_{j=1}^{\infty} \lambda_j^{(1)} \int |f_j(x,\sigma)|^2 dx$$

$$\leq \|\phi\|^2 \sum_{j=1}^{\infty} \lambda_j^{(1)} \int \sum_{\sigma'=1}^{q} \int |f_j(x,\sigma')|^2 dx$$

$$= \|\phi\|^2 \sum_{j=1}^{\infty} \lambda_j^{(1)} \underbrace{\int |f_j(z^{(1)})|^2 dz^{(1)}}_{\|f_j\|^2 = 1} = \|\phi\|^2 \sum_{j=1}^{\infty} \lambda^{(1)} = N\|\phi\|^2 < \infty$$

Thus (9) is verified. We have

$$(8) = \sum_{j=1}^{\infty} \lambda_j^{(1)} \sum_{\sigma=1}^{q} \iint f_j(x,\sigma) \bar{f}_j(x',\sigma)\, \phi(x')\, \bar{\phi}(x)\, dx' dx$$

$$= \sum_{j=1}^{\infty} \lambda_j^{(1)} \sum_{\sigma=1}^{q} \int f_j(x,\sigma) \bar{\phi}(x) dx \int \bar{f}_j(x',\sigma) \phi(x') dx'$$

$$= \sum_{j=1}^{\infty} \lambda_j^{(1)} \sum_{\sigma=1}^{q} \left| \int f_j(x,\sigma) \bar{\phi}(x) dx \right|^2 \geq 0$$

Hence $\gamma^{(1)}$ is positive semidefinite and therefore has nonnegative eigenvalues. Next, we show that

$$(\psi, \gamma^{(1)} \psi) \leq \|\gamma^{(1)}\|_\infty \qquad (10)$$

for every ~~no~~ normalized function $\psi \in L^2(\mathbb{R}^3; \mathbb{C}^q)$. Since $\{f_j\}$ is an orthonormal basis of $L^2(\mathbb{R}^3; \mathbb{C}^q)$, there exist a sequence $(\alpha_k) \in \ell^2$ such that

$$\psi = \sum_{k=1}^{\infty} \alpha_k f_k$$

As f_k is ~~the~~ a k'th eigenfunction of $\gamma^{(1)}$, we have $\gamma^{(k)} f_k = \lambda_k^{(1)} f_k$. Then

$$(\psi, \gamma^{(1)}\psi) = \Big(\sum_k \alpha_k f_k , \sum_j \alpha_j \gamma^{(1)} f_j \Big) = \Big(\sum_k \alpha_k f_k , \sum_j \lambda_j^{(1)} \alpha_j f_j \Big)$$

$$= \sum_k \alpha_k \overline{\lambda_k^{(1)} \alpha_k} = \sum_{k=1}^{\infty} \lambda_k^{(1)} |\alpha_k|^2$$

$$\leq \|\gamma^{(1)}\|_\infty \sum_{k=1}^{\infty} |\alpha_k|^2 = \|\gamma^{(1)}\|_\infty \|\psi\|^2 = \|\gamma^{(1)}\|_\infty$$

Thus, (10) is verified. We know that

$$L^2(\mathbb{R}^3; \mathbb{C}^q) = L^2(\mathbb{R}^3) \hat{\otimes} L^2(\{1,\dots,q\})$$

The space $L^2(\{1,\dots,q\})$ contains all square integrable functions from $\{1,\dots,q\}$ to $\mathbb{C}$. Hence all functions from $\{1,\dots,q\}$ to $\mathbb{C}$ belong to $L^2(\{1,\dots,q\})$. That is why this space is usually denoted as $\mathbb{C}^{\{1,\dots,q\}}$ or simply $\mathbb{C}^q$. The inner product on $L^2(\{1,\dots,q\})$ is as usual

$$(f,g) = \sum_{k=1}^{q} \overline{f(k)}\, g(k)$$

An orthonormal basis of its is $\{\chi_k\}_{k=1}^{q}$, where

$$\chi_k(\sigma) = \begin{cases} 1 & \text{if } \sigma = k \\ 0 & \text{otherwise} \end{cases}$$

Let $\{g_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^3)$ consisting of eigenfunctions of $\mathring{\gamma}^{(1)}$, i.e.

$$\mathring{\gamma}^{(1)} g_i = \mathring{\lambda}_i^{(1)} g_i$$

Then by Point 2, Part IV, report on "HS operator and...", $\{\phi_{ik}\}_{i\in\mathbb{N}, k=\overline{1,q}}$ is an orthonormal basis of $L^2(\mathbb{R}^3; \mathbb{C}^q)$ where

$$\phi_{ik}(x,\sigma) = g_i(x)\,\chi_k(\sigma).$$

Apply (10) for $\Psi = \phi_{ik}$, we have

$$\|\gamma^{(1)}\|_\infty \geq (\phi_{ik}, \gamma^{(1)}\phi_{ik}) = \sum_{\sigma=1}^{q} \int \gamma^{(1)}\phi_{ik}(x,\sigma)\,\overline{\phi_{ik}}(x,\sigma)\,dx$$

$$= \sum_{\sigma=1}^{q}\sum_{\sigma'=1}^{q} \iint \gamma^{(1)}(x,\sigma;x',\sigma')\,\phi_{ik}(x',\sigma')\,dx'\,\bar{\phi}_{ik}(x,\sigma)\,dx$$

$$= \sum_{\sigma,\sigma'=1}^{q} \iint \gamma^{(1)}(x,\sigma;x',\sigma')\,\phi_{ik}(x',\sigma')\,\bar{\phi}_{ik}(x,\sigma)\,dx\,dx'$$

$$= \sum_{\sigma,\sigma'=1}^{q} \iint \gamma^{(1)}(x,\sigma;x',\sigma')\,g_i(x')\,\chi_k(\sigma')\,\bar{g}_i(x)\,\chi_k(\sigma)\,dx\,dx'$$

$$= \cancel{\sum_{\sigma=1}^{q}} \iint \gamma^{(1)}(x,k;x',k)\,g_i(x')\,\bar{g}_i(x)\,dx\,dx'$$

$$= \cancel{\iint \mathring{\gamma}^{(1)}(x,x')\,g_i(x')\,\bar{g}_i(x)\,dx\,dx'}$$

$$= \cancel{\int \mathring{\gamma}^{(1)} g_i(x)\,\bar{g}_i(x)\,dx} = \left(g_i, \mathring{\gamma}^{(1)} g_i\right) = \mathring{\lambda}_i^{(1)}$$

Thus

$$\sum_{k=1}^{q} \|\gamma^{(1)}\|_\infty \geq \sum_{k=1}^{q} \iint \gamma^{(1)}(x,k;x',k)\,g_i(x')\,\bar{g}_i(x)\,dx\,dx'$$

$$= \iint \mathring{\gamma}^{(1)}(x,x')\,g_i(x')\,\bar{g}_i(x)\,dx\,dx'$$

$$= \int \mathring{\gamma}^{(1)} g_i(x)\,\bar{g}_i(x)\,dx = \left(g_i, \mathring{\gamma}^{(1)} g_i\right) = \mathring{\lambda}_i^{(1)}$$

Thus, $\quad q\,\|\gamma^{(1)}\|_\infty \geq \mathring{\lambda}_i^{(1)} \quad \forall i \in \mathbb{N}$

Thus, $\quad q\,\|\gamma^{(1)}\|_\infty \geq \max_{i\in\mathbb{N}} \mathring{\lambda}_i^{(1)} = \|\mathring{\gamma}^{(1)}\|_\infty$. Therefore,

$$\|\mathring{\gamma}^{(1)}\|_\infty \leq q\,\|\gamma^{(1)}\|_\infty .$$

8 Verify the first two equalities at (3.1.26) for bounded H

Because H is bounded and Γ is a trace class operator on $H^1(\mathbb{R}^{3N};\mathbb{C}^{q^N})$, we know by Number 8, Point 8, the report on "Compact operators on Hilbert space" that $tr(\Gamma H) = tr(H\Gamma)$. Next, we show that

$$Tr(H\Gamma) = \sum_{j=1}^{\infty} \lambda_j \mathcal{E}(\psi_j)$$

We have
$$H\Gamma = H \sum_j \lambda_j \Gamma_{\psi_j} = \sum_j \lambda_j H \Gamma_{\psi_j}$$

Since Tr is continuous with respect to the trace class norm, we have

$$Tr(H\Gamma) = \sum_j \lambda_j \, Tr(H\Gamma_{\psi_j})$$

By definition,
$$Tr(H\Gamma_{\psi_j}) = \sum_{k=1}^{\infty} (\psi_k, H\Gamma_{\psi_j}\psi_k) = \sum_{k=1}^{\infty} (\psi_k, H(\psi_j,\psi_k)\psi_j)$$
$$= (\psi_j, H\psi_j) = \mathcal{E}(\psi_j)$$

thus, $$Tr(H\Gamma) = \sum_j \lambda_j \mathcal{E}(\psi_j).$$

9 Verify the equation right below Eq. (3.1.34)

By definition,
$$\gamma^{(1)}(z_1, z_1') = N \int \Gamma(z_1, \underline{z}^{(N-1)}; z_1', \underline{z}^{(N-1)})\, d\underline{z}^{(N-1)}$$

$$\gamma^{(2)}(z_1, z_2; z_1', z_2') = N(N-1) \int \Gamma(z_1, z_2, \underline{z}^{(N-2)}; z_1', z_2', \underline{z}^{(N-2)})\, d\underline{z}^{(N-2)}$$

Then

$$\gamma^{(2)}(z_1, z_2; z_1', z_2) = N(N-1) \int \Gamma(z_1, z_2, \underline{z}^{(N-2)}; z_1', z_2, \underline{z}^{(N-2)})\, d\underline{z}^{(N-2)}$$

and thus

$$\int \gamma^{(2)}(z_1, z_2; z_1', z_2)\, dz_2 = N(N-1) \iint \Gamma(z_1, z_2, \underline{z}^{(N-2)}; z_1', z_2, \underline{z}^{(N-2)})\, dz_2\, d\underline{z}^{(N-2)}$$

$$= N(N-1) \int \Gamma(z_1, \underline{z}^{(N-1)}; z_1', \underline{z}^{(N-1)})\, d\underline{z}^{(N-1)}$$

$$= (N-1)\, \gamma^{(1)}(z_1, z_1')$$

Thus, $\gamma^{(1)}(z_1, z_1') = \frac{1}{N-1} \int \gamma^{(2)}(z_1, z_2; z_1', z_2)\, dz_2 = \frac{1}{N-1} \operatorname{Tr}^{(1)} \gamma^{(2)}$

$\boxed{10}$ <u>Proof of Theorem 3.1</u>

Because γ is self-adjoint, positive semidefinite on $L^2(\mathbb{R}^3; \mathbb{C}^q)$, and $\operatorname{Tr} \gamma = N$, there exists $\lambda_1, \lambda_2, \ldots \geq 0$ with sum N, and corresponding eigenvectors $f_1, f_2, \ldots$, i.e.

$$\gamma \phi = \sum_{j=1}^{\infty} \lambda_j (f_j, \phi) f_j \qquad \forall \phi \in L^2(\mathbb{R}^3; \mathbb{C}^q)$$

where

$$\sum_{j=1}^{\infty} \lambda_j = N$$

We will find a totally symmetric wavefunction $\Psi \in L^2(\mathbb{R}^{3N}; \mathbb{C}^{q^N})$ such that

$$\gamma = N\, \mathrm{Tr}^{(N-1)} \Gamma \quad \text{where } \Gamma = \Gamma_\Psi .$$

For each $k \in \mathbb{N}$, we put

$$\Psi_k(z_1, \dots, z_N) = f_k(z_1) \cdots f_k(z_N)$$

Then $(\Psi_k)_{k\in\mathbb{N}}$ is an orthonormal set in $L^2(\mathbb{R}^{3N}; \mathbb{C}^{q^N})$ and each Ψ_k is totally symmetric. We will find Ψ as the form

$$\Psi = \sum_{k=1}^{\infty} \alpha_k \Psi_k$$

where $$\sum_{k=1}^{\infty} |\alpha_k|^2 = 1 .$$

The kernel of the N-particle density matrix Γ is

$$\Gamma(z, z') = \Psi(z)\, \overline{\Psi(z')}$$

Then the reduced one-particle density matrix of Γ is

$$\gamma^{(1)}(z_1, z_1') = N \int \Gamma(z_1, z_2, \dots, z_N; z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

$$= N \int \Psi(z_1, z_2, \dots, z_N)\, \overline{\Psi}(z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

$$= N \int \sum_k \alpha_k \Psi_k(z_1, \dots, z_N) \sum_\ell \overline{\alpha_\ell}\, \overline{\Psi_\ell}(z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

For $k \neq \ell$

$$\int \Psi_k(z_1 \dots, z_N)\, \overline{\Psi_\ell}(z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

$$= \int f_k(z_1) \overline{f_\ell}(z_1')\, f_k(z_2) \overline{f_\ell}(z_2) \cdots f_k(z_N) \overline{f_\ell}(z_N)\, dz_2 \dots dz_N$$

$$= f_k(z_1) \overline{f_\ell}(z_1') \underbrace{\int f_k(z_2) \overline{f_\ell}(z_2)\, dz_2}_{0} \cdots \int f_k(z_N) \overline{f_\ell}(z_N)\, dz_N = 0$$

Thus, $$\gamma^{(1)}(z_1, z_1') = N \int \sum_k |\alpha_k|^2 \Psi_k(z_1, z_2, \dots, z_N)\, \overline{\Psi}_k(z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

$$= N \sum_k |\alpha_k|^2 \int f_k(z_1) f_k(z_2) \cdots f_k(z_N)\, \overline{f}_k(z_1')\, \overline{f}_k(z_2) \cdots \overline{f}_k(z_N)\, dz_2 \dots dz_N$$

$$= N \sum_k |\alpha_k|^2 f_k(z_1)\, \overline{f}_k(z_1') \Big(\underbrace{\int |f_k(z)|^2 dz}_{1}\Big)^{N-1}$$

$$= N \sum_k |\alpha_k|^2 f_k(z_1)\, \overline{f}_k(z_1')$$

Then
$$\gamma^{(1)}\phi(z_1) = \int \gamma^{(1)}(z_1, z_1')\, \phi(z_1')\, dz_1'$$

$$= N \int \sum_k |\alpha_k|^2 f_k(z_1)\, \overline{f}_k(z_1')\, \phi(z_1')\, dz_1'$$

$$= N \sum_k |\alpha_k|^2 \int f_k(z_1)\, \overline{f}_k(z_1')\, \phi(z_1')\, dz_1'$$

$$= N \sum_k |\alpha_k|^2 (f_k, \phi) f_k(z_1)$$

Thus
$$\gamma^{(1)}\phi = N \sum_{k=1}^{\infty} |\alpha_k|^2 (f_k, \phi) f_k$$

We know that
$$\gamma\phi = \sum_{j=1}^{\infty} \lambda_j (f_j, \phi) f_j$$

Thus, to make $\gamma^{(1)} = \gamma$, we choose
$$\alpha_k = \left(\frac{\lambda_j}{N}\right)^{1/2}.$$

$\boxed{11}$ Proof of the backward part of Theorem 3.2

Let Γ be a fermionic N-particle density matrix, i.e.

$$\Gamma = \sum_{j=1}^{\infty} \lambda_j \Gamma_{\psi_j}$$

where each $\lambda_j \geq 0$, $\sum \lambda_j = 1$ anad each ψ_j is fermionic. Put

$\gamma = \gamma^{(1)} = N \operatorname{Tr}^{(N-1)} \Gamma$. By Point 4 and Point 5, γ is self-adjoint semidefinite and $\operatorname{Tr} \gamma = N$. We now have to show that $\gamma \leq I$, i.e.

$$(\phi, \gamma\phi) \leq 1, \quad \forall \phi \in L^2(\mathbb{R}^3; \mathbb{C}^q), \ \|\phi\| = 1$$

Let $(f_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^3; \mathbb{C}^q)$ consisting of eigenfunctions of γ. Suppose that we have

$$(f_i, \gamma f_i) \leq 1 \qquad \forall i \in \mathbb{N}$$

For each $\phi \in L^2(\mathbb{R}^3; \mathbb{C}^q)$ with unit norm, there exists a sequence of complex numbers (α_k) such that

$$\phi = \sum_k \alpha_k f_k \quad \text{and} \quad \sum |\alpha_k|^2 = 1$$

Then

$$(\phi, \gamma\phi) = \Big(\sum_k \alpha_k f_k, \sum_j \alpha_j \gamma f_j\Big) = \sum_{k,j} \alpha_k \bar{\alpha}_j (f_k, \gamma f_j)$$

$$= \sum_{k,j} \alpha_k \bar{\alpha}_j (f_k, \lambda_j^{(1)} f_j)$$

$$= \sum_{k,j} \alpha_k \bar{\alpha}_j \lambda_j^{(1)} \delta_{kj} = \sum_k |\alpha_k|^2 \lambda_k^{(1)}$$

(20)

Since $\lambda_k^{(1)} = (f_k, \gamma f_k) \leq 1$, we have

$$(\phi, \gamma\phi) \leq \sum_{k=1}^{\infty} |\alpha_k|^2 = 1$$

Thus, it is sufficient to show that $(\phi, \gamma\phi) \leq 1$ for every ϕ of the form $\phi = f_i$. ~~Put~~ For each $i_1 < i_2 < \dots < i_N$, we put

$$k_{i_1 \dots i_N}(z_1, \dots, z_N) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma f_{i_1}(z_{\sigma(1)}) \cdots f_{i_N}(z_{\sigma(N)})$$

$$= \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma f_{i_{\sigma(1)}}(z_1) \cdots f_{i_{\sigma(N)}}(z_N)$$

Then by ~~Number 9,~~ Point 9, Part III, report on "Wedge Product", the space of all fermionic (totally antisymmetric) wave functions is simply the sub Hilbert space of $L^2(\mathbb{R}^{3N}; \mathbb{C}^{q^N})$ which has an orthonormal basis $\{k_{i_1 \dots i_N} / i_1 < \dots < i_N\}$. Hence, for each $j \in \mathbb{N}$, there exists a sequence $\{\alpha^{(j)}_{i_1 \dots i_N}\}_{i_1 < \dots < i_N}$ such that

$$\psi_j = \sum_{i_1 < \dots < i_N} \alpha^{(j)}_{i_1 \dots i_N} k_{i_1 \dots i_N} \quad \text{and} \quad \sum_{i_1 < \dots < i_N} |\alpha^{(j)}_{i_1 \dots i_N}|^2 = 1$$

We have

$$\Gamma(\underline{z}, \underline{z}') = \sum_j \lambda_j \psi_j(\underline{z}) \overline{\psi_j}(\underline{z}')$$

and $$\gamma\phi(z_1) = N \int \Gamma(z_1, z_2, \dots, z_N; z_1', z_2, \dots, z_N) \phi(z_1') \, dz_1' \, dz_2 \dots dz_N$$

$$= N \sum_j \lambda_j \int \psi_j(z_1, z_2, \ldots, z_N) \overline{\psi_j}(z_1', z_2, \ldots, z_N) \phi(z_1') \, dz_1' \, dz_2 \ldots dz_N$$

Thus,

$$(\phi, \gamma\phi) = \int \overline{\phi}(z_1) \gamma\phi(z_1) \, dz_1$$

$$= N \sum_j \lambda_j \int \psi_j(z_1, z_2, \ldots, z_N) \overline{\psi_j}(z_1', z_2, \ldots, z_N) \phi(z_1') \overline{\phi}(z_1) \, dz_1 \, dz_1' \, dz_2 \ldots dz_N$$

$$= N \sum_j \lambda_j \int \left| \int \psi_j(z_1, z_2, \ldots, z_N) \overline{\phi}(z_1) \, dz_1 \right|^2 dz_2 \ldots dz_N \qquad (11)$$

We have

$$\int \psi_j(z_1, z_2, \ldots, z_N) \overline{\phi}(z_1) \, dz_1 = \sum_{i_1 < \ldots < i_N} \alpha^{(j)}_{i_1 \ldots i_N} \int k_{i_1 \ldots i_N}(z_1, z_2, \ldots, z_N) \overline{\phi}(z_1) \, dz_1$$

$$= \frac{1}{\sqrt{N!}} \sum_{i_1 < \ldots < i_N} \alpha^{(j)}_{i_1 \ldots i_N} \sum_{\sigma \in S_N} (-1)^\sigma f_{i_{\sigma(2)}}(z_2) \cdots f_{i_{\sigma(N)}}(z_N) \int f_{i_{\sigma(1)}}(z_1) \overline{\phi}(z_1) \, dz_1$$

$$= \frac{1}{\sqrt{N!}} \sum_{i_1 < \ldots < i_N} \alpha^{(j)}_{i_1 \ldots i_N} \sum_{\sigma \in S_N} (-1)^\sigma (\phi, f_{i_{\sigma(1)}}) f_{i_{\sigma(2)}}(z_2) \cdots f_{i_{\sigma(N)}}(z_N)$$

Thus,

$$\left| \int \psi_j(z_1, z_2 \ldots, z_N) \overline{\phi}(z_1) \, dz_1 \right|^2 = \frac{1}{N!} \sum_{\substack{r_1 < \ldots < r_N \\ s_1 < \ldots < s_N}} \alpha^{(j)}_{r_1 \ldots r_N} \overline{\alpha}^{(j)}_{s_1 \ldots s_N} \sum_{\sigma, \sigma' \in S_N} (-1)^\sigma (-1)^{\sigma'}$$

$$\times (\phi, f_{r_{\sigma(1)}}) \overline{(\phi, f_{s_{\sigma'(1)}})} f_{r_{\sigma(1)}}(z_2) \overline{f}_{s_{\sigma'(1)}}(z_2) \ldots$$

$$\times f_{r_{\sigma(N)}}(z_N) \overline{f}_{s_{\sigma'(N)}}(z_N)$$

Then

$$\int \left| \int \Psi_j(z_1, \dots, z_N)\, \overline{\Phi(z_1)}\, dz_1 \right|^2 dz_2 \dots dz_N$$

$$= \frac{1}{N!} \sum_{\substack{r_1 < \dots < r_N \\ s_1 < \dots < s_N}} \alpha^{(j)}_{r_1 \dots r_N} \overline{\alpha}^{(j)}_{s_1 \dots s_N} \sum_{\sigma, \sigma' \in S_N} (-1)^{\sigma} (-1)^{\sigma'} (\phi, f_{r_{\sigma(1)}}) \overline{(\phi, f_{s_{\sigma'(1)}})}\, \delta_{r_{\sigma(2)}, s_{\sigma'(2)}} \dots \delta_{r_{\sigma(N)}, s_{\sigma'(N)}} \quad (12)$$

We note again that ϕ is equal to some f_i. Thus, the summand is nonzero if and only if

$$\begin{cases} r_{\sigma(1)} = s_{\sigma'(1)} \\ r_{\sigma(2)} = s_{\sigma'(2)} \\ \vdots \qquad \vdots \\ r_{\sigma(N)} = s_{\sigma'(N)} \end{cases}$$

Thus $r_k = s_k \ \forall k$ and $\sigma = \sigma'$. Then

$$(12) = \frac{1}{N!} \sum_{r_1 < \dots < r_N} |\alpha^{(j)}_{r_1 \dots r_N}|^2 \sum_{\sigma \in S_N} |(\phi, f_{r_{\sigma(1)}})|^2 \qquad (12')$$

Now we write $\phi = f_i$. If $i \notin \{r_1, \dots, r_N\}$ then $(\phi, f_{r_{\sigma(1)}}) = 0 \ \forall \sigma \in S_N$.

If there exists $r_k = i$ then $(\phi, f_{r_{\sigma(1)}}) = 1$ only for $\sigma \in S_N$ such that $\sigma(1) = k$ (otherwise $(\phi, f_{r_{\sigma(1)}}) = 0$). Thus,

$$\sum_{\sigma \in S_N} |(\phi, f_{r_{\sigma(1)}})|^2 \leq (N-1)!$$

and $(12) \leq \frac{1}{N!} (N-1)! = \frac{1}{N}$.

Then $~~(11) \leq N \frac{1}{N} = 1~~$.

$$(11) \leq N \sum_j \lambda_j \frac{1}{N} = \sum_j \lambda_j = 1$$

$\boxed{12}$ Verify the equation right below Eq. (3.1.38)

We correct the formula of the annihilation operator and creation operator

$$C_{N,\phi} \psi(z_1, \dots, z_{N-1}) = N^{1/2} \int \psi(z_1, \dots, z_{N-1}, z_N) \overline{\phi}(z_N) dz_N$$

$$C_{N,\phi}^{\dagger} \chi(z_1, \dots, z_N) = [(N-1)!]^{-1} N^{-1/2} \mathcal{A}\{\chi(z_1, \dots, z_{N-1}) \phi(z_N)\}$$

$$= [(N-1)!]^{-1} N^{-1/2} \sum_{\sigma \in S_N} (-1)^{\sigma} \chi(z_{\sigma(1)}, \dots, z_{\sigma(N-1)}) \phi(z_{\sigma(N)})$$

Now we show that $(\chi, C_{N,\phi} \psi)_{H_{N-1}} = (C_{N,\phi}^{\dagger} \chi, \psi)_{H_N}$ for every $\chi \in H_{N-1}$ and $\psi \in H_N$. We have

$$(\chi, C_{N,\phi} \psi)_{H_{N-1}} = \int \overline{\chi}(z_1, \dots, z_{N-1}) C_{N,\phi} \psi(z_1, \dots, z_{N-1}) dz_1 \dots dz_{N-1}$$

$$= (N)^{1/2} \int \psi(z_1, \dots, z_{N-1}, z_N) \overline{\phi}(z_N) \overline{\chi}(z_1, \dots, z_{N-1}) dz_1 \dots dz_N \qquad (13)$$

And

$$(C_{N,\phi}^{\dagger} \chi, \psi)_{H_N} = \int \overline{C_{N,\phi}^{\dagger} \chi}(z_1, \dots, z_N) \psi(z_1, \dots, z_N) dz_1 \dots dz_N$$

$$= [(N-1)!]^{-1} N^{-1/2} \sum_{\sigma \in S_N} (-1)^{\sigma} \int \overline{\chi}(z_{\sigma(1)}, \dots, z_{\sigma(N-1)}) \overline{\phi}(z_{\sigma(N)}) \psi(z_1, \dots, z_N) dz_1 \dots dz_N$$

Using the equality $(-1)^\sigma \psi(z_1,\dots,z_N) = \psi(z_{\sigma(1)},\dots,z_{\sigma(N)})$, we get

$$(C^\dagger_{N,\phi}\chi,\psi)_{H_N} = [(N-1)!]^{-1} N^{-1/2} \sum_{\sigma\in S_N} \int \bar{\chi}(z_{\sigma(1)},\dots,z_{\sigma(N-1)})\, \bar{\phi}(z_{\sigma(N)})\, \psi(z_{\sigma(1)},\dots,z_{\sigma(N)})\, dz_1\dots dz_N$$

$$= [(N-1)!]^{-1} N^{-1/2} \sum_{\sigma\in S_N} \int \bar{\chi}(z_{\sigma(1)},\dots,z_{\sigma(N-1)})\, \bar{\phi}(z_{\sigma(N)})\, \psi(z_{\sigma(1)},\dots,z_{\sigma(N)})\, dz_{\sigma(1)}\dots dz_{\sigma(N)}$$

(rearrange the order of integration)

$$= [(N-1)!]^{-1} N^{-1/2} \sum_{\sigma\in S_N} \int \bar{\chi}(z_1,\dots,z_{N-1})\, \bar{\phi}(z_N)\, \psi(z_1,\dots,z_N)\, dz_1\dots dz_N$$

$$\overset{(13)}{=} [(N-1)!]^{-1} N^{-1/2} \sum_{\sigma\in S_N} (N)^{1/2} \cancel{(C_{N,\phi}\psi,\chi)_{H_{N-1}}}\ (\chi, C_{N,\phi}\psi)_{H_{N-1}}$$

$$= (\chi, C_{N,\phi}\psi)_{H_{N-1}}$$

$\boxed{13}$ Verify Equation (3.1.40)

For each $\psi \in H_N$, we have

$$C_{N+1,\phi}\, C^\dagger_{N+1,\phi}\, \psi = (N+1)^{1/2} \int C^\dagger_{N+1,\phi}\, \psi(z_1,\dots,z_N, z_{N+1})\, \bar{\phi}(z_{N+1})\, dz_{N+1}$$

$$= (N!)^{-1} \int A\{\psi(z_1,\dots,z_N)\, \phi(z_{N+1})\}\, \bar{\phi}(z_{N+1})\, dz_{N+1}$$

$$= (N!)^{-1} \sum_{\sigma\in S_{N+1}} (-1)^\sigma \int \psi(z_{\sigma(1)},\dots,z_{\sigma(N)})\, \phi(z_{\sigma(N+1)})\, \bar{\phi}(z_{N+1})\, dz_{N+1} \qquad (14)$$

Put $S'_{N+1} = \{\sigma \in S_{N+1} \,/\, \sigma(N+1) \neq N+1\}$

$S''_{N+1} = \{\sigma \in S_{N+1} \,/\, \sigma(N+1) = N+1\}$

Then

$$(N!)^{-1}\sum_{\sigma\in S''_{N+1}}(-1)^{\sigma}\int \psi(z_{\sigma(1)},\dots,z_{\sigma(N)})\,\phi(z_{\sigma(N+1)})\,\bar{\phi}(z_{N+1})\,dz_{N+1}$$

$$=(N!)^{-1}\sum_{\sigma\in S_N}(-1)^{\sigma}\int \psi(z_{\sigma(1)},\dots,z_{\sigma(N)})\,\phi(z_{N+1})\,\bar{\phi}(z_{N+1})\,dz_{N+1}$$

$$\doteq (N!)^{-1}\sum_{\sigma\in S_N}\int \psi(z_1,\dots,z_N)\,|\phi(z_{N+1})|^2\,dz_{N+1}$$

$$=(N!)^{-1}N!\,\psi(z_1,\dots,z_N)\,(\phi,\phi)$$

$$=(\phi,\phi)\,\psi(z_1,\dots,z_N)$$

Thus,

$$(14)=(\phi,\phi)\psi+\underbrace{(N!)^{-1}\sum_{\sigma\in S'_{N+1}}(-1)^{\sigma}\int\psi(z_{\sigma(1)},\dots,z_{\sigma(N)})\,\phi(z_{\sigma(N+1)})\,\bar{\phi}(z_{N+1})\,dz_{N+1}}_{C} \qquad (15)$$

We will use the following notation

$$(a_1,a_2,\dots,a_n)\setminus(a_i):=(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n)$$

We introduce an equivalence relation on S'_{N+1}:

$$\sigma\sim\sigma'\iff(\sigma(1),\dots,\sigma(N+1))\setminus(N+1)=(\sigma'(1),\dots,\sigma'(N+1))\setminus(N+1)$$

Then the number of elements of each equivalence class is N. We define a map from $S'_{N+1}/\sim$ to S_N such that each $\tilde{\sigma}\in S'_{N+1}/\sim$ corresponds to $\pi\in S_N$ given by

$$(\pi(1),\dots,\pi(N))=(\sigma(1),\dots,\sigma(N+1))\setminus(N+1)\quad\text{for arbitrary } \sigma\in\tilde{\sigma}.$$

This map is well-defined and bijective. Then

$$C=\sum_{\sigma\in S'_{N+1}/\sim}\ \sum_{\sigma\in\tilde{\sigma}}(-1)^{\sigma}\int\psi(z_{\sigma(1)},\dots,z_{\sigma(N)})\,\phi(z_{\sigma(N+1)})\,\bar{\phi}(z_{N+1})\,dz_{N+1} \quad (16)$$

Since $\sigma(N+1)\neq N+1$, $\sigma(N+1)=\pi(N)$. The number of inversions of σ is equal to the number of inversion of π plus $N+1-1$ provided that $\sigma(i)=N+1$. Thus,

$$(-1)^{\sigma}\psi(z_{\sigma(1)},\dots,z_{\sigma(N)})=(-1)^{\pi}(-1)^{N+1-i}\psi(z_{\sigma(1)},\dots,\underset{\substack{\|\\ z_{N+1}}}{z_{\sigma(i)}},\dots,z_{\sigma(N)})$$

$$=(-1)^{\pi}(-1)^{N+1-i}(-1)^{N-i}\psi(z_{\sigma(1)}\cdots z_{\sigma(i-1)},z_{\sigma(i+1)}\cdots z_{\sigma(N)},z_{\sigma(i)})$$

$$=-(-1)^{\pi}\psi(z_{\pi(1)},\dots,z_{\pi(N-1)},z_{N+1})$$

Hence, from (16),

$$C=\sum_{\sigma\in S'_{N+1}/\sim}\ \sum_{\sigma\in\tilde{\sigma}}-(-1)^{\pi}\int\psi(z_{\pi(1)}\cdots,z_{\pi(N-1)},z_{N+1})\,\phi(z_{\pi(N)})\,\bar{\phi}(z_{N+1})\,dz_{N+1}$$

$$=-\sum_{\pi\in S_N}N(-1)^{\pi}\int\psi(z_{\pi(1)},\dots,z_{\pi(N-1)},z_{N+1})\,\phi(z_{\pi(N)})\,\bar{\phi}(z_{N+1})\,dz_{N+1}$$

Then

$$(15)=(\phi,\phi)\psi-[(N-1)!]^{-1}\sum_{\pi\in S_N}(-1)^{\pi}\int\psi(z_{\pi(1)},\dots,z_{\pi(N-1)},z_{N+1})\,\phi(z_{\pi(N)})\,\bar{\phi}(z_{N+1})\,dz_{N+1} \quad (17)$$

We have

$$C^{\dagger}_{N,\phi} C_{N,\phi} \Psi = [(N-1)!]^{-1} N^{-1/2} A \{ C_{N,\phi} \Psi(z_1, \ldots, z_{N-1}) \phi(z_N) \}$$

$$= [(N-1)!]^{-1} N^{-1/2} \sum_{\pi \in S_N} (-1)^{\pi} C_{N,\phi} \Psi(z_{\pi(1)}, \ldots, z_{\pi(N-1)}) \phi(z_{\pi(N)})$$

$$= [(N-1)!]^{-1} N^{-1/2} \sum_{\pi \in S_N} (-1)^{\pi} N^{1/2} \int \Psi(z_{\pi(1)}, \ldots, z_{\pi(N-1)}, z_{N+1}) \overline{\Phi}(z_{N+1}) dz_{N+1}$$
$$\times \phi(z_{\pi(N)}))$$

$$= [(N-1)!]^{-1} \sum_{\pi \in S_N} (-1)^{\pi} \int \Psi(z_{\pi(1)}, \ldots, z_{\pi(N-1)}, z_{N+1}) \overline{\Phi}(z_{N+1}) \phi(z_{\pi(N)}) dz_{N+1} \quad (18)$$

From (17) and (18),

$$C_{N+1,\phi} C^{\dagger}_{N+1,\phi} \Psi + C^{\dagger}_{N,\phi} C_{N,\phi} \Psi = (\phi, \phi) \Psi$$

Thus,

$$C_{N+1,\phi} C^{\dagger}_{N+1,\phi} + C^{\dagger}_{N,\phi} C_{N,\phi} = (\phi, \phi) \mathbb{I}_N$$

$\boxed{14}$ Proof of $(\phi, \gamma\phi) \le (\phi, \phi)$ (Eq. (3.1.41)) if Γ is not pure

Because Γ is in general not pure, we only have the representation

$$\Gamma = \sum_{j=1}^{\infty} \lambda_j \Gamma_{\Psi_j}$$

for $\lambda_j \ge 0$, $\sum \lambda_j = 1$, $\Psi_j \in H_N$. The corresponding kernel is

$$\Gamma(\underline{z}, \underline{z}') = \sum \lambda_j \Gamma_{\Psi_j}(\underline{z}, \underline{z}')$$

The one-body ~~ker~~ kernel is

(28)

$$\gamma(z_1, z_1') = \int \Gamma(z_1, z_2, \dots, z_N ; z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

$$= \sum_j \lambda_j \int \Gamma_{\psi_j}(z_1, z_2, \dots, z_N ; z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

$$= \sum_j \lambda_j \int \psi_j(z_1, z_2, \dots, z_N)\, \overline{\psi_j}(z_1', z_2, \dots, z_N)\, dz_2 \dots dz_N$$

Thus,

$$(\phi, \gamma\phi) = \int \gamma(z_1, z_1')\, \phi(z_1')\, \overline{\phi}(z_1)\, dz_1\, dz_1'$$

$$= \sum_j \lambda_j \underbrace{\int \left| \int \psi_j(z_1, \dots, z_N)\, \overline{\phi}(z_1)\, dz_1 \right|^2 dz_2 \dots dz_N}_{(C_{N,\phi}\psi_j,\ C_{N,\phi}\psi_j)}$$

$$= \sum_j \lambda_j \left(\psi_j,\ C^\dagger_{N,\phi} C_{N,\phi} \psi_j \right)$$

$$\overset{(3.1.40)}{=} \sum_j \lambda_j \left(\psi_j,\ (\phi, \phi)\psi_j - C_{N+1,\phi} C^\dagger_{N+1,\phi} \psi_j \right)$$

$$= \sum_j \left[(\phi, \phi)\, \lambda_j \underbrace{(\psi_j, \psi_j)}_{1} - \lambda_j \left(\psi_j, C_{N+1,\phi} C^\dagger_{N+1,\phi} \psi_j \right) \right]$$

$$= (\phi, \phi) \underbrace{\sum_j \lambda_j}_{1} - \sum \lambda_j \underbrace{\left(C^\dagger_{N+1,\phi} \psi_j,\ C^\dagger_{N+1,\phi} \psi_j \right)}_{\geq 0}$$

$$\leq (\phi, \phi).$$

$\boxed{15}$ Verify that if γ has $N-1$ eigenvalues equal to 1 and at least $N+1$ positive eigenvalues then Γ cannot be pure

We will prove an even stronger statement: if γ has exactly $N-1$ eigenvalues equal to 1 then Γ cannot be pure. Let $f_1, \dots, f_{N-1}$ be orthonormal eigenvectors of γ corresponding to the eigenvalue 1. For each $\phi = f_i$, $i = 1, \dots, N-1$, by (11) and (12') we have

$$(\phi, \gamma\phi) = N \int \left| \int \psi(z_1, z_2, \dots, z_N)\, \bar{\phi}(z_1)\, dz_1 \right|^2 dz_2 \dots dz_N \qquad (19)$$

(provided the Γ is pure, i.e. $\Gamma = \Gamma_\psi$)

$$= \frac{1}{(N-1)!} \sum_{r_1 < \dots < r_N} |\alpha_{r_1, r_2 \dots r_N}|^2 \sum_{\sigma \in S_N} |(\phi, f_{\sigma(1)})|^2 \qquad (20)$$

where

$$\psi = \sum_{r_1 < \dots < r_N} \alpha_{r_1 \dots r_N}\, k_{r_1 \dots r_N}$$

and

$$\sum_{r_1 < \dots < r_N} |\alpha_{r_1 \dots r_N}|^2 = 1$$

As mentioned in the end of Point 11, $(\phi, \gamma\phi) = 1$ only if $i \in \{r_1, \dots, r_N\}$

~~Thus, $\{1, \dots, N-1\} \subset \{r_1, \dots, r_N\}$.~~ and $\displaystyle\sum_{\substack{r_1 < \dots < r_N \\ i \in \{r_1, \dots, r_N\}}} |\alpha_{r_1 \dots r_N}|^2 = 1$

Thus $\alpha_{r_1 \dots r_N} = 0$ if $i \in \{r_1, \dots, r_N\}$.

Because i is arbitrarily chosen in $\{1,2\ldots,N-1\}$, we must have $\alpha_{r_1\cdots r_N}=0$ if $\{1,2\ldots,N-1\}\not\subset\{r_1,\ldots,r_N\}$. That means ~~the~~ a non-zero coefficient has the form $\alpha_{1,2,\ldots,N-1,n}$ where $n\geq N$. Thus,

$$\sum_{n=N}^{\infty}|\alpha_{1\ldots,N-1,n}|^2=1 \qquad (20')$$

and hence

$$\begin{aligned}\psi&=\sum_{n=N}^{\infty}\alpha_{1,\ldots,N-1,n}\,k_{1,\ldots,N-1,n}\\&=\sum_{n=N}^{\infty}\alpha_{1,\ldots,N-1,n}\,\mathrm{Alt}(f_1\otimes\cdots\otimes f_{N-1}\otimes f_n)\\&=\mathrm{Alt}\Big(f_1\otimes\cdots\otimes f_{N-1}\otimes\underbrace{\Big(\sum_{n=N}^{\infty}\alpha_{1,\ldots,N-1,n}f_n\Big)}_{g_N}\Big)\end{aligned}$$

We rename $f_i=g_i\ \ \forall i=1,\ldots,N-1$. Then

$$\psi=\mathrm{Alt}(g_1\otimes\cdots\otimes g_N)$$

The condition (20') ensures that g_N is normalized. It is also obvious from the definition of g_N that $(g_N,g_i)=0\ \ \forall i<N$. Thus the set $\{g_1,\ldots,g_N\}$ is othonormal. To get a contradiction, we will show that g_N is also an eigenfunction of γ corresponding to eigenvalue 1, i.e.

$$\gamma g_N=g_N$$

Because $\gamma\leq\mathbf{I}$, all of the eigenvalues of γ is less ~~of~~ equal to 1. Thus $|\gamma v|\leq 1\ \ \forall v$ with unit norm. Then it is sufficient that we show

$(g_N, \gamma g_N) = 1$. We have

$$\psi(z_1, \dots, z_N) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma g_{\sigma(1)}(z_1) \cdots g_{\sigma(N)}(z_N)$$

and

$$\int \psi(z_1 \dots, z_N)\, \bar{\phi}(z_1)\, dz_1 = \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma \int g_{\sigma(1)}(z_1) \cdots g_{\sigma(N)}(z_N)\, \bar{\phi}(z_1)\, dz_1$$

$$= \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma g_{\sigma(2)}(z_2) \cdots g_{\sigma(N)}(z_N) \left(\cancel{g_{\sigma(1)}, \phi}\right) (\phi, g_{\sigma(1)})$$

Thus

$$\left| \int \psi(z_1, \dots, z_N)\, \bar{\phi}(z_1) dz_1 \right|^2 = \frac{1}{N!} \sum_{\sigma, \sigma' \in S_N} (-1)^\sigma (-1)^{\sigma'} (\phi, g_{\sigma(1)}) \overline{(\phi, g_{\sigma'(1)})}$$

$$\times\, g_{\sigma(2)}(z_2)\, \overline{g_{\sigma'(2)}}(z_2) \cdots g_{\sigma(N)}(z_N)\, \overline{g_{\sigma'(N)}}(z_N)$$

Hence

$$\int \left| \int \psi(z_1 \dots, z_N)\, \bar{\phi}(z_1)\, dz_1 \right|^2 dz_2 \dots dz_N = \frac{1}{N!} \sum_{\sigma, \sigma' \in S_N} (-1)^\sigma (-1)^{\sigma'} (\phi, g_{\sigma(1)}) \overline{(\phi, g_{\sigma'(1)})}$$

$$\times\, \delta_{\sigma(2), \sigma'(2)} \cdots \delta_{\sigma(N), \sigma'(N)}$$

For $\phi = g_N$, we have

$$\int \left| \int \psi(z_1 \dots, z_N)\, \bar{\phi}(z_1)\, dz_1 \right|^2 dz_2 \dots dz_N = \frac{1}{N!} \sum_{\substack{\sigma \in S_N \\ \sigma(1) = N}} \underbrace{|(\phi, g_{\sigma(1)})|^2}_{1} = \frac{(N-1)!}{N!} = \frac{1}{N}.$$

Thus, by (19), $(g_N, \gamma g_N) = N \int \left| \int \psi(z_1 \dots, z_N)\, \bar{\phi}(z_1)\, dz_1 \right|^2 dz_2 \dots dz_N = 1$.

16 Two important formula

$$C_{N,g_1}(\mathrm{Alt}(g_1\otimes\cdots\otimes g_N)) = \mathrm{Alt}(g_2\otimes\cdots\otimes g_N)$$

$$C^{\dagger}_{N,g_1}(\mathrm{Alt}(g_2\otimes\cdots\otimes g_N)) = \mathrm{Alt}(g_1\otimes\cdots\otimes g_N)$$

where

$$\mathrm{Alt}(f_1\otimes\cdots\otimes f_N) = \frac{1}{\sqrt{N!}}\sum_{\sigma\in S_N} f_{\sigma(1)}(z_1)\cdots f_{\sigma(N)}(z_N)$$

17 Proof of theorem 3.2, the forward part

Let γ be a self-adjoint ~~semi-defi~~ positive semidefinite operator on $L^2(\mathbb{R}^3;\mathbb{C}^q)$ and $\mathrm{Tr}\,\gamma = N$, and $\gamma \le \mathbb{I}$. We will show that there exists an N-particle density matrix Γ on $L^2(\mathbb{R}^{3N};\mathbb{C}^{q^N})$ such that $\gamma = N\,\mathrm{Tr}^{(N-1)}\Gamma$. Let $\lambda_1^{(1)} \ge \lambda_2^{(1)} \ge \lambda_3^{(1)} \ge \cdots \ge 0$ be the sequence of eigenvalues of γ. Then

$$\sum_{j=1}^{\infty} \lambda_j^{(1)} = N$$

Because $\gamma \le \mathbb{I}$, each $\lambda_j^{(1)}$ is less or equal 1. Let f_j be an eigenfunction of γ corresponding to eigenvalue $\lambda_j^{(1)}$ such that $\{f_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^3;\mathbb{C}^q)$. The kernel of Γ must satisfy

$$\gamma\phi(z_1) = N\int \Gamma(z_1,z_2,\ldots,z_N;\, z_1',z_2,\ldots,z_N)\,\phi(z_1')\,dz_1'\,dz_2\ldots dz_N$$

$$\forall \phi \in L^2(\mathbb{R}^3;\mathbb{C}^q)$$

It is equivalent that Γ satisfies

$$\gamma f_j(z_1) = N \int \Gamma(z_1, z_2, \dots, z_N; z_1', z_2, \dots, z_N) f_j(z_1') dz_1' dz_2 \dots dz_N$$

or

$$\lambda_j^{(1)} f_j(z_1) = N \int \Gamma(z_1, z_2 \dots, z_N; z_1', z_2 \dots, z_N) f_j(z_1') dz_1' dz_2 \dots dz_N \qquad (21)$$

Put $k_{i_1 \dots i_N} = Alt(f_{i_1} \otimes \dots \otimes f_{i_N})$

$$= \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma f_{i_{\sigma(1)}}(z_1) \cdots f_{i_{\sigma(N)}}(z_N)$$

for each $i_1 < i_2 < \dots < i_N$. We will find Γ of the form

$$\Gamma = \sum_{i_1 < \dots < i_N} \lambda_{i_1 \dots i_N} \Gamma_{k_{i_1 \dots i_N}}$$

where each $\lambda_{i_1 \dots i_N} \geq 0$ and $\sum_{i_1 < \dots < i_N} \lambda_{i_1 \dots i_N} = 1$. Then the kernel of Γ is

$$\Gamma(\underline{z}, \underline{z}') = \sum_{i_1 < \dots < i_N} \lambda_{i_1 \dots i_N} k_{i_1 \dots i_N}(z_1, z_2, \dots, z_N) \overline{k_{i_1 \dots i_N}}(z_1', z_2, \dots, z_N)$$

Then

$$(21) = N \sum_{i_1 < \dots < i_N} \lambda_{i_1 \dots i_N} \int k_{i_1 \dots i_N}(z_1, z_2 \dots, z_N) \overline{k_{i_1 \dots i_N}}(z_1', z_2, \dots, z_N) f_j(z_1') dz_1' dz_2 \dots dz_N$$

$$= N \sum_{i_1 < \dots < i_N} \lambda_{i_1 \dots i_N} \int \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} (-1)^\sigma f_{i_{\sigma(1)}}(z_1) \cdots f_{i_{\sigma(N)}}(z_N) \frac{1}{\sqrt{N!}} \sum_{\sigma' \in S_N} (-1)^{\sigma'} \overline{f}_{i_{\sigma'(1)}}(z_1') \overline{f}_{i_{\sigma'(2)}}(z_2) \cdots \overline{f}_{i_{\sigma'(N)}}(z_N)$$

$$\times f_j(z_1') dz_1' dz_2 \dots dz_N$$

$$= \frac{1}{N!} N \sum_{i_1 < \dots < i_N} \lambda_{i_1 \dots i_N} \sum_{\sigma, \sigma' \in S_N} (-1)^\sigma (-1)^{\sigma'} \int f_j(z_1') \overline{f}_{i_{\sigma'(1)}}(z_1') f_{i_{\sigma(2)}}(z_2) \overline{f}_{i_{\sigma'(2)}}(z_2) \cdots f_{i_{\sigma(N)}}(z_N) \overline{f}_{i_{\sigma'(N)}}(z_N)$$

$$f_{i_{\sigma(1)}}(z_1) dz_1' dz_2 \dots dz_N$$

$$= \frac{1}{(N-1)!} \sum_{i_1<\dots<i_N} \lambda_{i_1\dots i_N} \sum_{\sigma,\sigma'\in S_N} (-1)^{\sigma}(-1)^{\sigma'} (f_{i_{\sigma'(1)}}, f_j)(f_{i_{\sigma'(2)}}, f_{i_{\sigma(2)}})\dots(f_{i_{\sigma'(N)}}, f_{i_{\sigma(N)}})\, f_{i_{\sigma(1)}}(x_1)$$

Note that if $\sigma'(i)=\sigma(i)$ $\forall i=2,\dots,N$ then $\sigma'(i)=\sigma(i)$ $\forall i=1,\dots N$; i.e. $\sigma'=\sigma$.

Thus

$$\lambda_j^{(1)} f_j(x_1) = \frac{1}{(N-1)!} \sum_{i_1<\dots<i_N} \lambda_{i_1\dots i_N} \sum_{\sigma\in S_N} (f_{i_{\sigma(1)}}, f_j)\, f_{i_{\sigma(1)}}(x_1) \qquad (22)$$

(22) is equivalent to

$$\lambda_j^{(1)} (f_j, f_r) = \frac{1}{(N-1)!} \sum_{i_1<\dots<i_N} \lambda_{i_1\dots i_N} \sum_{\sigma\in S_N} (f_{i_{\sigma(1)}}, f_j)(f_{i_{\sigma(1)}}, f_r), \quad \forall r\in\mathbb{N}. \qquad (23)$$

If $r\neq j$ then the right hand side of (23) automatically vanishes and is thus equal to the left hand side. If $r=j$, (23) becomes

$$\lambda_j^{(1)} = \frac{1}{(N-1)!} \sum_{i_1<\dots<i_N} \lambda_{i_1\dots i_N} \sum_{\sigma\in S_N} (f_{i_{\sigma(1)}}, f_j)^2$$

$$= \frac{1}{(N-1)!} \sum_{\substack{i_1<\dots<i_N \\ j\in\{i_1,\dots,i_N\}}} \lambda_{i_1\dots i_N} \underbrace{\sum_{\substack{\sigma\in S_N \\ \sigma(1)=s \\ \text{where } i_s=j}} 1}_{(N-1)!}$$

$$= \sum_{\substack{i_1<\dots<i_N \\ j\in\{i_1,\dots,i_N\}}} \lambda_{i_1\dots i_N}$$

Let χ_A be the characteristic function of a set A, i.e.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x\in A \\ 0 & \text{if } x\notin A \end{cases}$$

Then

$$\sum_{i_1<\dots<i_N} \lambda_{i_1\dots i_N} \chi_{\{i_1\dots i_N\}}(j) = \lambda_j^{(1)}$$

or

$$\sum_{\substack{A\subset \mathbb{N} \\ |A|=N}} \lambda_A \chi_A(j) = \lambda_j^{(1)}$$

Therefore, all what we need to show is that the sequence $(\lambda_j^{(1)})$ is a convex combination of characteristic functions of N elements in $\mathbb{N}$.

This is justified by the first statement of Remark 3.2.