Connection between two formalisms of logarithm and exponential functions

Last time: we defined logarithm of complex variables by power series. The method is the analytic extension. Then we define the exponential function as the inverse of logarithm. Now we try to connect two formalisms to see that they are equivalent in sense that we can choose to begin with one way of defining logarithm and then reconstruct all results of the other way. Let's restate the results obtained last time.

1) We defined $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$

2) We defined $\log : \Omega \to \{w : -\pi < \operatorname{Im} w < \pi\}$, where $\Omega = \mathbb{C} \setminus \{z : \operatorname{Re}(z) \leq 0, \operatorname{Im} z = 0\}$

Then log is bijective, analytic and

(i) $\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots$

(ii) $\log z$ is the maximal analytic extension of the power series on the disk $\{z : |z-1| < 1\}$

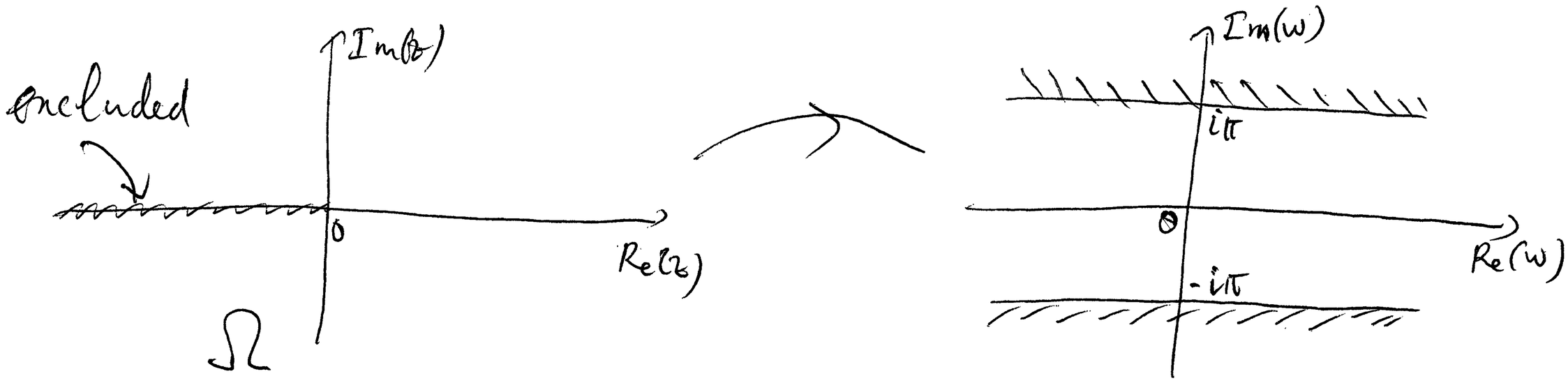
(iii) $\operatorname{Re}(\log z) = \underbrace{\log|z|}_{\text{usual logarithm on } \mathbb{R}}$

(iv) $\log i = \frac{\pi}{2} i$

$\boxed{2}$

(v) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ for all z_1, z_2 having positive real parts.

(vi) $(\log z)' = \dfrac{1}{z}$



3) We defined $g : \{w : -\pi < \operatorname{Im} w < \pi\} \longrightarrow \Omega$ to be the inverse of logarithm. Then we showed that

(i) g is bijective and analytic

(ii) $g'(w) = g(w)$ $\forall w$ in the domain

(iii) $g(0) = 1$

<u>Next goals</u>

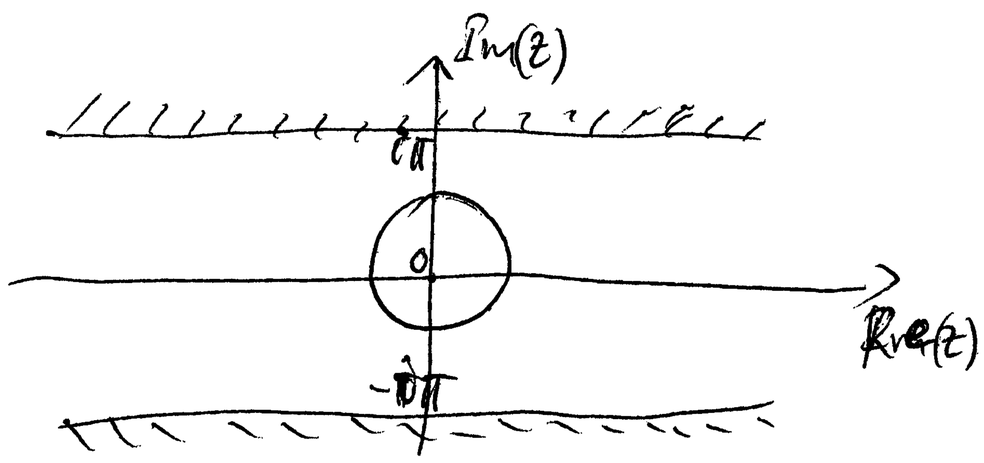
1. Show that g has an analytic extension over $\mathbb{C}$, called $h : \mathbb{C} \to \mathbb{C}$. Then with π given by 1), we show that h is periodic with period $2\pi i$.

2. Create a family of analytic functions from the logarithm (these are called "single-valued branches" in Ahlfors' when we started from the exponential function).

To do so, we will follow the following steps:

4) Because the disk $\{w : |w| < 1\}$ is contained in the domain of g, g can be represented as a power series $g(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n$ on that disk. By the condition $g'(z) = g(z)$, we get

$$g(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \quad \forall z \in \{w : |w| < 1\}$$



5) The above series has radius of convergence $R = \infty$. Thus, we can define $h : \mathbb{C} \to \mathbb{C}$, $h(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$

Then h is the unique analytic extension of g over $\mathbb{C}$. Now

~~6) h has~~ h is exactly the familiar exponential function but we still use the name "h" instead of "exp" to avoid possible confusions of what we already have and what we have to prove.

6) h has the following properties:

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(i) h is injective on $\{w : -\pi < \operatorname{Im} w < \pi\}$

(ii) $h(z_1+z_2) = h(z_1)h(z_2) \quad \forall z_1, z_2 \in \mathbb{C}$

(iii) $h(z+i2\pi) = h(z), \quad \forall z \in \mathbb{C}$

(iv) $\pm i2\pi$ are the only prime periods of h. Any other period of h must be an integer multiple of $\pm i2\pi$

(v) $h(i\pi) = -1$, and $|h(i\theta)| = 1 \quad \forall \theta \in \mathbb{R}$.

The map $\theta \in (-\pi, \pi] \longmapsto h(i\theta) \in \{z : |z| = 1\}$ is bijective. As a result of this, we can define $\phi : \mathbb{C}\setminus\{0\} \longrightarrow (-\pi, \pi]$

$$\phi(z) = -i h^{-1}\left(\frac{z}{|z|}\right)$$

ϕ will be called the <u>argument</u> of z. (See page 9 for more comment on argument)

(vi) The map $R : (-\pi, \pi] \times \mathbb{C} \longrightarrow \mathbb{C}$

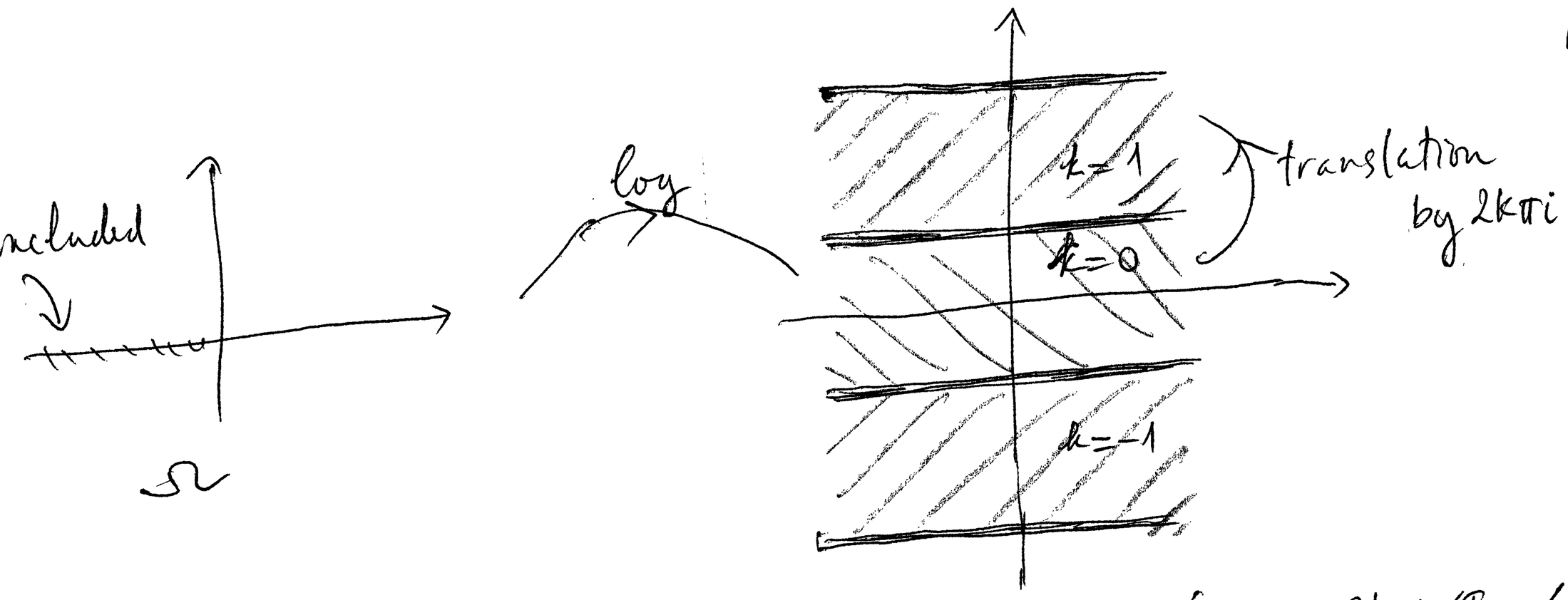
$$(\theta, z) \longmapsto h(i\theta)z$$

is the <u>rotation</u> on $\mathbb{C}$ in sense that for every $\theta \in (-\pi, \pi]$, we have

- $|R_\theta(z)| = |z|$
- $R_\theta(tz) = tR_\theta(z) \quad \forall t \in \mathbb{R}$

Moreover, <u>all</u> rotations on $\mathbb{C}$ must be of this form.

7) Explain the phenomenon of multiple-valued of logarithm in Ahlfors' formalism: From log, we can construct many analytic bijective function from Ω to a <u>fundamental domain</u> of h.

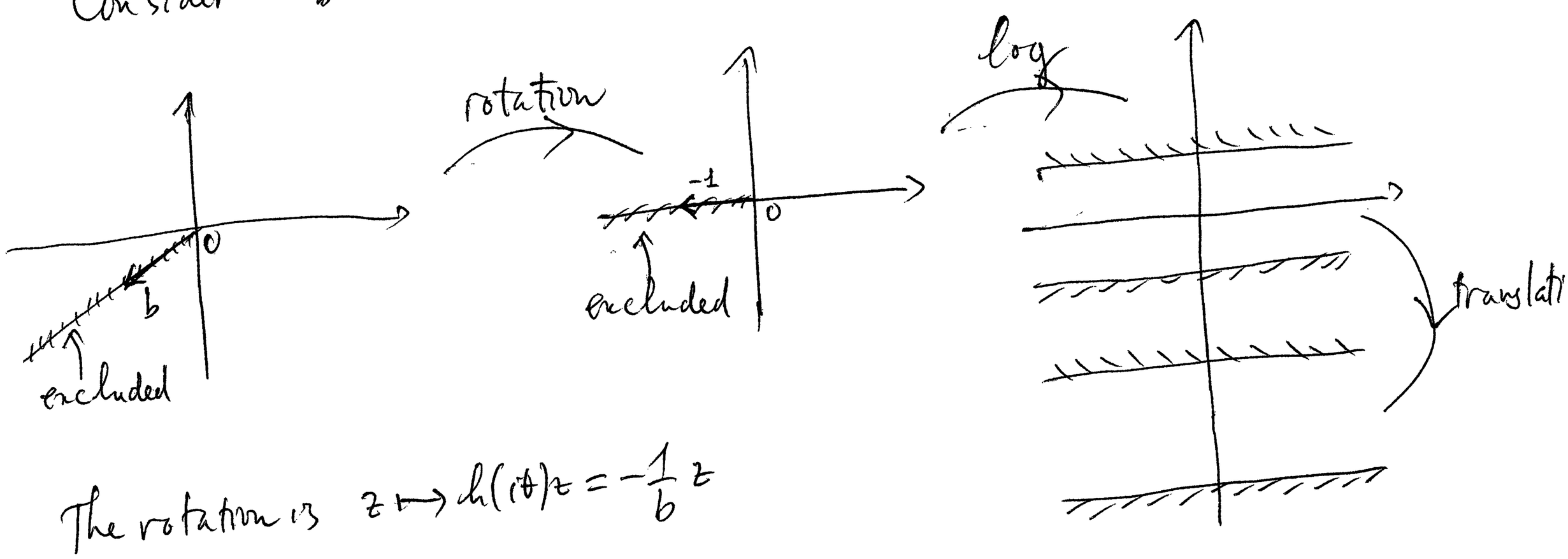


For each $k \in \mathbb{Z}$, we have the function $f_k : \Omega \to \{w : -\pi + 2k\pi i < \operatorname{Im} w < \pi + 2k\pi i\}$

$$f_k(z) = \log z + 2k\pi i$$

8) We can create even more analytic bijective functions from logarithm.

Consider $\Omega_b = \mathbb{C} \setminus \{z : z = bt, t \geq 0\}$ where $|b| = 1$.



The rotation is $z \mapsto h(i\theta)z = -\frac{1}{b} z$

The translation is $w \mapsto w + 2k\pi i$

Therefore for each b with $|b| = 1$ and $k \in \mathbb{Z}$, we have an analytic bijective function $f : \Omega_b \to \{w : -\pi + 2k\pi < \operatorname{Im} w < \pi + 2k\pi\}$

$$f(z) = \log\left(-\frac{z}{b}\right) + 2k\pi i$$

And this is the full picture of multiplicity and analytic branch cut in Ahlfors' formalism.

Details of proofs

Step 5 As a property of power series, h is analytic on $\mathbb{C}$. Since g and h coincide on the open disk $\{w : |w| < 1\}$, they coincide on the whole domain of g. Moreover, if k is another analytic extension of g over $\mathbb{C}$, then k and h coincide on the disk $\{w : |w| < 1\}$. Thus they coincide everywhere on the plane.

Step 6

(i) Because $h(z) = g(z)$ $\forall z \in \{w : -\pi < \operatorname{Im} w < \pi\}$, and g is injective in that domain, so is h.

(ii) Because g, h coincide in the domain of g, we have $h'(z) = g'(z)$
$= g(z)$

in the domain of g. That means h' is also an analytic extension of g over $\mathbb{C}$. Thus h and h' are the same, i.e. $h(z) = h'(z)$ $\forall z \in \mathbb{C}$.

~~For each $c \in \mathbb{C}$, We put~~ For each $c \in \mathbb{C}$, we define $k(z) = h(z)h(c-z)$, $\forall z \in \mathbb{C}$.

Then $k'(z) = h'(z)h(c-z) - h(z)h'(c-z) = h(z)h(c-z) - h(z)h(c-z) = 0$.

Thus k is a constant function. Therefore

$$h(z)h(c-z) = k(z) = k(0) = \underbrace{h(0)}_{g(0)=1}h(c) = h(c)$$

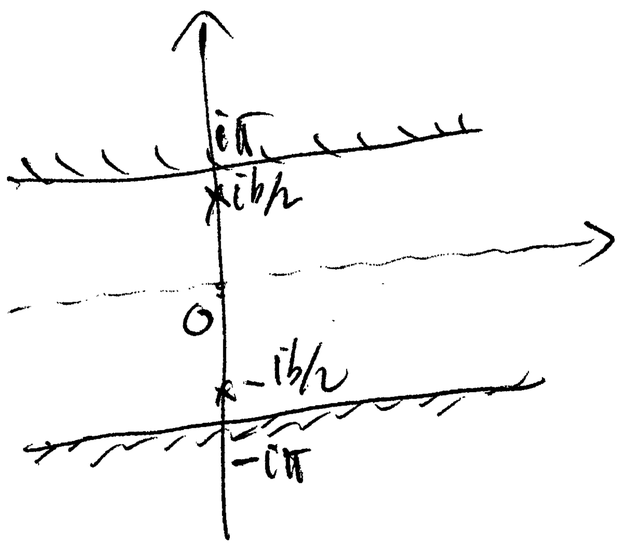
Thus $h(z_1 + z_2) = h(z_1) + h(z_2)$ $\forall z_1, z_2 \in \mathbb{C}$.

(iii) By part (ii), we have $h(z+i2\pi) = h(z) * h(i2\pi)$. We only need to show $h(i2\pi) = 1$. We have

$$h(i2\pi) = h\left(i\frac{\pi}{2} + i\frac{\pi}{2} + i\frac{\pi}{2} + i\frac{\pi}{2}\right) = \left(h\left(i\frac{\pi}{2}\right)\right)^4 = \left(g\left(i\frac{\pi}{2}\right)\right)^4$$

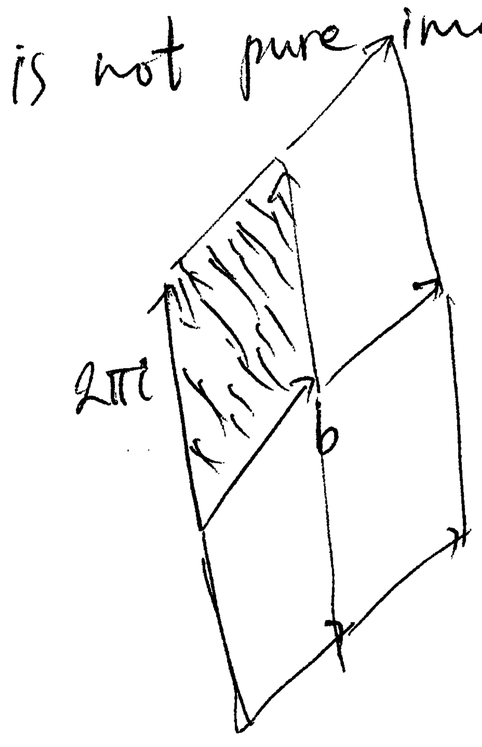
By (iv) of part 2), we already have $\log i = \frac{\pi}{2} i$. Thus $g\left(\frac{\pi}{2} i\right) = i$. Therefore $h(i2\pi) = i^4 = 1$

(iv) First, suppose by contradiction that there exists $0 < b < 2\pi$ such that $h(z+ib) = h(z)$ $\forall z \in \mathbb{C}$.



Then $i\frac{b}{2}$ and $-i\frac{b}{2}$ belongs to the domain of g. Thus by part (i), h is injective in that domain and $h\left(i\frac{b}{2}\right) \neq h\left(-i\frac{b}{2}\right)$. This contradict our assumption that ib is a period of h.

Second, suppose that there exists $b \in \mathbb{C}$, another period of h, that is not pure imaginary. This is the case of double-periodicity.



Then there exists a finite parallellogram that contai all value of function h. This is a contradiction because h is injective in $\{w : -\pi < \operatorname{Im} w < \pi\}$, whic is an unbounded domain.

(v) $h(i\pi) = h\left(i\frac{\pi}{2} + i\frac{\pi}{2}\right) = 2h\left(i\frac{\pi}{2}\right) = 2g\left(i\frac{\pi}{2}\right)$.

Because $\log i = i\frac{\pi}{2}$, we have $g\left(i\frac{\pi}{2}\right) = i$. Thus $h(i\pi) = i^2 = -1$.

We will show that $|h(i\theta)| = 1$ $\forall \theta \in \mathbb{R}$. Since h is periodic of period $2\pi i$, it suffices to consider only $\theta \in (-\pi, \pi]$. Since we already know that $|h(i\pi)| = |-1| = 1$, we will only consider $\theta \in (-\pi, \pi)$.

Then $h(i\theta) = g(i\theta)$. We have

$$\log g(i\theta) = i\theta \quad \text{(because } g \text{ is the inverse function of log)}$$

Moreover, by part 2), (iii), $\mathrm{Re}(\log g(i\theta)) = \log |g(i\theta)|$.

Therefore $\log |g(i\theta)| = 0$, or $|g(i\theta)| = 1$

therefore $|h(i\theta)| = 1$

Next, we'll show that the map $\theta \in (-\pi, \pi] \longmapsto h(i\theta) \in \{z : |z| = 1\}$ is bijective. Because h is injective in $\{w : -\pi < \mathrm{Im}\, w < \pi\}$, we only need to check if $h(i\pi) \neq h(i\theta)$ $\forall \theta \in (-\pi, \pi)$ for the injectivity of h.

We have $h(i\pi) = -1$ and $h(i\theta) = g(i\theta)$ $\forall \theta \in (-\pi, \pi)$.

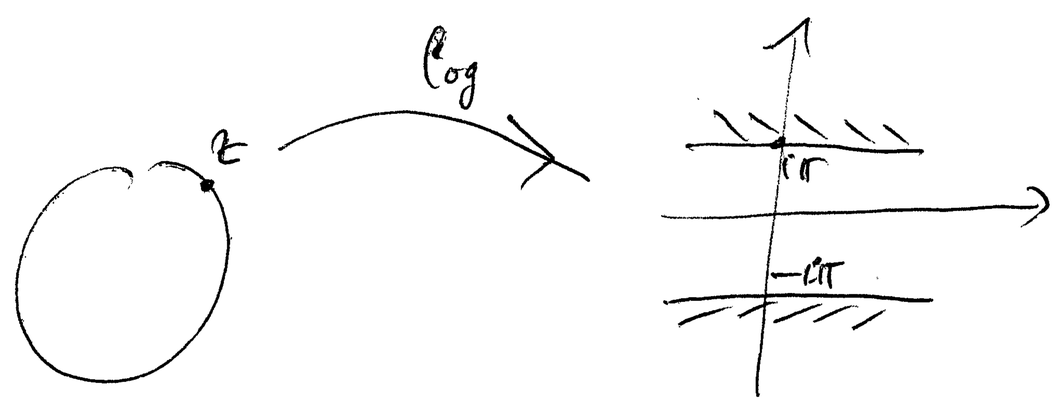
Since $g : \{w : -\pi < \mathrm{Im}\, w < \pi\} \to \Omega$, $g(i\theta) \neq -1$. Thus $h(i\pi) \neq h(i\theta)$ $\forall \theta \in (-\pi, \pi)$

Therefore the map $\theta \in (-\pi, \pi] \to h(i\theta) \in \{z : |z| = 1\}$ is injective.

For surjectivity, we take $z \in \mathbb{C}$ such that $|z| = 1$. If $z = -1$, then we already know that $h(i\pi) = -1$. Otherwise $z \in \Omega$.

Put $\theta = \operatorname{Im}(\log z)$ then $-\pi < \theta < \pi$. We have

$$\operatorname{Re}(\log z) = \log|z| = 1$$



Therefore $\theta = \operatorname{Im}(\log z) = -i \log z$. Thus $i\theta = \log z$, and thus

$$g(i\theta) = g(\log z) = z$$

thus $h(i\theta) = z$. Therefore h is surjective.

Now we can replace 'h' by 'the exponential function' $h(it) = e^{it}$.

We see that $\{e^{i\theta} : -\pi < \theta \leq \pi\} = \{z : |z| = 1\}$. We can say, each point $z \in \{w : |w| = 1\}$ corresponds uniquely to a number $\theta \in (-\pi, \pi]$, called its <u>argument</u>, in the manner $z = e^{i\theta}$.

(vi) $R_\theta(z) = h(i\theta) z = e^{i\theta} z$ has $|R_\theta(z)| = |e^{i\theta} z| = |e^{i\theta}| |z| = |z|$

and $R_\theta(tz) = (e^{i\theta})(tz) = t(e^{i\theta} z) = t R_\theta(z)$.

Thus R_θ, which can be used interchangeably with $e^{i\theta}$, is a rotation on $\mathbb{C}$.

Next, we will show that all rotations on $\mathbb{C}$ must be of the form $z \mapsto e^{i\theta} z$. A rotation on $\mathbb{C}$ is defined as an analytic function $f : \mathbb{C} \to \mathbb{C}$ such that $\begin{cases} |f(z)| = |z| \\ f(tz) = t f(z) \end{cases}$ $\forall z \in \mathbb{C}, t \in \mathbb{R}$

At 0, we have $|f(0)| = |0|$, and thus $f(0) = 0$. ~~We define~~

~~$g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$~~ If $f(z) = -z$, $\forall z \neq 0$ then

~~$g(z) = -i \log \frac{f(z)}{z}$~~ $f(z) = e^{i\pi} z$.

If there exists $z_0 \neq 0$ such that $f(z_0) \neq -z_0$, then $\frac{f(z_0)}{z_0} \neq -1$. Because the map $z \mapsto \frac{f(z)}{z}$ is continuous on $\mathbb{C} \setminus \{0\}$, there exists an open ball $B(z_0, r)$ in $\mathbb{C} \setminus \{0\}$ such that $f(z)/z \neq -1$ for every $z \in B(z_0, r)$.

We define $g : B(z_0, r) \to \mathbb{C}$, $g(z) = -i \log \frac{f(z)}{z}$

Then g is analytic on $B(z_0, r)$. Moreover, $\operatorname{Im} g(z) = \operatorname{Im}\left(-i \log \frac{f(z)}{z}\right)$

$$= \operatorname{Re}\left(\log \frac{f(z)}{z}\right) = \log \left|\frac{f(z)}{z}\right|$$

Thus $\operatorname{Im} g(z) = 0$, and therefore $g : B(z_0, r) \to \mathbb{R}$. Because g is analytic it must be a constant (note that $B(z_0, r)$ is connected). Thus the function $f(z)/z$ coincide a constant on an open disk, it must be constant throughout $\mathbb{C} \setminus \{0\}$. Thus, there exists $a \in \mathbb{C}$ s.t. $f(z) = az$ $\forall z \in \mathbb{C}$.

Since $|a| = |f(1)| = 1$, there exists $\theta \in (-\pi, \pi]$ s.t. $a = e^{i\theta}$.