Linear Partial Differential Equations with Constant Coefficients

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1 Introduction

The paper is an investigation of the solvability of linear partial differential equations with constant coefficients in the whole space $\mathbb{R}^n$. The work was motivated by the Poisson's equation

$$\Delta u = f \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where f is a given smooth function. Here the smoothness can be continuity, Lipschitz continuity, C^1, C^2... Its meaning will be made clear when needed. A natural question is whether Problem (1.1) has a solution in $C^2(\mathbb{R}^n)$. For $n = 1$, it certainly has a classical solution obtained by taking antiderivatives of both sides twice. In this case, f only needs to be continuous. For $n \geq 2$, however, the question is nontrivial. One reason is that the continuity of f no longer guarantees the existence of u in $C^2(\mathbb{R}^2)$. Indeed, consider $n = 2$: put $u(x_1, x_2) = x_1 x_2 (-\log|x|)^\gamma \chi(x_1, x_2)$ where $\gamma \in (0, 1)$ and χ is some smooth cutoff function supported in the unit disk; define $f(x_1, x_2) = \Delta u(x_1, x_2) \in C(\mathbb{R}^2)$; then $u \notin C^2(\mathbb{R}^2)$ and $\Delta u = f$ in $\mathbb{R}^2$; any function $v \in C^2(\mathbb{R}^2)$ such that $\Delta v = f$ would satisfy $\Delta(v - u) = 0$, which would imply $v - u \in C^2(\mathbb{R}^2)$; this is a contradiction. Another reason is that the Newtonian potential does not always exist unless f has certain decay at infinity.

The paper does not attempt to give a criterion on the smoothness of f so that Problem (1.1) is solvable. We simply assume that f is as smooth as we want. Instead, without any assumption on the decay of f at infinity, we show the existence of a classical solution to Problem (1.1). Our method is similar to the method of Mittag-Leffler for constructing a meromorphic function with infinitely many prescribed poles. It is done in Proposition 2.2 of Section 2. In the same section, some properties of the solutions are discussed, for example the lack of an *a priori* estimate (Proposition 2.3) and the order of growth of the solutions at infinity (Proposition 2.4). A consequence is that every smooth vector field in $\mathbb{R}^3$ has a Helmholtz decomposition (Proposition 2.5).

It is then natural to ask if the existence result is still true for the heat equation, wave equation or even a general linear partial differential equation with constant coefficients

$$P(D)u = f \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where P is a nonzero polynomial of n variables and f is a given smooth function. Specifically, given a function $f \in C^\infty(\mathbb{R}^n)$, does there exist $u \in C^\infty(\mathbb{R}^n)$ such that $P(D)u = f$? The answer is yes and was proved quite early by L. Ehrenpreis [Ehr54, Theorem 10] and B. Malgrange [Mal56, Theorem 3, p.294]. Interestingly, these are also the papers in which the authors independently proved a famous result saying that every linear partial differential equation with constant coefficients has a fundamental solution. It was later known as the Malgrange-Ehrenpreis theorem. J.P. Rosay [Ros91] gave an elementary proof of this theorem without using Fourier transform or any complex functions. The proof is based on Hörmander's inequality, the Riesz representation theorem for L^2, and little background in the space of distributions. Section 3 of this paper is a detail exposition for Rosay's paper.

Following the theorem on the existence of a fundamental solution, Ehrenpreis and Malgrange established several general theorems which imply the following identities.

1°. $P(D)(\mathscr{D}'(\mathbb{R}^n)) = \mathscr{D}'(\mathbb{R}^n)$ [Ehr56, Theorem 3].

2°. $P(D)(C^\infty(\mathbb{R}^n)) = C^\infty(\mathbb{R}^n)$ [Ehr54, Theorem 10], [Mal56, Theorem 3].

3°. $P(D)(C^k(\mathbb{R}^n)) \supset C^{k+[\frac{n}{2}]+1}(\mathbb{R}^n)$ [Mal56, Remark of Theorem 3].

Later, Hörmander [Hor58] showed that

4°. $P(D)(\mathscr{S}'(\mathbb{R}^n)) = \mathscr{S}'(\mathbb{R}^n)$,

where $\mathscr{S}'(\mathbb{R}^n)$ is the space of tempered distributions in $\mathbb{R}^n$. The theory of linear partial differential operators was largely developed during the 1950s and 1960s. For example, 2° and 3° are still true when $\mathbb{R}^n$ is replaced by a convex open subset Ω [Mal56]; with the same replacement, 1° is still true [Mal59]; Hörmander [Hor62] introduced the notion of strong $P(D)$-convexity and showed that $P(D)(\mathscr{D}'(\Omega)) = \mathscr{D}'(\Omega)$ if and only if the open set Ω is strongly $P(D)$-convex; Agranovich [Agr61, Theorem 5] gave a constructive proof of 2°: an explicit solution of Problem (1.2) was given as a contour integral in $\mathbb{C}^n$, called *Hörmander's steps.* Trèves [Tre67, p.61] pointed out that there was no need to restrict our consideration of 2° to $\mathbb{R}^n$ or its open subsets: the existence result extends trivially to C^{∞}-manifolds. Much of the literature on linear partial differential operators during the 1950s and 1960s is systematized in the book [Tre66]. There the author showed 1°, 2°, 3°, 4° and other identities for C^{∞}-manifolds that satisfy the countability at infinity, $P(D)$-convexity and some other conditions. It is interesting to note that the identity

5°. $P(D)(H) = H$,

where H is the space of all real analytic functions in $\mathbb{R}^n$, is false in general. E. De Giorgi and L. Cattabriga [GC71] showed that 5° is true for $n = 2$. L. Piccinini [Pic73] showed that 5° is false if $n \geq 3$. One of his counterexamples is the heat operator $P(x) = x_n - x_1^2 - \ldots - x_{n-1}^2$. Hörmander [Hor73] gave a necessary and sufficient condition for the polynomial $P(x)$ in order to get 5°. Particularly, if $P(0) = 0$ and $P(x) > 0$ for all $x \neq 0$ then 5° is true.

Our concern in this paper is completely restricted to proving the identity 2°. Malgrange [Mal56] proved the existence of solutions to Problem (1.2) by a method analogous to the method we use for the Poisson's equation in Section 2. The key step is to show that a function v satisfying $P(D)v = 0$ in an open bounded subset Ω of $\mathbb{R}^n$ is the limit of a sequence (v_m) satisfying $P(D)v_m = 0$ in $\mathbb{R}^n$. If $P(D)$ is the Laplacian, this result is known as Walsh's theorem [Gar95, p.8] (quoted in Proposition 2.1). For a general operator $P(D)$, Malgrange [Mal56, Theorem 2, p.292] showed that v is the limit of a sequence of linear combinations of exponential polynomials. After this step, one can construct a solution in $C^{\infty}(\mathbb{R}^n)$ to Problem (1.2) by the Mittag-Leffler approximation procedure. Trèves [Tre66] gave a different method to achieve the first step without resorting the exponential polynomials. His method requires some background in the topology of the dual space of $C^{\infty}(\mathbb{R}^n)$.

Section 4 of the paper is an exposition for [Tre66, Chapter 1]. Much simplification is made because we are considering Problem (1.2) in $\mathbb{R}^n$ instead of a C^{∞}-manifold. However, as mentioned earlier, we do not lose any key ideas due to this restriction. Section 5 consists of two simple applications of 2°: the solvability of a system of linear differential equations and the solvability of linear Stokes equations without the initial condition. Section 6 is a collection of basic properties of topological vector spaces (TVS), test-functions and distributions in $\mathbb{R}^n$ that are used in this paper.

2 Poisson's equation in $\mathbb{R}^n$

Let us recall the following definitions.

In [GT98, p.52], a function f defined on an open subset Ω of $\mathbb{R}^n$ is said to be *locally Hölder continuous* with exponent $\alpha \in (0, 1]$ if it is Hölder continuous with that exponent in every compact subset of Ω, i.e. the quantity

$$[f]_{\alpha;D} = \sup_{\substack{x,y\in D \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite for every compact subset D of Ω. By this definition, it is clear that every function in $C^1(\mathbb{R}^n)$ is locally Hölder continuous with exponent $\alpha = 1$.

Next, the function $\Gamma : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$,

$$\Gamma(z) = \begin{cases} \dfrac{1}{2\pi} \log |z|, & n = 2 \\ \dfrac{1}{n(2 - n)|B_1|} |z|^{2-n}, & n \geq 3 \end{cases}$$

where $|B_1|$ denotes the volume of the unit ball in $\mathbb{R}^n$, is called *the fundamental solution of Laplace's equation.*

Given a function f defined on an open bounded subset Ω of $\mathbb{R}^n$. Suppose that f is bounded in Ω. Then the function

$$v(x) = \int_\Omega \Gamma(x - y)f(y)dy \quad \forall x \in \Omega$$

is well-defined and called the *Newtonian potential* of f on Ω.

Proposition 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $0 < a < b < c < \infty$. Denote $A_{b,c} = \{x \in \mathbb{R}^n : b \leq |x| \leq c\}$ and $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ for every $r > 0$. Then for every $\epsilon > 0$ and for every function $f : \mathbb{R}^n \to \mathbb{R}$ that is locally Hölder continuous with exponent $\alpha \in (0, 1]$ in $\mathbb{R}^n$ and supported in $A_{b,c}$, there exists a function $u_\epsilon \in C^2(\mathbb{R}^n)$ such that $\Delta u_\epsilon = f$ in $\mathbb{R}^n$ and $|u_\epsilon(x)| \leq \epsilon$ in $\overline{B_a}$.*

Proof. Let $v : \mathbb{R}^n \to \mathbb{R}$ be the Newtonian potential of f, namely

$$v(x) = \int_{\mathbb{R}^n} \Gamma(x - y)f(y)dy \quad \forall x \in \mathbb{R}^n.$$

Then $v \in C^2(\mathbb{R}^n)$ and $\Delta v = f$ in $\mathbb{R}^n$ by Lemma 2.6. We have $\Delta v = 0$ in B_b because $f(x) = 0$ in B_b. Thus v is a harmonic function in B_b, which is an open set containing $\overline{B_a}$. We know that the complement of $\overline{B_a}$ in $\mathbb{R}^n$ is connected. Walsh's theorem in [Gar95, p.8] states that:

> Let K be a compact subset of $\mathbb{R}^n$, $n \geq 2$, such that $\mathbb{R}^n\backslash K$ is connected. Then for each function u which is harmonic on an open set containing K and for each positive number ϵ, there exists a harmonic polynomial w such that $|w - u| < \epsilon$ in K.

For $n = 2$, this is known as Runge's theorem [GK06, p.363]. Applying Walsh's theorem for $K = \overline{B_a}$ and $u = v$, we conclude that for every $\epsilon > 0$, there exists a harmonic polynomial v_ϵ such that $|v_\epsilon - v| < \epsilon$ in $\overline{B_a}$. Define

$$u_\epsilon(x) = v(x) - v_\epsilon(x) \quad \forall x \in \mathbb{R}^n.$$

Then $u_\epsilon \in C^2(\mathbb{R}^n)$ and $\Delta u_\epsilon = \Delta v - \Delta v_\epsilon = f$ in $\mathbb{R}^n$. Moreover, $|u_\epsilon(x)| = |v(x) - v_\epsilon(x)| \leq \epsilon$ for all $x \in \overline{B_a}$. □

Proposition 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Hölder continuous with exponent $\alpha \in (0, 1]$ in $\mathbb{R}^n$ where $n \geq 2$. Then there exists a function $u \in C^2(\mathbb{R}^n)$ such that $\Delta u = f$ in $\mathbb{R}^n$.*

Proof. Denote

$$\begin{aligned} A_0 &= \{x \in \mathbb{R}^n : |x| < 2\}, \\ A_k &= \{x \in \mathbb{R}^n : k < |x| < k+2\} \quad \forall k \in \mathbb{N}. \end{aligned}$$

Then $\{A_k : k = 0, 1, 2\ldots\}$ is an open cover of $\mathbb{R}^n$. Let $\{\phi_k : k = 0, 1, 2\ldots\}$ be a smooth partition of unity subordinate to this cover. Put $f_k(x) = f(x)\phi_k(x)$ for all $x \in \mathbb{R}^n$ and $k = 0, 1, 2, \ldots$ Because f and ϕ_k are locally Hölder continuous with exponents ≤ 1 in $\mathbb{R}^n$, we conclude that f_k is also locally Hölder continuous with exponent ≤ 1 in $\mathbb{R}^n$ by Lemma 2.7. Morever, each f_k is supported in A_k. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be the Newtonian potential of f_0, namely

$$u_0(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f_0(y) dy \quad \forall x \in \mathbb{R}^n.$$

Then $u_0 \in C^2(\mathbb{R}^n)$ and $\Delta u_0 = f_0$ in $\mathbb{R}^n$ by Lemma 2.6.

For each $k \in \mathbb{N}$, we have $\operatorname{supp} f \subset A_k \subset A_{k,k+2}$, which is the closed annulus whose inner and outer radii are k and $k+2$ respectively. Then by Proposition 2.1, there exists a function $u_k \in C^2(\mathbb{R}^n)$ such that

$$\begin{cases} \Delta u_k = f_k & \text{in } \mathbb{R}^n, \\ |u_k(x)| \leq \frac{1}{k^2} & \text{in } B_{k-\frac{1}{2}}, \end{cases}$$

where B_r denotes the open ball of radius r centered at the origin.

For each compact subset A of $\mathbb{R}^n$, there exists a number $k_0 \in \mathbb{N}$ such that $A \subset B_{k_0}$. Thus $A \subset B_{k-\frac{1}{2}}$ for all $k > k_0$. Hence,

$$|u_k(x)| \leq \frac{1}{k^2} \quad \forall x \in B_{k-\frac{1}{2}}, \forall k > k_0.$$

This implies that the series $\sum_{k=0}^{\infty} u_k(x)$ converges uniformly on A. We thus conclude that the series $\sum_{k=0}^{\infty} u_k(x)$ converges uniformly on every compact subset of $\mathbb{R}^n$ to a (continuous) function $u : \mathbb{R}^n \to \mathbb{R}$.

We have $\Delta u_k = f_k = 0$ in B_k. Thus u_k is a harmonic function in B_k. Theorem 2.10 in [GT98, p.23] states that:

Let u be harmonic in Ω [which is an open subset of $\mathbb{R}^n$] and let Ω' be any compact subset of Ω. Then for any multi-index α we have

$$\sup_{\Omega'} |D^\alpha u| \leq \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_{\Omega} |u|,$$

where $d = \operatorname{dist}(\Omega', \partial\Omega)$.

Applying this theorem for $u_k, B_{k-\frac{1}{2}}, B_{k-1}$ in place of u, Ω, Ω' respectively, we get

$$\sup_{B_{k-1}} |D^\alpha u_k| \leq \left(\frac{n|\alpha|}{1/2}\right)^{|\alpha|} \sup_{B_{k-\frac{1}{2}}} |u_k| \quad \forall k \geq 2.$$

For $1 \leq |\alpha| \leq 2$, we have

$$\sup_{B_{k-1}} |D^\alpha u_k| \leq (2n|\alpha|)^{|\alpha|} \sup_{B_{k-\frac{1}{2}}} |u_k| \leq \frac{(4n)^2}{k^2} \quad \forall k \geq 2.$$

Therefore,

$$|D^\alpha u_k(x)| \leq \frac{(4n)^2}{k^2} \quad \forall x \in B_{k-1},\ \forall 1 \leq |\alpha| \leq 2,\ \forall k \geq 2.$$

This implies that the series $\sum_{k=0}^\infty D^\alpha u_k(x)$ converges uniformly on every compact subset of $\mathbb{R}^n$ for all multi-indicies $1 \leq |\alpha| \leq 2$. Thus $u \in C^2(\mathbb{R}^n)$ and

$$D^\alpha u(x) = \sum_{k=0}^\infty D^\alpha u_k(x) \quad \forall x \in \mathbb{R}^n,\ \forall 1 \leq |\alpha| \leq 2.$$

In particular,

$$\Delta u(x) = \sum_{k=0}^\infty \Delta u_k(x) = \sum_{k=0}^\infty f_k(x) = f(x) \quad \forall x \in \mathbb{R}^n.$$

$\square$

Proposition 2.3. *Denote $A_{1,2} = \{x \in \mathbb{R}^2 : 1 \leq |x| \leq 2\}$ and $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$ for any $r > 0$. There exists a function $f \in C^\infty(\mathbb{R}^2)$ such that* $\operatorname{supp} f \subset A_{1,2}$ *and that there is no function $u \in C^2(\mathbb{R}^2)$ satisfying simultaneously the following conditions.*

(i) $\Delta u = f$ *in* $\mathbb{R}^2$,

(ii) $u = 0$ *in* B_ϵ *for some* $0 < \epsilon < 1$.

Such a counterexample shows that it is impossible to find an estimate of the form

$$\sup_{B_r} |u_f| \leq C(r) \sup_{B_r} |f| \quad \forall r > 0,\ \forall f \in C_0^\infty(\mathbb{R}^n),$$

where u_f is a classical solution to the problem $\Delta u = f$ in $\mathbb{R}^n$.

Proof of Proposition 2.3. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\begin{cases} \eta(t) = 0, & |t| \geq 1 \\ \eta(t) = 1, & |t| \leq \frac{1}{2} \\ 0 \leq \eta(t) \leq 1, & \text{otherwise.} \end{cases}$$

Then the function $t \mapsto \eta(2t-3)$ is supported in [1,2] and $\eta(2t-3) = 1$ if $t \in \left[\frac{5}{4}, \frac{7}{4}\right]$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x) = f(x_1, x_2) = \eta(2r-3) \sum_{k=1}^{\infty} \frac{\cos k\theta}{2^k}, \tag{2.1}$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $r \geq 0$, $\theta \in \mathbb{R}$. For each $m = 0, 1, 2, \ldots$ the series $\sum_{k=1}^{\infty} \frac{k^m}{2^k}$ converges. Thus the function

$$g(\theta) = \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{2^k}$$

belongs to $C^\infty(\mathbb{R})$ and

$$\frac{d^m g}{d\theta^m} = \sum_{k=1}^{\infty} \frac{k^m \cos\left(k\theta + \frac{m\pi}{2}\right)}{2^k} \qquad \forall m \in \mathbb{N}.$$

Equation (2.1) is the representation of f in polar coordinates. Such a representation in general may have a singularity at $r = 0$, but it is not the case here because $\eta(2r-3) = 0$ for all $0 \leq r \leq 1$. Thus, $f \in C^\infty(\mathbb{R}^2)$ and $\operatorname{supp} f \subset A_{1,2}$.

Suppose by contradiction that there exists a function $u \in C^2(\mathbb{R}^2)$ satisfying the conditions (i) and (ii) mentioned above. Put

$$v(x) = \int_{\mathbb{R}^2} \Gamma(x-y) f(y) dy = \int_{A_{1,2}} \Gamma(x-y) f(y) dy, \quad \text{where} \quad \Gamma(z) = \frac{1}{2\pi} \log|z|.$$

Then $v \in C^2(\mathbb{R}^2)$ and $\Delta v = f$ in $\mathbb{R}^2$ by Lemma 2.6. Put $w = v - u$. Then $\Delta w = 0$ in $\mathbb{R}^2$. Consequently, w is real analytic in $\mathbb{R}^2$. Thus, the restriction of w to the real line, namely $w(\cdot, 0)$, is given by a power series

$$w(t, 0) = a_0 + a_1 t + a_2 t^2 + \ldots \quad \forall t \in \mathbb{R}. \tag{2.2}$$

The radius of convergence of this series is equal to infinity, so

$$\limsup_{k \to \infty} \sqrt[k]{|a_k|} = 0. \tag{2.3}$$

In addition, $w = v$ in B_ϵ because $u = 0$ in B_ϵ. For each $k \in \mathbb{N}$, we define

$$f_k(x) = f_k(x_1, x_2) = \eta(2r-3) \frac{\cos(k\theta)}{2^k}$$

where $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $r \geq 0$, $\theta \in \mathbb{R}$. Then $f_k \in C^\infty(\mathbb{R}^2)$ and $\mathrm{supp} f_k \subset A_{1,2}$. Put

$$v_k(x) = \int\limits_{\mathbb{R}^2} \Gamma(x-y) f_k(y)dy = \int\limits_{A_{1,2}} \Gamma(x-y) f_k(y)dy.$$

Then $v_k \in C^2(\mathbb{R}^2)$ and $\Delta v_k = f_k$ in $\mathbb{R}^2$ by Lemma 2.6. Note that $|f_k(x)| \leq \frac{1}{2^k}$. Thus, the series $\sum\limits_{k=1}^{\infty} f_k(x)$ converges to $f(x)$ uniformly in $\mathbb{R}^2$. Because $\Gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, the series

$$\sum_{k=1}^{\infty} v_k(x) = \sum_{k=1}^{\infty} \int\limits_{A_{1,2}} \Gamma(x-y) f_k(y)dy$$

converges to $v(x)$ uniformly on every compact subset of $\mathbb{R}^2$.

Next, we compute v_k in B_ϵ. Take $x = (x_1, x_2) = (\rho\cos\phi, \rho\sin\phi)$ where $0 < \rho < \epsilon < 1$. For each $y \in A_{1,2}$, we write $y = (y_1, y_2) = (r\cos\theta, r\sin\theta)$ where $1 \leq r \leq 2$. Then $|x-y|^2 = \rho^2 + r^2 - 2\rho r\cos(\theta - \phi)$. We have

$$\begin{aligned} v_k(x) &= \int\limits_{A_{1,2}} \Gamma(x-y) f_k(y)dy = \frac{1}{4\pi} \int\limits_{A_{1,2}} \log(|x-y|^2)\eta(2r-3)\frac{\cos(k\theta)}{2^k}dy \\ &= \frac{1}{4\pi} \int\limits_{A_{2,3}} \log(\rho^2 + r^2 - 2\rho r\cos(\theta-\phi))\eta(2r-3)\frac{\cos(k\theta)}{2^k}dy \\ &= \frac{1}{4\pi 2^k} \int_1^2 \left[\int_0^{2\pi} \log(\rho^2 + r^2 - 2\rho r\cos(\theta-\phi))\cos(k\theta)d\theta\right] r\eta(2r-3)dr. \end{aligned}$$

For $\phi = 0$, we have $x = (\rho, 0)$ and

$$v_k(\rho, 0) = \frac{1}{4\pi 2^k} \int_1^2 \underbrace{\left[\int_0^{2\pi} \log(\rho^2 + r^2 - 2\rho r\cos\theta)\cos(k\theta)d\theta\right]}_{\{1\}} r\eta(2r-3)dr \tag{2.4}$$

We have

$$\begin{aligned} \{1\} &= \int_0^{2\pi} \left[\log(r^2) + \log\left(\left(\frac{\rho}{r}\right)^2 + 1 - 2\frac{\rho}{r}\cos\theta\right)\right]\cos(k\theta)d\theta \\ &= \int_0^{2\pi} \log(r^2)\cos(k\theta)d\theta + \int_0^{2\pi} \log(t^2 + 1 - 2t\cos\theta)\cos(k\theta)d\theta \end{aligned}$$

where $t = \rho/r \in (0,1)$. The first integral is equal to zero. The second integral is equal to $-\frac{2\pi}{k}t^k$ by Lemma 2.9 below. Thus,

$$\{1\} = -\frac{2\pi}{k}\left(\frac{\rho}{r}\right)^k.$$

Then (2.4) becomes

$$v_k(\rho,0) = \frac{1}{4\pi 2^k}\int_1^2 -\frac{2\pi}{k}\left(\frac{\rho}{r}\right)^k r\eta(2r-3)dr = -\frac{\rho^k}{2^{k+1}k}\int_1^2 \frac{\eta(2r-3)}{r^{k-1}}dr.$$

Put

$$a_k = -\frac{1}{2^{k+1}k}\int_1^2 \frac{\eta(2r-3)}{r^{k-1}}dr < 0. \tag{2.5}$$

We get $v_k(\rho,0) = a_k\rho^k$ for all $0<\rho<\epsilon$. Hence,

$$v(\rho,0) = \sum_{k=1}^{\infty} v_k(\rho,0) = \sum_{k=1}^{\infty} a_k\rho^k \quad \forall \rho\in(0,\epsilon).$$

Because $w=v$ in B_ϵ, we have

$$w(\rho,0) = v(\rho,0) = \sum_{k=1}^{\infty} a_k\rho^k \quad \forall \rho\in(0,\epsilon). \tag{2.6}$$

Thus, the equality (2.3) must be satisfied. By (2.5) we have

$$\begin{aligned} |a_k| &= \frac{1}{2^{k+1}k}\int_1^2 \frac{\eta(2r-3)}{r^{k-1}}dr \\ &\geq \frac{1}{2^{k+1}k}\int_{5/4}^{7/4} \frac{1}{r^{k-1}}dr = \frac{1}{2^{k+1}k}\frac{\left(\frac{5}{4}\right)^{2-k} - \left(\frac{7}{4}\right)^{2-k}}{k-2} \quad \forall k\geq 3. \end{aligned}$$

There exists a number $k_0\geq 3$ such that $(7/4)^{2-k}\leq \frac{1}{2}(5/4)^{2-k}$ for all $k>k_0$. Then for $k>k_0$ we have

$$|a_k| \geq \frac{1}{2^{k+1}k}\frac{\left(\frac{5}{4}\right)^{2-k} - \frac{1}{2}\left(\frac{5}{4}\right)^{2-k}}{k-2} = \frac{25}{64k(k-2)}\left(\frac{5}{4}\right)^{-k}.$$

Then

$$\limsup_{k\to\infty}\sqrt[k]{|a_k|} \geq \limsup_{k\to\infty}\sqrt[k]{\frac{25}{64k(k-2)}}\left(\frac{5}{4}\right)^{-1} = \frac{4}{5} > 0.$$

This is a contradiction. □

Proposition 2.4. *If f is smooth and bounded in $\mathbb{R}^2$ then there may NOT exist any function u whose Laplacian is f and which grows at most at quadratic order at infinity.*

Proof. Define the map $\eta:\mathbb{R}\to\mathbb{R}$,

$$\eta(t) = \begin{cases} \exp\left(\dfrac{t^2}{t^2-1}\right) & |t|<1, \\ 0 & |t|\geq 1. \end{cases}$$

Then $\eta \in C^\infty(\mathbb{R})$ with $0 \leq \eta(t) \leq 1$ for all $t \in \mathbb{R}$ and $\eta(0) = 1$. Put $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = \dfrac{1 - \eta(r)}{r^2}$ where $r = \sqrt{x^2 + y^2}$.

First we show that $g \in C^\infty(\mathbb{R}^2)$. It is clear that $g \in C^\infty(\mathbb{R}^2 \backslash \{0\})$. Thus it suffices to show that g is smooth in $B_{1/2}$, which is the disk of radius $1/2$ centered at the origin. The complex function $z \mapsto \dfrac{1 - \exp\left(\frac{z}{z-1}\right)}{z}$ is holomorphic in $B_{1/2}$ because the numerator is holomorphic in $B_{1/2}$ and vanishes at $z = 0$. This implies that the restriction of that function to the real segment $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is real analytic, i.e. the map $h : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{R}$,

$$h(t) = \frac{1 - \exp\left(\frac{t}{t-1}\right)}{t}$$

is real analytic. Thus, $h \in C^\infty((-\frac{1}{2}, \frac{1}{2}))$. For $(x, y) \in B_{1/2}$, we have

$$g(x, y) = \frac{1 - \eta(r)}{r^2} = \frac{1 - \exp\left(\frac{r^2}{r^2 - 1}\right)}{r^2} = h(r^2) = h(x^2 + y^2),$$

which is in $C^\infty(B_{1/2})$.

Next, we define a function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = \frac{1 - \eta(r)}{r^2}(x^2 - y^2).$$

By the first step, $f \in C^\infty(\mathbb{R}^2)$. Moreover, f is bounded because

$$|f(x, y)| \leq (1 - \eta(r))\frac{|x^2 - y^2|}{x^2 + y^2} \leq \frac{|x^2 - y^2|}{x^2 + y^2} \leq 1$$

We claim that there exists a function $w \in C^2(\mathbb{R}^2)$ such that $\Delta w = f$ in $\mathbb{R}^2$ and

$$\frac{1}{8}R^2 \log(R) \leq ||w||_{L^\infty(B_R)} \leq \frac{1}{2}R^2 \log(R) \tag{2.7}$$

for all R sufficiently large, where B_R denotes the disk of radius R centered at the origin.

With $x = r\cos\theta$ and $y = r\sin\theta$, we can write $f(x, y)$ in polar coordinates

$$f(x, y) = \frac{1 - \eta(r)}{r^2}(x^2 - y^2) = (1 - \eta(r))\cos(2\theta) = c(r)\cos(2\theta)$$

where $c(r) = 1 - \eta(r)$. By the first step, $c(r)/r \in C^\infty(\mathbb{R})$. Put

$$v_1(t) = \int_1^t \frac{1}{4}\frac{c(s)}{s}ds, \quad v_2(t) = -\int_0^t \frac{1}{4}c(s)s^3 ds.$$

Then v_1 and v_2 are smooth. Define $v(t) = v_1(t)t^2 + v_2(t)t^{-2}$. Because

$$\lim_{t\to 0} v_2(t)t^{-2} = \lim_{t\to 0} \frac{v_2(t)}{t^2} = \lim_{t\to 0} \frac{{v_2}'(t)}{2t} = \lim_{t\to 0} \frac{1}{4}c(t)\frac{t^2}{2} = 0,$$

we conclude that $v \in C(\mathbb{R})$. It is clear that $v \in C^2(\mathbb{R}^2\backslash\{0\})$. By direct computation, we get

$$v''(r) + \frac{1}{r}v'(r) - \frac{4}{r^2}v(r) = c(r) \quad \forall r > 0$$

For $r > 0$ and $\theta \in \mathbb{R}$, we define $\tilde{w}(r,\theta) = v(r)\cos(2\theta)$. Then $\tilde{w} \in C^2(\mathbb{R}_+ \times \mathbb{R})$ and

$$\frac{\partial^2 \tilde{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{w}}{\partial \theta^2} = \left(v''(r) + \frac{1}{r}v'(r) - \frac{4}{r^2}v(r)\right)\cos(2\theta) = c(r)\cos(2\theta)$$

Since $v(r) \to 0$ as $r \to 0$, there is $\epsilon > 0$ such that $|v(r)| < 1$ for all $0 < r < \epsilon$. Thus $|\tilde{w}(r,\theta)| \le |v(r)| < 1$ for all $0 < r < \epsilon,\ \theta \in \mathbb{R}$. Then by Lemma 2.8 below, $w \in C^2(\mathbb{R}^2)$ and $\Delta w = f$ in $\mathbb{R}^2$. We see that $0 \le c(s) \le 1$ for all $s > 0$ and $c(s) = 1$ for all $s \ge 1$. Hence, for $r > 1$,

$$|v_1(r)| = \int_1^r \frac{1}{4}\frac{c(s)}{s}ds = \int_1^r \frac{1}{4}\frac{1}{s}ds = \frac{\log(r)}{4}.$$

Also,

$$|v_2(r)r^{-2}| = r^{-2}\int_0^r \frac{1}{4}c(s)s^3 ds \le r^{-2}\int_0^r \frac{1}{4}s^3 ds = \frac{r^2}{16}.$$

We have

$$v(r) = v_1(r)r^2 + v_2(r)r^{-2} = \frac{1}{4}r^2\log(r) + v_2(r)r^{-2} \tag{2.8}$$

Then

$$|v(r)| \le \frac{1}{4}r^2\log(r) + |v_2(r)|r^{-2} \le \frac{1}{4}r^2\log(r) + \frac{r^2}{16} \le \frac{1}{2}r^2\log(r)$$

provided that $r > \exp(1/4)$. Also, (2.8) implies that

$$v(r) \ge \frac{1}{4}r^2\log(r) - |v_2(r)|r^{-2} \ge \frac{1}{4}r^2\log(r) - \frac{r^2}{16} \ge \frac{1}{8}r^2\log(r)$$

provided that $r > \exp(1/2)$. Therefore,

$$\frac{1}{8}r^2\log(r) \le |v(r)| \le \frac{1}{2}r^2\log(r) \quad \forall r > \exp(1/2),$$

and thus

$$|\tilde{w}(r,\theta)| \le \frac{1}{2}r^2\log(r) \quad \forall r > \exp(1/2),\ \forall \theta \in \mathbb{R}. \tag{2.9}$$

Then

$$||w||_{L^\infty(B_R)} \le \frac{1}{2}R^2\log(R) \quad \forall R > \exp(1/2).$$

On the other hand,

$$||w||_{L^\infty(B_R)} \geq |w(R,0)| = |\tilde{w}(R,0)| = |v(R)| \geq \frac{1}{8}R^2\log(R) \quad \forall R > \exp(1/2).$$

Thus the claim (2.7) is proved. Note that it is important that we have

$$|w(R,0)| \geq \frac{1}{8}R^2\log(R) \quad \forall R > \exp(1/2). \tag{2.10}$$

Now suppose by contradiction that there exists a function $u \in C^2(\mathbb{R}^2)$ such that $\Delta u = f$ in $\mathbb{R}^2$ and that $u(x)$ grows at most at quadratic order as x go to infinity. There exists $R_0 > 0$ such that

$$|u(x)| \leq |x|^2 \quad \forall |x| > R_0.$$

We can assume $R_0 > \exp(2)$. By (2.9),

$$|w(x)| \leq \frac{1}{2}|x|^2\log|x| \quad \forall |x| > R_0.$$

Put $u_0 = w - u$. Then $u_0 \in C^2(\mathbb{R}^2)$, $\Delta u_0 = 0$ in $\mathbb{R}^2$ and

$$|u_0(x)| \leq |u(x)| + |w(x)| \leq |x|^2 + \frac{1}{2}|x|^2\log|x| \leq |x|^2\log|x| \quad \forall |x| > R_0.$$

We now show that u_0 is a polynomial of degree at most 2. Because u_0 is harmonic in $\mathbb{R}^2$, it is the real part of a complex function $\phi(z)$ which is holomorphic in $\mathbb{C}$. For any $R > 0$ we have the Poisson's formula

$$\phi(z) = \frac{1}{2\pi}\int_0^{2\pi} u_0(\mathrm{Re}^{i\theta})\mathrm{Re}\left(\frac{\mathrm{Re}^{i\theta}+z}{\mathrm{Re}^{i\theta}-z}\right)d\theta \quad \forall |z| < R.$$

For each $|z| > R_0$, we choose $R = 2|z|$. Then we get the estimation

$$\begin{aligned} |\phi(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |u_0(\mathrm{Re}^{i\theta})|\left|\frac{\mathrm{Re}^{i\theta}+z}{\mathrm{Re}^{i\theta}-z}\right|d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} R^2(\log R)\frac{R+|z|}{R-|z|}d\theta \\ &= 3R^2\log R = 12|z|^2\log(2|z|). \end{aligned}$$

Thus,

$$|\phi(z)| \leq 12|z|^2\log(2|z|) \quad \forall |z| > R_0. \tag{2.11}$$

Since $\phi(z)$ is holomorphic in $\mathbb{C}$, it has a Taylor expansion $\phi(z) = a_0 + a_1 z + a_2 z^2 + \ldots$, where

$$a_m = \frac{1}{2\pi i}\int_\gamma \frac{\phi(\xi)}{\xi^{m+1}}d\xi$$

and γ is any circle centered at the origin. We can choose γ to be a circle of radius $r > R_0$. By (2.11) we have the estimation

$$|a_m| \leq \frac{1}{2\pi}\int_\gamma \frac{|\phi(\xi)|}{|\xi|^{m+1}}|d\xi| \leq \frac{1}{2\pi}(2\pi r)\frac{12r^2\log(2r)}{r^{m+1}} = 12r^{2-m}\log(2r) \tag{2.12}$$

If $m > 3$ then $a_m = 0$ because $RHS(2.12)$ goes to 0 as r goes to infinity. Hence $\phi(z) = a_0 + a_1 z + a_2 z^2$. Because $u_0(z) = \mathrm{Re}(\phi(z))$, u_0 is a polynomial (of degree at most 2) in two real variables. Thus there exists $R_1 > 0$ such that

$$|u_0(x) \leq |x|^2 \quad \forall |x| > R_1.$$

Then

$$|w(x)| \leq |u(x)| + |u_0(x)| \leq 2|x|^2 \quad \forall |x| > \max\{R_0, R_1\}.$$

In particular,

$$|w(R,0)| \leq 2R^2 \quad \forall R > \max\{R_0, R_1\}.$$

This contradicts (2.10). $\square$

Proposition 2.5 (Helmholtz decomposition)**.** *Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a locally Hölder continuous map with exponent $\alpha \in (0,1]$. Then there exist $\phi \in C^1(\mathbb{R}^3)$ and $\vec{A} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ such that $\vec{F} = -\nabla\phi + \nabla \times \vec{A}$.*

Proof. By Proposition 2.2, there exists a vector field $\vec{G} \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ such that $\Delta\vec{G} = -\vec{F}$. Using the identity $\nabla \times (\nabla \times \vec{G}) = \nabla(\nabla \cdot \vec{G}) - \Delta\vec{G}$, we get

$$\vec{F} = -\Delta\vec{G} = -\nabla(\nabla \cdot \vec{G}) + \nabla \times (\nabla \times \vec{G}).$$

Therefore, we can choose $\phi = \nabla \cdot \vec{G} \in C^1(\mathbb{R}^3)$ and $\vec{A} = \nabla \times \vec{G} \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. $\square$

Lemma 2.6. *Let f be a locally Hölder continuous function with exponent $\alpha \in (0,1]$ in $\mathbb{R}^n$ where $n \geq 2$. Suppose that f is also compactly supported in $\mathbb{R}^n$. Then the Newtonian potential of f, namely*

$$v(x) = \int\limits_{\mathbb{R}^n} \Gamma(x-y) f(y) dy,$$

belongs to $C^2(\mathbb{R}^n)$ and $\Delta v = f$ in $\mathbb{R}^n$.

Proof. There exists a number $r_0 > 0$ such that $\mathrm{supp} f \subset B_{r_0}$. Then

$$v(x) = \int\limits_{B_r} \Gamma(x-y) f(y) dy$$

for all $r \geq r_0$ and $x \in \mathbb{R}^n$. Lemma 4.2 in [GT98, p.55] states that:

> If f is bounded and locally Hölder continuous with exponent $\alpha \leq 1$ in Ω [which is an open bounded set], and let w be the Newtonian potential of f [on Ω]. Then $w \in C^2(\Omega)$ and $\Delta w = f$ in Ω.

Applying this result for $\Omega = B_r$ and $w = v|_{B_r}$, we get $v|_{B_r} \in C^2(B_r)$ and $\Delta v = f$ in B_r. Because r can be chosen arbitrarily large, we conclude that $v \in C^2(\mathbb{R}^n)$ and $\Delta v = f$ in $\mathbb{R}^n$. $\square$

Lemma 2.7. *Suppose that f and g are locally Hölder continuous functions with exponents ≤ 1 in $\mathbb{R}^n$. Then so is the product fg.*

Proof. Let K be any compact subset of $\mathbb{R}^n$. We want to show that the product function fg is Hölder continuous on K. Because f and g are continuous in $\mathbb{R}^n$, they are bounded in K. Thus there exists a number $M = M(K) > 0$ such that $|f(x)|,\ |g(x)| \leq M$ for all $x \in K$. Because f and g are locally Hölder continuous in $\mathbb{R}^n$, there exist $\alpha_1, \alpha_2 \in (0, 1]$ and $C_1(K), C_2(K) > 0$ such that

$$\begin{aligned} \frac{|f(x) - f(y)|}{|x-y|^{\alpha_1}} &\leq C_1 \quad \forall x, y \in K, x \neq y, \\ \frac{|g(x) - g(y)|}{|x-y|^{\alpha_2}} &\leq C_2 \quad \forall x, y \in K, x \neq y. \end{aligned}$$

Denote $\alpha = \min(\alpha_1, \alpha_2)$ and $d = \operatorname{diam}(K) < \infty$. For any $x, y \in K$, $x \neq y$, we have

$$\begin{aligned} \frac{f(x)g(x) - f(y)g(y)}{|x-y|^{\alpha}} &= f(x)\frac{g(x)-g(y)}{|x-y|^{\alpha}} + g(x)\frac{f(x)-f(y)}{|x-y|^{\alpha}} \\ &= f(x)\frac{g(x)-g(y)}{|x-y|^{\alpha_2}}|x-y|^{\alpha_2-\alpha} + g(x)\frac{f(x)-f(y)}{|x-y|^{\alpha_1}}|x-y|^{\alpha_1-\alpha} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{|f(x)g(x) - f(y)g(y)|}{|x-y|^{\alpha}} &\leq |f(x)|\frac{|g(x)-g(y)|}{|x-y|^{\alpha_2}}|x-y|^{\alpha_2-\alpha} + |g(x)|\frac{|f(x)-f(y)|}{|x-y|^{\alpha_1}}|x-y|^{\alpha_1-\alpha} \\ &\leq MC_2 d^{\alpha_2-\alpha} + MC_1 d^{\alpha_1-\alpha}. \end{aligned}$$

This means fg is Hölder continuous with exponent $\alpha \in (0, 1]$ in every compact subset K of $\mathbb{R}^n$. ☐

Lemma 2.8. *Denote $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$. Let $f \in C^2(\mathbb{R}^2)$ and define*

$$\tilde{f}(r, \theta) = f(r\cos\theta, r\sin\theta) \quad \forall r \in \mathbb{R}_+,\ \theta \in \mathbb{R}.$$

Suppose that there is a function $\tilde{u} \in C^2(\mathbb{R}_+ \times \mathbb{R})$ satisfying

(i) $\tilde{u}(r, \theta + 2\pi) = \tilde{u}(r, \theta) \quad \forall r \in \mathbb{R}_+,\ \theta \in \mathbb{R}$,

(ii) there exist $M, \epsilon > 0$ such that $|\tilde{u}(r, \theta)| \leq M \quad \forall 0 < r < \epsilon,\ \theta \in \mathbb{R}$,

(iii) $\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \theta^2} = \tilde{f}(r, \theta) \quad \forall r \in \mathbb{R}_+,\ \theta \in \mathbb{R}$.

Define a function $u : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ by $u(r\cos\theta, r\sin\theta) = \tilde{u}(r, \theta)$ for $(r, \theta) \in \mathbb{R}_+ \times \mathbb{R}$. Then $u \in C^2(\mathbb{R}^2)$ and $\Delta u = f$ in $\mathbb{R}^2$.

Proof. Note that u is well-defined because $\tilde{u}$ is periodic in θ with period 2π. For each $(x,y) \in \mathbb{R}^2 \backslash \{0\}$, we have

$$u(x,y) = \begin{cases} \tilde{u}\left(\sqrt{x^2+y^2},\ \arctan\left(\dfrac{y}{x}\right)\right), & x > 0 \\ \tilde{u}\left(\sqrt{x^2+y^2},\ \pi + \arctan\left(\dfrac{y}{x}\right)\right), & x < 0 \\ \tilde{u}\left(\sqrt{x^2+y^2},\ \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{y}\right)\right), & y > 0 \\ \tilde{u}\left(\sqrt{x^2+y^2},\ \dfrac{\pi}{2} - \arctan\left(\dfrac{x}{y}\right)\right), & y < 0 \end{cases}$$

Since $\tilde{u} \in C^2(\mathbb{R}_+ \times \mathbb{R})$, we have $u \in C^2(\mathbb{R}^2 \backslash \{0\})$. By the chain rule, we have

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \theta^2} = \tilde{f}(r,\theta) = f(x,y),$$

for all $(x,y) \in \mathbb{R}^2 \backslash \{0\}$, $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, $\theta \in \mathbb{R}$. For $(x,y) \in B(0,\epsilon)\backslash\{0\}$, we have $|u(x,y)| = |\tilde{u}(r,\theta)| \le M$. Thus, u is bounded in $B(0,\epsilon)\backslash\{0\}$.

Put $g = u|_{\partial B_1} \in C(\partial B_1)$ where B_1 is the unit disk centered at the origin. Then the problem

$$\begin{cases} \Delta v = f & \text{in } \ B_1, \\ v = g & \text{on } \ \partial B_1 \end{cases}$$

has a solution $v \in C^2(B_1) \cap C(\overline{B_1})$ which is given via a Green function [GT98, p.56, Theorem 4.3]. Put $w = u - v$. Then $w \in C^2(B_1\backslash\{0\}) \cap C(\overline{B_1}\backslash\{0\})$ and is bounded in B_1. Moreover,

$$\begin{cases} \Delta w = 0 & \text{in } \ B_1\backslash\{0\}, \\ w = 0 & \text{on } \ \partial B_1. \end{cases}$$

By the theorem of removable singularity of harmonic functions [ABR00, p.32, Theorem 2.3], w is a harmonic function in B_1. Then $w = 0$ in B_1 by the maximum principle. Thus $u = v$, which is in $C^2(B_1)$. Therefore, $u \in C^2(\mathbb{R}^2)$ and $\Delta u = f$ in $\mathbb{R}^2$. $\square$

Lemma 2.9. *For* $0 < t < 1$ *and* $k \in \mathbb{N}$*, we have*

$$\int_0^{2\pi} \log(t^2 + 1 - 2t\cos\theta)\cos(k\theta)d\theta = -\frac{2\pi}{k}t^k$$

Proof. Denote

$$J(t) = \int_0^{2\pi} \log(t^2 + 1 - 2t\cos\theta)\cos(k\theta)d\theta.$$

Then $J(t)$ is continuously differentiable on [0,1). We denote

$$\begin{aligned} I(t) = J'(t) &= \int_0^{2\pi} \frac{\partial}{\partial t}\left[\log(t^2 + 1 - 2t\cos\theta)\right]\cos(k\theta)d\theta \\ &= \int_0^{2\pi} \frac{2t - 2\cos\theta}{t^2 + 1 - 2t\cos\theta}\cos(k\theta)d\theta. \end{aligned}$$

On the complex plane, we consider the unit circle $z = e^{i\theta}$. Then

$$\begin{aligned}
\cos\theta &= \frac{z+\bar{z}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right), \\
\cos(k\theta) &= \frac{z^k + \bar{z}^k}{2} = \frac{1}{2}\left(z^k + \frac{1}{z^k}\right), \\
d\theta &= -i\frac{dz}{z}.
\end{aligned}$$

Therefore,

$$I = \frac{-i}{2} \int\limits_{|z|=1} \underbrace{\frac{z^{2k}+1}{z^{k+1}} \frac{z^2 - 2tz + 1}{t(z^2+1) - (t^2+1)z}}_{f(z)} dz. \tag{2.13}$$

We have

$$f(z) = \frac{z^{2k}+1}{z^{k+1}} \frac{z^2 - 2tz + 1}{(tz-1)(z-t)},$$

which is a meromorphic function in $\mathbb{C}$. The poles of $f(z)$ are at $z = 1/t$, $z = t$ (both with multiplicity one), and $z = 0$ (with multiplicity $k+1$). Only t and 0 are the poles enclosed in the unit circle. Thus,

$$\int\limits_{|z|=1} f(z)dz = 2\pi i \left(\mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=t} f(z)\right).$$

Substituting this equality into (2.13) we get

$$I(t) = \pi i \left(\mathrm{Res}_{z=0} f(z) + \mathrm{Res}_{z=t} f(z)\right). \tag{2.14}$$

Compute the residue of $f(z)$ at $z = t$.

Because $z = t$ is a simple pole,

$$\mathrm{Res}_{z=t} f(z) = \lim_{z\to t} f(z)(z-t) = \lim_{z\to t} \frac{z^{2k}+1}{z^{k+1}} \frac{z^2 - 2tz + 1}{tz - 1} = -\frac{t^{2k}+1}{t^{k+1}}. \tag{2.15}$$

Compute the residue of $f(z)$ at $z = 0$.

Denote $t_1 = t$ and $t_2 = 1/t$. Then $t_1 t_2 = 1$, $0 < t_1 < 1$, $t_2 > 1$. Then

$$\begin{aligned}
f(z) &= \frac{z^{2k}+1}{t_1 z^{k+1}} \frac{z^2 - 2t_1 z + 1}{(z-t_2)(z-t_1)} \\
&= \frac{1}{t_1(t_2 - t_1)} \left[\underbrace{\frac{z^{2k}+1}{z^{k+1}} \frac{z^2 - 2t_1 z + 1}{z - t_2}}_{f_1(z)} - \underbrace{\frac{z^{2k}+1}{z^{k+1}} \frac{z^2 - 2t_1 z + 1}{z - t_1}}_{f_2(z)} \right]
\end{aligned} \tag{2.16}$$

When z is near 0,

$$\begin{aligned}
\frac{z^2 - 2t_1 z + 1}{z - t_2} &= z - t_2 - 2(t_1 - t_2) - \frac{t_2 - t_1}{1 - t_1 z} \\
&= z - t_2 - 2(t_1 - t_2) - (t_2 - t_1)(1 + t_1 z + t_1^2 z^2 + \ldots)
\end{aligned}$$

Thus,

$$\begin{aligned} f_1(z) &= \frac{z^{2k}+1}{z^{k+1}}\left[z-t_2-2(t_1-t_2)-(t_2-t_1)(1+t_1z+t_1^2z^2+...)\right] \\ &= \left(z^{k-1}+z^{-k-1}\right)\left[z-t_2-2(t_1-t_2)-(t_2-t_1)\sum_{j=0}^{\infty}t_1^jz^j\right]. \end{aligned}$$

The coefficient of z^{-1} in the Laurent series of $f_1(z)$ is equal to the coefficient of z^k in the square bracket. That is,

$$\mathrm{Res}_{z=0}f_1(z) = \begin{cases} 1-(t_2-t_1)t_1, & k=1 \\ -(t_2-t_1)t_1^k, & k\geq 2 \end{cases} \tag{2.17}$$

When z is near 0,

$$\frac{z^2-2t_1z+1}{z-t_1} = z-t_1-\frac{t_2-t_1}{1-\frac{z}{t_1}} = z-t_1-(t_2-t_1)\left(1+\frac{z}{t_1}+\frac{z^2}{{t_1}^2}+\ldots\right).$$

Thus,

$$\begin{aligned} f_2(z) &= \frac{z^{2k}+1}{z^{k+1}}\left[z-t_1-(t_2-t_1)\left(1+\frac{z}{t_1}+\frac{z^2}{{t_1}^2}+...\right)\right] \\ &= \left(z^{k-1}+z^{-k-1}\right)\left[z-t_1-(t_2-t_1)\sum_{j=0}^{\infty}t_1^{-j}z^j\right] \end{aligned}$$

The coefficient of z^{-1} in the Laurent series of $f_2(z)$ is equal to the coefficient of z^k in the square bracket. That is,

$$\mathrm{Res}_{z=0}f_2(z) = \begin{cases} 1-(t_2-t_1)t_1^{-1}, & k=1 \\ -(t_2-t_1)t_1^{-k}, & k\geq 2 \end{cases} \tag{2.18}$$

By (2.17) and (2.18), we have $\mathrm{Res}_{z=0}f_1(z)-\mathrm{Res}_{z=0}f_2(z) = -(t_2-t_1)(t_1^k-t_1^{-k})$. Replacing this identity into (2.16), we get

$$\begin{aligned} \mathrm{Res}_{z=0}f(z) &= \frac{1}{t_1(t_2-t_1)}(\mathrm{Res}_{z=0}f_1(z)-\mathrm{Res}_{z=0}f_2(z)) \\ &= -\frac{{t_1}^k-{t_1}^{-k}}{t_1} = -\frac{t^k-t^{-k}}{t} = -\frac{t^{2k}-1}{t^{k+1}} \end{aligned} \tag{2.19}$$

Now substituting (2.15) and (2.19) into (2.14), we get

$$I(t) = \pi\left(-\frac{t^{2k}+1}{t^{k+1}}-\frac{t^{2k}-1}{t^{k+1}}\right) = -2\pi t^{k-1}.$$

Recall that $I(t)=J'(t)$. Since $J(0)=\int_0^{2\pi}\log(1)\cos(k\theta)d\theta=0$, we get

$$J(t) = -\frac{2\pi}{k}t^k.$$

$\square$

3 A proof of the Malgrange-Ehrenpreis theorem

This section gives a detail proof for the Malgrange-Ehrenpreis theorem according to the method in [Ros91]. This is an elementary proof which requires little background in the space of distributions. The necessary background is collected in Section 6.4. Throughout the section, P always denotes a non-identically-zero polynomial in n real variables, where $n \in \mathbb{N}$. Write

$$P(x) = \sum_{|\alpha| \le N} c_\alpha x^\alpha = \sum_{|\alpha| \le N} c_\alpha {x_1}^{\alpha_1} \dots {x_n}^{\alpha_n}.$$

If the set $\{c_\alpha : \ |\alpha| = N\}$ has at least one nonzero element, we say that P has *degree* N and each nonzero element of that set is called a *highest coefficient* of P. Define the differential operators $P(D)$ and $P(-D)$ as follows.

$$\begin{aligned} P(D) &= \sum_{|\alpha| \le N} c_\alpha D^\alpha = \sum_{|\alpha| \le N} c_\alpha D_1^{\alpha_1} \dots D_n^{\alpha_n}, \\ P(-D) &= \sum_{|\alpha| \le N} (-1)^{|\alpha|} c_\alpha D^\alpha = \sum_{|\alpha| \le N} (-1)^{|\alpha|} c_\alpha D_1^{\alpha_1} \dots D_n^{\alpha_n}. \end{aligned}$$

$P(-D)$ is called the *conjugate differential operator* of $P(D)$. By the integration-by-part formula, we have

$$\langle P(D)u, Q(D)\phi \rangle = \langle Q(-D)u, P(-D)\phi \rangle \quad \forall u, \phi \in \mathscr{D}(\mathbb{R}^n), \tag{3.1}$$

where the brackets $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(\mathbb{R}^n)$. Consequently,

$$||P(D)\phi||_{L^2(\mathbb{R}^n)} = ||P(-D)\phi||_{L^2(\mathbb{R}^n)} \quad \forall \phi \in \mathscr{D}(\mathbb{R}^n). \tag{3.2}$$

Below is the outline of the proof of the Malgrange-Ehrenpreis theorem.

Step 1. Hörmander's inequality: for each open bounded set $\Omega \subset \mathbb{R}^n$, there exists a number $C = C(P, N) > 0$ such that

$$||P(D)\phi||_{L^2(\Omega)} \ge C||\phi||_{L^2(\Omega)} \quad \forall \phi \in \mathscr{D}(\Omega).$$

Step 2. If Ω is an open bounded subset of $\mathbb{R}^n$ and $g \in L^2(\Omega)$, then there exists $u \in L^2(\Omega)$ such that $P(D)u = g$ in sense of $\mathscr{D}'(\Omega)$.

Step 3. For each $\phi \in \mathscr{D}(\mathbb{R}^n)$, $[\text{supp} P(D)\phi] = [\text{supp}\phi]$. Here $[K]$ denotes the *convex hull* of a set $K \subset \mathbb{R}^n$ in $\mathbb{R}^n$.

Step 4. For each $u \in \mathscr{E}'(\mathbb{R}^n)$, $[\text{supp} P(D)u] = [\text{supp} u]$.[†]

Step 5. Suppose we have $0 < s < r < R$ and $v \in L^2(B_r)$ such that $P(D)v = 0$ in sense of $\mathscr{D}'(B_r)$. Then there exists a sequence (v_k) in $L^2(B_R)$ such that $P(D)v_k = 0$ in sense of $\mathscr{D}'(B_R)$ and $||v_k - v||_{L^2(B_s)} \to 0$ as $k \to \infty$.

[†]See the definition of the space $\mathscr{E}'(\mathbb{R}^n)$ in the remark preceding Proposition 6.16.

Step 6. For each $g \in L^2_{\text{loc}}(\mathbb{R}^n)$, there exists $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that $P(D)u = g$ in sense of $\mathscr{D}'(\mathbb{R}^n)$.

Step 7. Consider the Heaviside function $H : \mathbb{R}^n \to \mathbb{R}$,

$$H(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_1, \dots, x_n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Q(D)H = \delta_0$, where $Q(x) = x_1 x_2 \dots x_n$ and δ_0 is the Dirac measure defined in Proposition 6.22.

Step 8. There exists $E \in \mathscr{D}'(\mathbb{R}^n)$ such that $P(D)E = \delta_0$.

Proposition 3.1 (Hörmander's Inequality)**.** *Let Ω be a nonempty open bounded subset of $\mathbb{R}^n$, and $P \in \mathbb{R}[x_1, \dots, x_n] \backslash \{0\}$. Then there exists a number $C > 0$ depending only on Ω, the degree of P and the highest coefficients of P such that*

$$||P(D)\phi||_{L^2(\Omega)} \geq C||\phi||_{L^2(\Omega)} \quad \forall \phi \in \mathscr{D}(\Omega). \tag{3.3}$$

Proof. Put $r = \sup\{|x| : x \in \Omega\}$. First, we show by induction in $m = 0, 1, 2, \dots$ that

$$||P(D)(x_j\phi) - x_j P(D)\phi||_{L^2(\Omega)} \leq 2mr||P(D)\phi||_{L^2(\Omega)} \tag{3.4}$$

for all $1 \leq j \leq n$, $\phi \in \mathscr{D}(\Omega)$ and $P \in \mathbb{R}[x_1, \dots, x_n]$, $\deg P = m$.

For $m = 0$, P is a constant c. Then (3.4) is true because the left hand side is zero. Suppose that (3.4) is true for all $m = 0, 1, \dots, N-1$ where $N \geq 1$. Let $j \in \{1, 2, \dots, n\}$, $\phi \in \mathscr{D}(\Omega)$ and $P \in \mathbb{R}[x_1, \dots, x_n]$, $\deg P = N$. Write $P(D) = \sum_{|\alpha| \leq N} c_\alpha D^\alpha$. Recall the generalized Leibniz formula

$$D^\alpha(uv) = \sum\nolimits_{\{\beta : \beta \leq \alpha\}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (D^{\alpha-\beta}u)(D^\beta v). \tag{3.5}$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_2)$ are multi-indices. We write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all $1 \leq i \leq n$. Also, we define

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

Hence, $P(D)(x_j\phi) = x_j P(D)\phi + P_j(D)\phi$ where

$$P_j(D) = \sum\nolimits_{|\alpha| \leq N} c_\alpha \sum\nolimits_{\substack{\{\beta:\ \beta \leq \alpha\} \\ |\beta| \geq 1}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta(x_j) D^{\alpha-\beta}. \tag{3.6}$$

We have

$$D^\beta(x_j) = \begin{cases} 1 & \text{if } \beta = e_j, \\ 0 & \text{otherwise,} \end{cases}$$

where e_j is the j'th vector of the standard basis of $\mathbb{R}^n$. By the definition of P_j, $\deg P_j < N$. If $P_j \not\equiv 0$ then by the induction hypothesis,

$$\begin{aligned} ||P_j(D)(x_j\phi) - x_j P_j(D)\phi||_{L^2(\Omega)} &\leq 2r \deg(P_j)||P_j(D)\phi||_{L^2(\Omega)} \\ &\leq 2r(N-1)||P_j(D)\phi||_{L^2(\Omega)}. \end{aligned}$$

By the triangle inequality,

$$\begin{aligned}||P_j(D)(x_j\phi)||_{L^2(\Omega)} &\leq ||x_jP_j(D)\phi||_{L^2(\Omega)} + ||P_j(D)(x_j\phi) - x_jP_j(D)\phi||_{L^2(\Omega)}\\ &\leq r||P_j(D)\phi||_{L^2(\Omega)} + 2r(N-1)||P_j(D)\phi||_{L^2(\Omega)}\\ &= r(2N-1)||P_j(D)\phi||_{L^2(\Omega)}.\end{aligned}$$

Thus,

$$||P_j(D)(x_j\phi)||_{L^2(\Omega)} \leq r(2N-1)||P_j(D)\phi||_{L^2(\Omega)}. \tag{3.7}$$

Note that (3.7) is also true when $P_j \equiv 0$. Taking the inner product of both sides of the identity $P(D)(x_j\phi) - x_jP(D)\phi = P_j(D)\phi$ with $P_j(D)\phi$, we get

$$\underbrace{\langle P(D)(x_j\phi), P_j(D)\phi\rangle}_{\{1\}} - \underbrace{\langle x_jP(D)\phi, P_j(D)\phi\rangle}_{\{2\}} = ||P_j(D)\phi||^2_{L^2(\Omega)}.$$

Hence,

$$||P_j(D)\phi||^2_{L^2(\Omega)} \leq |\{1\}| + |\{2\}|. \tag{3.8}$$

By (3.1), $\{1\} = \langle P_j(-D)(x_j\phi), P(-D)\phi\rangle$, Thus,

$$\begin{aligned}|\{1\}| &\leq ||P_j(-D)(x_j\phi)||_{L^2(\Omega)}\ ||P(-D)\phi||_{L^2(\Omega)} \quad \text{(Schwarz)}\\ &= ||P_j(D)(x_j\phi)||_{L^2(\Omega)}\ ||P(D)\phi||_{L^2(\Omega)} \quad \text{(by (3.1))}\\ &\leq r(2N-1)||P_j(D)\phi||_{L^2(\Omega)}\ ||P(D)\phi||_{L^2(\Omega)} \quad \text{(by (3.7))}.\end{aligned}$$

Also we have

$$\begin{aligned}|\{2\}| &\leq ||x_jP(D)\phi||_{L^2(\Omega)}\ ||P_j(D)\phi||_{L^2(\Omega)} \quad \text{(Schwarz)}\\ &\leq r||P(D)\phi||_{L^2(\Omega)}\ ||P_j(D)\phi||_{L^2(\Omega)}.\end{aligned}$$

With the estimation of $\{1\}$ and $\{2\}$ above, (3.8) implies

$$\begin{aligned}||P_j(D)\phi||^2_{L^2(\Omega)} &\leq r(2N-1)||P_j(D)\phi||_{L^2(\Omega)}\ ||P(D)\phi||_{L^2(\Omega)}\\ &+ r||P(D)\phi||_{L^2(\Omega)}\ ||P_j(D)\phi||_{L^2(\Omega)}\\ &= 2Nr||P_j(D)\phi||_{L^2(\Omega)}\ ||P(D)\phi||_{L^2(\Omega)}.\end{aligned}$$

Thus, $||P_j(D)\phi||_{L^2(\Omega)} \leq 2Nr||P(D)\phi||_{L^2(\Omega)}$. Thus, we have proved (3.4) for $m = N$. Thus (3.4) is true for all $m = 0, 1, 2, \ldots$ We rewrite (3.4) as follows.

$$||Q_j(D)(x_j\phi) - x_jQ_j(D)\phi||_{L^2(\Omega)} \leq 2r\deg(Q)||Q(D)\phi||_{L^2(\Omega)}, \tag{3.9}$$

for all $1 \leq j \leq n$, $\phi \in \mathscr{D}(\Omega)$ and $Q \in \mathbb{R}[x_1, \ldots, x_n]$. Now return to the problem. We prove (3.3) by induction in m = degP. If deg$P$=0 then P is a constant c. Then (3.3) is true by taking $C = c$. Suppose that (3.3) is true if deg$P = N-1$, for some $N \geq 1$. Consider a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ with deg$P = N$. Write $P(D) = \sum_{|\alpha|\leq N} c_\alpha D^\alpha$. Then $c_{\alpha_0} \neq 0$ for some multi-index α_0 with $|\alpha_0| = N$. Write $\alpha_0 = (\alpha_1, \ldots, \alpha_n)$. Then $\alpha_j \geq 1$ for some $1 \leq j \leq n$. We define a polynomial P_j as in (3.6). Then deg$P_j = N-1$ and

$$P(D)(x_j\phi) - x_jP(D)\phi = P_j(D)\phi \quad \forall\phi \in \mathscr{D}(\Omega).$$

Applying (3.9) for $Q = P$, we get

$$||P_j(D)\phi||_{L^2(\Omega)} \leq 2Nr||P(D)\phi||_{L^2(\Omega)} \quad \forall \phi \in \mathscr{D}(\Omega). \tag{3.10}$$

The highest coefficients of P_j correspond to $\beta = e_j$ and $|\alpha| = N$. Thus, these coefficients are in the set

$$A = \left\{ c_\alpha \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \ |\alpha| = N, \ \beta = e_j \right\}.$$

By the induction hypothesis, there exists a number $C_1 > 0$ depending only on Ω, $\deg P_j = N - 1$ and the elements of A such that

$$||P_j(D)\phi||_{L^2(\Omega)} \geq C_1||\phi||_{L^2(\Omega)} \quad \forall \phi \in \mathscr{D}(\Omega). \tag{3.11}$$

Choose $C = (2Nr)^{-1}C_1$. Then C depends only on Ω, $\deg P = N$ and the elements of the set $\{c_\alpha : \ |\alpha| = N\}$. From (3.10) and (3.11) we get

$$||P(D)\phi||_{L^2(\Omega)} \geq (2Nr)^{-1}C_1||\phi||_{L^2(\Omega)} = C||\phi||_{L^2(\Omega)} \quad \forall \phi \in \mathscr{D}(\Omega).$$

This means (3.3) is true for the case $\deg P = N$. $\square$

Proposition 3.2. *Let $P \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$ and Ω be a nonempty open bounded subset of $\mathbb{R}^n$. Then for each $g \in L^2(\Omega)$, there exists $u \in L^2(\Omega)$ such that $P(D)u = g$ in sense of $\mathscr{D}'(\Omega)$. Moreover, there is a number $C > 0$ depending only on Ω, the degree of P, and the highest coefficients of P such that $||g||_{L^2(\Omega)} \geq C||u||_{L^2(\Omega)}$.*

Proof. The identity $P(D)u = g$ in sense of $\mathscr{D}'(\Omega)$ means

$$\langle u, P(-D)\phi \rangle = \langle g, \phi \rangle \quad \forall \phi \in \mathscr{D}(\Omega). \tag{3.12}$$

Define a map $T_1 : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$, $T_1(\phi) = P(-D)\phi$. This is a linear map. Thus, the range of T_1, denoted by E, is a vector subspace of $\mathscr{D}(\Omega)$. Hence, E is also a vector subspace of $L^2(\Omega)$. Because $P(x) \not\equiv 0$, $P(-x) \not\equiv 0$. By Proposition 3.1, there exists a number $C > 0$ depending only on Ω, $\deg P$ and the highest coefficients of P such that

$$||P(-D)\phi||_{L^2(\Omega)} \geq C||\phi||_{L^2(\Omega)} \quad \forall \phi \in \mathscr{D}(\Omega).$$

This means $||T_1\phi||_{L^2(\Omega)} \geq C||\phi||_{L^2(\Omega)}$. Thus, T_1 is injective. Given a function $g \in L^2(\Omega)$, we define a map $T_2 : E \to \mathbb{R}$, $T_2\psi = \langle g, T_1^{-1}\psi \rangle$. Then T_2 is linear. Also, for every $\psi \in E$,

$$|T_2\psi| = |\langle g, T_1^{-1}\psi \rangle| \leq ||g||_{L^2(\Omega)}||T_1^{-1}\psi||_{L^2(\Omega)} \leq C^{-1}||g||_{L^2(\Omega)}||\psi||_{L^2(\Omega)}.$$

Thus, T_2 is a linear continuous functional on $(E, ||.||_{L^2(\Omega)})$ and $||T_2|| \leq C^{-1}||g||_{L^2(\Omega)}$. By Hahn-Banach theorem, T_2 can extend to a linear continuous functional $\tilde{T}_2$ on $L^2(\Omega)$ with $||\tilde{T}_2|| \leq C^{-1}||g||_{L^2(\Omega)}$. By Riesz representation theorem, there exists $u \in L^2(\Omega)$ such that

$$\tilde{T}_2(\psi) = \langle u, \psi \rangle \quad \forall \psi \in L^2(\Omega),$$

and $||u||_{L^2(\Omega)} = ||\tilde{T}_2|| \le C^{-1}||g||_{L^2(\Omega)}$. Thus, $\left\langle g, T_1^{-1}\psi \right\rangle = \langle u, \psi \rangle$ for all $\psi \in E$. Write $T_1^{-1}\psi = \phi$. Then

$$\langle g, \phi \rangle = \langle u, T_1\phi \rangle = \langle u, P(-D)\phi \rangle \quad \forall \phi \in \mathscr{D}(\Omega).$$

Therefore, (3.12) is proved. $\square$

For each subset K of $\mathbb{R}^n$, we denote by $[K]$ the *convex hull* of K in $\mathbb{R}^n$, i.e. the smallest convex subset of $\mathbb{R}^n$ containing K.

Proposition 3.3. *Let* $P \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$ *and* $\phi \in \mathscr{D}(\mathbb{R}^n)$. *Then* $[\mathrm{supp}P(D)\phi] = [\mathrm{supp}\phi]$.

Proof. Take any $x \in \mathbb{R}^n\backslash\mathrm{supp}\phi$. Then $\phi = 0$ in a neighborhood U of x. Then $P(D)\phi = 0$ in U. Thus, $x \in \mathbb{R}^n\backslash\mathrm{supp}P(D)\phi$. This means $\mathbb{R}^n\backslash\mathrm{supp}\phi$ is contained in $\mathbb{R}^n\backslash\mathrm{supp}P(D)\phi$. Hence, $\mathrm{supp}P(D)\phi \subset \mathrm{supp}\phi$ and $[\mathrm{supp}P(D)\phi] \subset [\mathrm{supp}\phi]$. Suppose by contradiction that $[\mathrm{supp}P(D)\phi] \neq [\mathrm{supp}\phi]$. Then there exists a point $a \in \mathrm{supp}\phi\backslash[\mathrm{supp}P(D)\phi]$. Since $\mathrm{supp}\phi$ is closed, we can assume $a \in \{x \in \mathbb{R}^n : \ \phi(x) \neq 0\}$. Then there exists a hyperplane (H) separating a and $\mathrm{supp}P(D)\phi$. We choose a new Cartesian coordinate system $y = (y_1, \ldots, y_n)$ by translating and rotating the old one, i.e. $y = Ax + b$ where A and b are a matrix and a vector of real constant coefficients. They are chosen so that $(H) = \{y_1 = 0\}$, $a \in \{y_1 > 0\}$ and $\mathrm{supp}P(D)\phi \subset \{y_1 < 0\}$.

Under this change of variables, the differential operator $P(D) = P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ becomes a differential operator $Q(D) = Q\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)$ which also has constant coefficients. The function $\phi(x)$ becomes $\psi(y)$ with $\psi(a) \neq 0$. Therefore, all hypotheses in the problem still hold after the change of variables. Thus, we could have assumed from the beginning that $(H) = \{x_1 = 0\}$, $a \in \{x_1 > 0\}$ and $\mathrm{supp}P(D)\phi \subset \{x_1 < 0\}$.

Write $a = (a_1, \ldots, a_n)$ with $a_1 > 0$. Since $\mathrm{supp}P(D)\phi$ is compact, there exists $\epsilon > 0$ such that $\mathrm{supp}P(D)\phi \subset \{x_1 < -\epsilon\}$. Let Ω be an open bounded subset of $\mathbb{R}^n$ that contains $\mathrm{supp}\phi$. Since $\phi(a) \neq 0$ and $a \in \mathrm{supp}\phi \subset \Omega$, there exists a number $r \in (0, \frac{a_1}{2})$ such that $B_r(a) \subset \Omega$ and

$$|\phi(x)| > \frac{|\phi(a)|}{2} \quad \forall x \in B_r(a). \tag{3.13}$$

For every $x \in B_r(a)$,

$$x_1 \in (a_1 - r, a_1 + r) \subset \left(\frac{a_1}{2}, \frac{3a_1}{2}\right).$$

Thus,

$$x_1 > \frac{a_1}{2} \quad \forall x \in B_r(a). \tag{3.14}$$

For each $\psi \in \mathscr{D}(\mathbb{R}^n)$ and $s \in \mathbb{R}$, $s > 0$, we define

$$Q_s(D)\psi = e^{\frac{sx_1}{2}} P(D)\left(e^{\frac{-sx_1}{2}}\psi\right). \tag{3.15}$$

It is important to note that $Q_s(D)$ is also a differential operator with constant coefficients. Put $N = \text{deg} P \geq 0$ and write

$$P(D) = \sum\nolimits_{|\alpha|=N} c_\alpha D^\alpha + \sum\nolimits_{|\alpha|<N} c_\alpha D^\alpha.$$

Then by the definition of $Q_s(D)$, we have

$$Q(D) = \sum\nolimits_{|\alpha|=N} c_\alpha D^\alpha + \sum\nolimits_{|\alpha|<N} \tilde{c}_\alpha D^\alpha$$

where $\tilde{c}_\alpha$ are numbers that may depend on s. In other words, the degree and the highest coefficients of Q_s are the same as those of P. In particular, they are independent of s. Put

$$\psi_s(x) = e^{\frac{sx_1}{2}} \phi(x) \in \mathscr{D}(\Omega). \tag{3.16}$$

By Proposition 3.1, there exists a number $C > 0$ depending only on Ω, N and the highest coefficients of P such that

$$||Q_s(D)\psi_s||_{L^2(\Omega)} \geq C||\psi_s||_{L^2(\Omega)} \quad \forall s > 0.$$

Taking the square of both sides, we get

$$\int_\Omega |Q_s(D)\psi_s|^2 dx \geq C^2 \int_\Omega |\psi_s|^2 dx \quad \forall s > 0. \tag{3.17}$$

Replacing (3.15) and (3.16) into (3.17) we get

$$\int_\Omega e^{sx_1} (P(D)\phi)^2 dx \geq C^2 \int_\Omega e^{sx_1} \phi^2 dx \quad \forall s > 0.$$

Thus,

$$\int_\Omega e^{sx_1} \phi^2 dx \leq C^{-2} \int_\Omega e^{sx_1} (P(D)\phi)^2 dx \leq C^{-2} \int_\Omega e^{-s\epsilon} (P(D)\phi)^2 dx = C^{-2} e^{-s\epsilon} \int_\Omega (P(D)\phi)^2 dx.$$

Thus,

$$\lim_{s\to\infty} \int_\Omega e^{sx_1} \phi^2 dx = 0. \tag{3.18}$$

On the other hand,

$$\begin{aligned}
\int_\Omega e^{sx_1} \phi^2 dx &\geq \int_{B_r(a)} e^{sx_1} \phi^2 dx \\
&\geq \int_{B_r(a)} e^{s\frac{a_1}{2}} \left(\frac{\phi(a)}{2}\right)^2 dx \quad \text{(by (3.13) and (3.14))} \\
&= e^{s\frac{a_1}{2}} \left(\frac{\phi(a)}{2}\right)^2 |B_r(a)| \to \infty \text{ as } s \to \infty.
\end{aligned}$$

This contradicts (3.18). □

Proposition 3.4. *Let* $P \in \mathbb{R}[x_1, \ldots, x_n] \backslash \{0\}$ *and* $u \in \mathscr{E}'(\mathbb{R}^n)$. *Then* $[\text{supp} P(D)u] = [\text{supp} u]$.

Proof. Let $\{\eta_\epsilon\}_{\epsilon>0}$ be the approximate identity defined on Page 73. Put $u_\epsilon = u * \eta_\epsilon$. Then $u_\epsilon \in \mathscr{D}(\mathbb{R}^n)$ by Part (ii) of Proposition 6.21. Then by Proposition 3.3,

$$[\text{supp} P(D)u_\epsilon] = [\text{supp} u_\epsilon] \quad \forall \epsilon > 0. \tag{3.19}$$

Put $v = P(D)u$. Then $v \in \mathscr{E}'(\mathbb{R}^n)$ and $\text{supp} v \subset \text{supp} u$. Then $[\text{supp} v] \subset [\text{supp} u]$. Now we want to show that $[\text{supp} u] \subset [\text{supp} v]$. For each $\epsilon > 0$, we put $v_\epsilon = v * \eta_\epsilon$. Then

$$\begin{aligned} P(D)u_\epsilon &= P(D)(u * \eta_\epsilon) \\ &= (P(D)u) * \eta_\epsilon \quad \text{(by Proposition 6.20)} \\ &= v * \eta_\epsilon = v_\epsilon. \end{aligned} \tag{3.20}$$

Take any $\delta > 0$. By Part (ii) of Proposition 6.24, there exists $\lambda > 0$ such that

$$\text{supp} u \subset (\text{supp} u_\epsilon) + \bar{B}_\delta \quad \forall 0 < \epsilon < \lambda. \tag{3.21}$$

Since $[\text{supp} u_\epsilon] + \bar{B}_\delta$ is a convex set containing $(\text{supp} u_\epsilon) + \bar{B}_\delta$, it also contains the convex hull of $(\text{supp} u_\epsilon) + \bar{B}_\delta$. Then (3.20) and (3.21) imply

$$\begin{aligned} [\text{supp} u] \subset [(\text{supp} u_\epsilon) + \bar{B}_\delta] &\subset [\text{supp} u_\epsilon] + \bar{B}_\delta \\ &= [\text{supp} P(D)u_\epsilon] + \bar{B}_\delta \\ &= [\text{supp} v_\epsilon] + \bar{B}_\delta \quad \forall 0 < \epsilon < \lambda. \end{aligned} \tag{3.22}$$

By Part (i) of Proposition 6.24, $\text{supp} v_\epsilon \subset (\text{supp} v) + \bar{B}_\epsilon$. Thus, $[\text{supp} v_\epsilon] \subset [\text{supp} v] + \bar{B}_\epsilon$. Together with (3.22) we have $[\text{supp} u] \subset [\text{supp} v] + \bar{B}_\epsilon + \bar{B}_\delta$ for $0 < \epsilon < \lambda$. Thus,

$$[\text{supp} u] \subset \bigcap_{0<\epsilon<\lambda} ([\text{supp} v] + \bar{B}_\epsilon + \bar{B}_\delta) = [\text{supp} v] + \bar{B}_\delta.$$

Because this is true for all $\delta > 0$, we have

$$[\text{supp} u] \subset \bigcap_{\delta>0} ([\text{supp} v] + \bar{B}_\delta) = [\text{supp} v].$$

□

For each $r > 0$, we denote by B_r the open ball in $\mathbb{R}^n$ centered at the origin with radius r. We know that $L^2(B_r)$ is a Hilbert space with the inner product

$$\langle u_1, u_2 \rangle_{B_r} = \int_{B_r} u_1 u_2 dx \quad \forall u_1, u_2 \in L^2(B_r).$$

The induced norm on $L^2(B_r)$ is

$$||u||_{B_r} = \left(\int_{B_r} u^2 dx \right)^{1/2}.$$

Proposition 3.5. *Let $0 < s < r < R$, $P \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$ and $v \in L^2(B_r)$ such that $P(D)v = 0$ in sense of $\mathscr{D}'(B_r)$. Then there exists a sequence (v_k) in $L^2(B_R)$ such that $P(D)v_k = 0$ in sense of $\mathscr{D}'(B_R)$ and $||v_k - v||_{B_s} \to 0$ as $k \to \infty$.*

Proof. Define a set

$$E = \{u \in L^2(B_s) : \exists \text{extension } \tilde{u} \in L^2(B_R) \text{ such that } P(D)\tilde{u} = 0 \text{ in sense of } \mathscr{D}'(B_R)\}.$$

Note that E is a vector subspace of $(L^2(B_s), ||.||_{B_s})$. We need to show $v|_{B_s} \in \bar{E}$. To do so, we take any linear continuous functional $T : L^2(B_s) \to \mathbb{R}$ such that $T(E) = \{0\}$ and show that $Tv = 0$. By Riesz representation theorem, there exists a function $g \in L^2(B_s)$ such that

$$T(u) = \langle g, u\rangle_{B_s} = \int_{B_s} gu\ dx \quad \forall u \in L^2(B_s).$$

We have

$$\langle g, u\rangle_{B_s} = 0 \quad \forall u \in E. \tag{3.23}$$

We want to show that $\langle g, v\rangle_{B_s} = 0$. First, we show that there exists $w \in L^2(B_R)$ such that

$$\langle g, \phi\rangle_{B_s} = \langle w, P(D)\phi\rangle_{B_R} \quad \forall \phi \in \mathscr{D}(\mathbb{R}^n). \tag{3.24}$$

Consider the map $T_1 : \mathscr{D}(B_{R+1}) \to \mathscr{D}(B_{R+1})$, $T_1(\phi) = P(D)\phi$ for all $\phi \in \mathscr{D}(B_{R+1})$. By Proposition 3.1, T_1 is injective. Put $F = T_1(\mathscr{D}(B_{R+1}))$. We can regard F as a vector subspace of $(L^2(B_R), ||.||_{B_R})$. Define a map $T_2 : F \to \mathbb{R}$, $T_2(\psi) = \langle g, T_1^{-1}(\psi)\rangle_{B_s}$ for all $\psi \in F$.

For each $\psi \in F$, there exists $\phi \in \mathscr{D}(B_{R+1})$ such that $\psi = T_1(\phi) = P(D)\phi$. By Proposition 3.2, there exists $u_0 \in L^2(B_R)$ such that $P(D)u_0 = \psi$ in sense of $\mathscr{D}'(B_R)$. Moreover, there exists a number $C > 0$ depending only on the domain B_R and the polynomial P such that

$$C||u_0||_{L^2(B_R)} \leq ||\psi||_{L^2(B_R)}. \tag{3.25}$$

We have $P(D)u_0 = \psi = P(D)\phi$ in sense of $\mathscr{D}'(B_R)$. Thus, $P(D)(u_0 - \phi) = 0$ in sense of $\mathscr{D}'(B_R)$. By (3.23), we have $\langle g, u_0 - \phi\rangle_{B_s} = 0$. Thus, $\langle g, u_0\rangle_{B_s} = \langle g, \phi\rangle_{B_s}$. Thus,

$$\begin{aligned} |T_2(\psi)| = |\langle g, T_1^{-1}(\psi)\rangle_{B_s}| = |\langle g, \phi\rangle_{B_s}| &= |\langle g, u_0\rangle_{B_s}| \\ &\leq ||g||_{L^2(B_s)}||u_0||_{L^2(B_s)} \\ &\leq C^{-1}||g||_{L^2(B_s)}||\psi||_{L^2(B_R)} \text{ (by (3.25))}. \end{aligned}$$

Because this estimation is true for all $\psi \in F$, T_2 is a linear continuous functional on $(F, ||.||_{L^2(B_R)})$. By Hahn-Banach theorem, T_2 can extend to a linear continuous functional $\tilde{T}_2$ on $L^2(B_R)$. By Riesz representation theorem, there exists $w \in L^2(B_R)$ such that

$$\tilde{T}_2(\psi) = \langle w, \psi\rangle_{B_R} \quad \forall \psi \in L^2(B_R).$$

Thus, $\left\langle g, T_1^{-1}(\psi)\right\rangle_{B_s} = \langle w, \psi\rangle_{B_R}$ for all $\psi \in F$. Thus,

$$\langle g, \phi\rangle_{B_s} = \langle w, P(D)\phi\rangle_{B_R} \quad \forall \phi \in \mathscr{D}(B_{R+1}). \tag{3.26}$$

Now take any $\psi \in \mathscr{D}(\mathbb{R}^n)$. Let χ be a function in $\mathscr{D}(B_{R+1})$ such that $\chi = 1$ in B_R. Then $\psi\chi \in \mathscr{D}(B_{R+1})$ and $\psi\chi = \psi$ in B_R. Applying (3.26) for $\phi = \psi\chi$, we have

$$\langle w, P(D)(\psi\chi)\rangle_{B_R} = \langle g, \psi\chi\rangle_{B_s}. \tag{3.27}$$

Because $\psi\chi = \psi$ in B_R, LHS(3.27)=$\langle w, P(D)\psi\rangle_{B_R}$ and RHS(3.27)=$\langle g, \psi\rangle_{B_s}$. Thus, (3.27) implies

$$\langle w, P(D)\psi\rangle_{B_R} = \langle g, \psi\rangle_{B_s} \quad \forall \psi \in \mathbb{D}(\mathbb{R}^n).$$

We have proved (3.24). Next, we define two functions $\tilde{g}, \tilde{w} : \mathbb{R}^n \to \mathbb{R}$,

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in B_s, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{w}(x) = \begin{cases} w(x) & \text{if } x \in B_R, \\ 0 & \text{otherwise.} \end{cases}$$

Then (3.24) implies $\langle \tilde{g}, \phi\rangle_{B_s} = \langle \tilde{w}, P(D)\phi\rangle_{B_R}$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$. Consequently, if $\tilde{g}$ and $\tilde{w}$ are viewed as distributions on $\mathbb{R}^n$, then $\tilde{g} = P(-D)\tilde{w}$. By the definition of $\tilde{g}$ and $\tilde{w}$, we have $\tilde{g}, \tilde{w} \in \mathscr{E}'(\mathbb{R}^n)$, $\text{supp}\tilde{g} \subset \bar{B}_s$ and $\text{supp}\tilde{w} \subset \bar{B}_R$. By Proposition 3.4, $[\text{supp}\tilde{w}] = [\text{supp}P(-D)\tilde{w}]$. Thus, $[\text{supp}\tilde{w}] = [\text{supp}\tilde{g}] \subset \bar{B}_s$. Thus, $\text{supp}w \subset \bar{B}_s$.

Extend v by zero outside of B_r. Let $\{\eta_\epsilon\}_{\epsilon>0}$ be the approximate identity on $\mathbb{R}^n$ as defined on Page 73. For $0 < \epsilon < \min\{R - r, r - s\}$, we put $v_\epsilon = v * \eta_\epsilon$. Then $v_\epsilon \in \mathscr{D}(B_R)$ and $\lim_{\epsilon\to 0} ||v_\epsilon - v||_{B_r} = 0$ according to [Adm75, Lemma 2.18]. For each $\phi \in \mathscr{D}_{B_s}$, we have

$$\begin{aligned}
\langle P(D)v_\epsilon, \phi\rangle_{B_r} &= \langle v_\epsilon, P(-D)\phi\rangle = \langle v * \eta_\epsilon, P(-D)\phi\rangle_{B_r} \\
&= \int_{B_r} (v * \eta_\epsilon)(x)(P(-D)\phi)(x)dx \\
&= \int_{B_r}\int_{R^n} v(y)\eta_\epsilon(x-y)(P(-D)\phi)(x)dydx \\
&= \int_{\mathbb{R}^n}\int_{R^n} v(y)\eta_\epsilon(y-x)(P(-D)\phi)(x)dxdy \\
&= \int_{\mathbb{R}^n} v(y)(\eta_\epsilon * (P(-D)\phi))(y)dy \\
&= \langle v, \eta_\epsilon * (P(-D)\phi)\rangle_{B_r} \\
&= \langle v, P(-D)(\eta_\epsilon * \phi)\rangle_{B_r}.
\end{aligned}$$

Thus,

$$\langle P(D)v_\epsilon, \phi\rangle_{B_r} = \langle v, P(-D)(\eta_\epsilon * \phi)\rangle_{B_s} \quad \forall \phi \in D(B_s). \tag{3.28}$$

We have $\text{supp}(\phi * \eta_\epsilon) \subset (\text{supp}\phi) + (\text{supp}\eta_\epsilon) \subset B_s + \bar{B}_\epsilon \subset B_r$. Thus, $\phi * \eta_\epsilon \in \mathscr{D}(B_r)$. Since $P(D)v = 0$ in sense of $\mathscr{D}'(B_r)$, we have

$$\langle v, P(-D)(\eta_\epsilon * \phi)\rangle_{B_s} = \langle v, P(-D)(\eta_\epsilon * \phi)\rangle_{B_r} = 0 \quad \forall \phi \in \mathscr{D}(B_s). \tag{3.29}$$

Applying (3.24) for $\phi = v_\epsilon$, we have

$$\langle g, v_\epsilon \rangle_{B_s} = \langle w, P(D)v_\epsilon \rangle_{B_R} = \langle P(D)v_\epsilon, w \rangle_{B_R}.$$

Because supp$w \subset \bar{B}_s$, we have

$$\langle g, v_\epsilon \rangle_{B_s} = \langle P(D)v_\epsilon, w \rangle_{B_s}. \tag{3.30}$$

Because $\mathscr{D}(B_s)$ is dense in $L^2(B_s)$, there exists a sequence (ϕ_n) in $\mathscr{D}(B_s)$ such that $||\phi_n - w||_{B_s} \to 0$ as $n \to \infty$. By (3.29) we have $\langle P(D)v_\epsilon, \phi_n \rangle_{B_s} = 0$ for all $n \in \mathbb{N}$. Let $n \to \infty$, we get $\langle P(D)v_\epsilon, w \rangle_{B_s} = 0$. Thus, (3.30) implies $\langle g, v_\epsilon \rangle_{B_s} = 0$. This is true for every $0 < \epsilon < \min\{R - r, r - s\}$. Therefore,

$$\langle g, v \rangle_{B_s} = \lim_{\epsilon \to 0} \langle g, v_\epsilon \rangle_{B_s} = 0.$$

□

Proposition 3.6. *Let $P \in \mathbb{R}[x_1, \ldots, x_n] \backslash \{0\}$ and $g \in L^2_{\text{loc}}(\mathbb{R}^n)$. Then there exists $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that $P(D)u = g$ in sense of $\mathscr{D}'(\mathbb{R}^n)$.*

Proof. Because $g \in L^2(B_2)$, by Proposition 3.2, there exists a function $u_1 \in L^2(B_2)$ such that $P(D)u_1 = g$ in sense of $\mathscr{D}'(B_2)$.

Suppose that $u_k \in L^2(B_{k+1})$ has been defined such that $P(D)u_k = g$ in sense of $\mathscr{D}'(B_{k+1})$. We define u_{k+1} as follows. Since $g \in L^2(B_{k+2})$, by Proposition 3.2, there exists a function $w \in L^2(B_{k+2})$ such that $P(D)w = g$ in sense of $\mathscr{D}'(B_{k+2})$. Put $v = w - u_k \in L^2(B_{k+1})$. By Proposition 3.5, there exists a sequence (v_k) in $L^2(B_{k+2})$ such that $P(D)v_k = 0$ in sense of $\mathscr{D}'(B_{k+2})$ and $||v_k - v||_{B_k} \to 0$ as $k \to \infty$. Thus, there exists $k_0 \in \mathbb{N}$ such that

$$||v_{k_0} - v||_{B_k} \leq 2^{-k}.$$

Define $u_{k+1} = w - v_{k_0}$. Then $u_{k+1} \in L^2(B_{k+2})$. Moreover, because $P(D)w = g$ and $P(D)v_{k_0} = 0$ in sense of $\mathscr{D}'(B_{k+2})$, we have $P(D)u_{k+1} = P(D)w - P(D)v_{k_0} = g$ in sense of $\mathscr{D}'(B_{k+2})$. Also, we have

$$||u_{k+1} - u_k||_{B_k} = ||(w - v_{k_0}) - u_k||_{B_k} = ||v - v_{k_0}||_{B_k} \leq 2^{-k}.$$

This induction process defines a sequence of functions (u_k) satisfying

$$\begin{cases} u_k \in L^2(B_{k+1}), \\ P(D)u_k = g \quad \text{in sense of } \mathscr{D}'(B_{k+1}), \\ ||u_{k+1} - u_k||_{B_k} \leq 2^{-k}. \end{cases}$$

Extend each function u_k by zero outside B_{k+1} so that it is defined in $\mathbb{R}^n$. Put

$$f(x) = \sum_{k=1}^{\infty} |u_{k+1}(x) - u_k(x)| \quad \forall x \in \mathbb{R}^n.$$

Put $A = \{x \in \mathbb{R}^n : \ f(x) = \infty\}$. We show that A is of measure zero in $\mathbb{R}^n$. For each $m \in \mathbb{N}$, we have $u_k \in L^2(B_m)$ for all $k \in \mathbb{N}$. Thus, $|u_{k+1} - u_k| \in L^2(B_m)$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned} ||f||_{L^2(B_m)} &= \lim_{N\to\infty} \left\| \sum_{k=1}^{N} |u_{k+1} - u_k| \right\|_{L^2(B_m)} \quad \text{(Lebesgue's monotone convergence)} \\ &\leq \lim_{N\to\infty} \sum_{k=1}^{N} \|u_{k+1} - u_k\|_{L^2(B_m)} \quad \text{(triangle inequality)} \\ &= \sum_{k=1}^{\infty} \|u_{k+1} - u_k\|_{L^2(B_m)}. \end{aligned} \tag{3.31}$$

For $k > m$, $B_m \subset B_k$. Thus, $||u_{k+1} - u_k||_{L^2(B_m)} \leq ||u_{k+1} - u_k||_{L^2(B_k)} \leq 2^{-k}$. This means the series at (3.31) converges. Thus, $||f||_{L^2(B_m)} < \infty$. This implies that the set $A \cap B_m = \{x \in B_m : \ f(x) = \infty\}$ is of measure zero. Because

$$A = A \cap \mathbb{R}^n = A \cap \left(\bigcup_{m=1}^{\infty} B_m \right) = \bigcup_{m=1}^{\infty} (A \cap B_m),$$

A is also of measure zero. Thus, the series $\sum_{k=1}^{\infty} (u_{k+1}(x) - u_k(x))$ converges absolutely for almost every $x \in \mathbb{R}^n$. This allows us to define a function

$$u(x) = u_1(x) + \sum_{k=1}^{\infty} (u_{k+1}(x) - u_k(x))$$

almost everywhere in $\mathbb{R}^n$. We have

$$||u||_{L^2(B_m)} \leq ||u_1||_{L^2(B_m)} + ||f||_{L^2(B_m)} < \infty \quad \forall m \in \mathbb{N}.$$

Thus, $u \in L^2_{\text{loc}}(\mathbb{R}^n)$. For almost every $x \in \mathbb{R}^n$, we have

$$u(x) = u_1(x) + \lim_{k\to\infty} \sum_{l=1}^{k-1} (u_{l+1}(x) - u_l(x)) = \lim_{k\to\infty} u_k(x).$$

For each $m \in \mathbb{N}$ and for every $x \in \mathbb{R}^n$,

$$\begin{aligned} |u_k(x)| &\leq |u_1(x)| + \sum_{l=1}^{k-1} |u_{l+1}(x) - u_l(x)| \\ &\leq |u_1(x)| + \sum_{l=1}^{\infty} |u_{l+1}(x) - u_l(x)| \\ &= |u_1(x)| + |f(x)| \quad \forall k \in \mathbb{N}. \end{aligned}$$

Because $|u_1| + |f| \in L^2(B_m)$, by Lebesgue's Dominated Convergence Theorem, we have $||u_k - u||_{L^2(B_m)} \to 0$ as $k \to \infty$.

Take any $\phi \in \mathscr{D}(\mathbb{R}^n)$. There exists $m \in \mathbb{N}$ such that $\phi \in \mathscr{D}(B_m)$. Then

$$\langle u, P(-D)\phi \rangle_{\mathbb{R}^n} = \langle u, P(-D)\phi \rangle_{B_m} = \lim_{k\to\infty} \langle u_k, P(-D)\phi \rangle_{B_m}. \tag{3.32}$$

Since $P(D)u_k = g$ in sense of $\mathscr{D}'(B_k)$, we have $P(D)u_k = g$ in sense of $\mathscr{D}'(B_m)$ whenever $k > m$. Thus, $\langle u_k, P(-D)\phi \rangle_{B_m} = \langle g, \phi \rangle_{B_m}$ for all $k > m$. Then (3.32) gives

$$\langle u, P(-D)\phi \rangle_{\mathbb{R}^n} = \lim_{k\to\infty} \langle g, \phi \rangle_{B_m} = \langle g, \phi \rangle_{B_m} = \langle g, \phi \rangle_{\mathbb{R}^n}.$$

Therefore, $P(D)u = g$ in sense of $\mathscr{D}'(\mathbb{R}^n)$. $\square$

Proposition 3.7. *Consider the Heaviside function $H : \mathbb{R}^n \to \mathbb{R}$,*

$$H(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_1, \dots, x_n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Q(D)H = \delta_0$, where $Q(x) = x_1 x_2 \dots x_n$.

Proof. We need to show that $\langle H, Q(-D)\phi \rangle = \phi(0)$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$. By the definition of the conjugate differential operator on Page 18, $Q(-D) = (-1)^n D_1 D_2 \dots D_n$. Therefore, we want to show that

$$\int_{\mathbb{R}^n} H(x)(D_1 D_2 \dots D_n)\phi(x)dx = (-1)^n \phi(0) \quad \forall \phi \in \mathscr{D}(\mathbb{R}^n). \tag{3.33}$$

For convenience, we denote the set of all points $x \in \mathbb{R}^n$ whose all coordinates are positive by $\{x_1, \dots, x_n > 0\}$. Showing (3.33) is equivalent to showing that

$$\int_{\{x_1, \dots, x_n > 0\}} (D_1 D_2 \dots D_n)\phi(x)dx = (-1)^n \phi(0) \quad \forall \phi \in \mathscr{D}(\mathbb{R}^n). \tag{3.34}$$

We prove (3.34) by induction in $n \in \mathbb{N}$. For $n = 1$, (3.34) becomes

$$\int_{\{x>0\}} \phi'(x)dx = -\phi(0) \quad \forall \phi \in \mathscr{D}(\mathbb{R}). \tag{3.35}$$

For each $\phi \in \mathscr{D}(\mathbb{R})$, there exists a number $M > 0$ such that $\phi(x) = 0$ for all $x \geq M$. Thus,

$$\text{LHS}(3.35) = \int_0^M \phi'(x)dx = \phi(M) - \phi(0) = -\phi(0) = \text{RHS}(3.35).$$

Thus, (3.35) is proved.

Suppose that (3.34) is true for $n = N - 1$. Take $\phi \in \mathscr{D}(\mathbb{R}^N)$ arbitrarily. We show that

$$\int_{\{x_1, \dots, x_N > 0\}} (D_1 D_2 \dots D_N)\phi(x)dx = (-1)^N \phi(0).$$

Since $\text{supp}\phi$ is bounded, there exists a number $M > 0$ such that $\max\{|x_1|, \dots, |x_n|\} < M$ for all $x \in \text{supp}\phi$. Put $\Omega = (0, M)^N$, which is a Lipschitz domain. According to [Nec67, p.117], the Green formula is still valid for the domain Ω. Namely,

$$\int_\Omega D_1 f dx = \int_{\partial\Omega} f n_1 d\sigma, \tag{3.36}$$

where f is a sufficiently regular function and $\vec{n} = (n_1, \dots, n_N)$ is the exterior normal vector. Applying (3.36) for $f = (D_2 \dots D_N)\phi$, we get

$$\int_\Omega (D_1 D_2 \dots D_N)\phi(x)dx = \int_{\partial\Omega} (D_2 \dots D_N)\phi(x) n_1 d\sigma. \tag{3.37}$$

Note that

$$\text{LHS}(3.37) = \int_{\{x_1,\ldots,x_N>0\}} (D_1 D_2 \ldots D_N)\phi(x)dx. \tag{3.38}$$

The boundary of Ω consists of 2^N faces. However, ϕ vanishes on all but (at most) N faces, namely

$$\sigma_i = \{(x_1, ..., x_N) : \ x_i = 0, \ 0 \le x_j \le M \ \forall 1 \le j \ne i \le N\} \quad \forall 1 \le i \le N.$$

Thus,

$$\text{RHS}(3.37) = \sum\nolimits_{i=1}^N \int_{\sigma_i} (D_2 ... D_N)\phi(x) n_1 d\sigma. \tag{3.39}$$

The exterior normal vector $\vec{n}$ on the face σ_i is $\vec{n} = -e_i$. Thus, $n_1 = -1$ on σ_1 and $n_1 = 0$ on other faces. Thus, (3.39) becomes

$$\text{RHS}(3.37) = -\int_{\sigma_1} (D_2 \ldots D_N)\phi(x) n_1 d\sigma = -\int\limits_{\{x_1=0, x_2,\ldots,x_N>0\}} (D_2 \ldots D_N)\phi(x) n_1 d\sigma. \tag{3.40}$$

On the plane $\{x_1 = 0\}$, we write $x = (0, y)$ with $y = (y_1, \ldots, y_{N-1}) = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$. Put $\psi(y) = \phi(0, y) \in \mathscr{D}(\mathbb{R}^n)$. Then (3.40) becomes

$$\begin{aligned} \text{RHS}(3.37) &= \int_{\{y_1,\ldots,y_{N-1}>0\}} (D_1 \ldots D_{N-1})\psi(y)dy \\ &= -(-1)^{N-1}\psi(0) \quad \text{(by the induction hypothesis)} \\ &= (-1)^N \phi(0). \end{aligned}$$

Then together with (3.38) we get

$$\int_{\{x_1,\ldots,x_N>0\}} (D_1 D_2 \ldots D_N)\phi(x)dx = (-1)^N \phi(0).$$

□

Proposition 3.8 (Malgrange-Ehrenpreis theorem)**.** *Let* $P \in \mathbb{R}[x_1, \ldots, x_n] \backslash \{0\}$. *Then there exists* $E \in \mathscr{D}'(\mathbb{R}^n)$ *such that* $P(D)E = \delta_0$ *in sense of* $\mathscr{D}'(\mathbb{R}^n)$.

The distribution E is called a *fundamental solution* of $P(D)$.

Proof. By Proposition 3.7, $Q(D)H = \delta_0$ where $H$is the Heaviside function on $\mathbb{R}^n$ and $Q(D) = D_1 D_2 \ldots D_N$. Since H is bounded, $H \in L^2_{\text{loc}}(\mathbb{R}^n)$. By Proposition 3.6 there exists a function $u \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that $P(D)u = H$ in sense of $\mathscr{D}'(\mathbb{R}^n)$. Put $E = Q(D)u$. Then $E \in \mathscr{D}'(\mathbb{R}^n)$ by Proposition 6.18. We have

$$\begin{aligned} P(D)E &= P(D)(Q(D)u) \\ &= Q(D)(P(D)u) \quad \text{(by Proposition 6.19)} \\ &= Q(D)H = \delta_0. \end{aligned}$$

□

4 Existence of smooth solutions to $P(D)u = f$

Let $P \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$ and $f \in C^\infty(\mathbb{R}^n)$. In this section, we show the existence of $u \in C^\infty(\mathbb{R}^n)$ satisfying $P(D)u = f$. We again use the idea which was used for the Poisson problem in Proposition 2.2. Specifically, we choose an open cover of $\mathbb{R}^n$ by annuli and take a smooth partition of unity subordinate to this cover. Then the function f is decomposed into the sum of compactly supported functions f_k's. By Malgrange-Ehrenpreis theorem, there exists a smooth function v_k such that $P(D)v_k = f_k$. In general, the series $\sum_k v_k$ does not converge in $C^\infty(\mathbb{R}^n)$ because $D^\alpha v_k(x)$ is not small as α and x are fixed and k increases.

The idea is to replace v_k by $u_k = v_k - w_k$ where w_k satisfies $P(D)w_k = 0$ in $\mathbb{R}^n$. Each function w_k has to be chosen so that for each α, the series $\sum_k D^\alpha u_k$ converges uniformly on every compact subset of $\mathbb{R}^n$. We expect that all derivatives up to order k of u_k should be bounded by $\frac{1}{k^2}$ in the ball $B_{k-\frac{1}{2}}$. If this can be done, the function $u = \sum_k u_k$ will be a smooth solution of $P(D)u = f$. This method is inspired by the method of Mittag-Leffler for constructing a meromorphic function with infinitely many prescribed poles. Note that we only prove the 'pure' existence of the functions w_k instead of constructing them. To do so, we need some properties of the dual space of $C^\infty(\mathbb{R}^n)$.

4.1 Some topological properties of the dual of a TVS

First we recall the definition of weak* topology. Let X be a TVS[†] and E be either the *dual* of X, i.e. the *set* of all linear continuous maps from X to $\mathbb{R}$, or the *algebraic dual* of X, i.e. the set of all linear maps from X to $\mathbb{R}$. For each $x \in X$, we define a map $p_x : E \to \mathbb{R}$, $p_x(g) = |\langle g, x\rangle|$ for all $g \in E$. Then p_x is a seminorm[‡] on E. For $g \in E\backslash\{0\}$, there exists $x \in X$ such that $\langle g, x\rangle \neq 0$. Thus, $p_x(g) \neq 0$. This means that $\{p_x\}_{x\in X}$ is a separating family of seminorms on E. By Part (i) of Proposition 6.5, this family gives rise to a locally convex TVS structure on E. Moreover, E has a local base consisting of open sets $\{U_\epsilon(x_1, \ldots, x_m) : \ \epsilon > 0, x_1, \ldots, x_m \in X\}$, where

$$U_\epsilon(x_1, \ldots, x_m) = \{g \in E : \ |\langle g, x_i\rangle| < \epsilon \ \forall 1 \leq i \leq m\}.$$

We call this topology the *weak* topology on E* and denote it by $\sigma(E, X)$. By this definition, for each $x \in X$, the map $g \in (E, \sigma(E, X)) \mapsto \langle g, x\rangle \in \mathbb{R}$ is linear and continuous.

Next, we define another topology on the dual of X. Let E be the *dual* of X, i.e. the *set* of all linear continuous maps from X to $\mathbb{R}$. For each compact, balanced[§], convex subset A of X, we define a map $p_A : E \to \mathbb{R}$,

$$p_A(g) = \max_{x\in A} |\, \langle g, x\rangle \,| \quad \forall g \in E.$$

[†]See the definition of a *topological vector space* on Page 46.
[‡]See the definition of a *seminorm* on Page 47.
[§]See the definition of a *balanced set* on Page 47.

Then p_A is a seminorm on E. For each $g \in E\backslash\{0\}$, there exists $x \in X$ such that $\langle g, x\rangle \neq 0$. Put $A = \{tx : \; -1 \leq t \leq 1\}$. Then A is a compact, balanced and convex subset of X. We have

$$p_A(g) = \max_{y\in A} |\langle g, y\rangle| \geq |\langle g, x\rangle| > 0.$$

Thus, the family $\{p_A : \; A \text{ is a compact, balanced, convex subset of } X\}$ is a separating family of seminorms on E. By Part (i) of Proposition 6.5, this family gives rise to a locally convex TVS structure on E. Moreover, E has a local base consisting of open subsets

$$U_\epsilon(A) = \left\{ g \in E : \max_{x\in A} |\langle g, x\rangle| < \epsilon \right\},$$

where $\epsilon > 0$ and A varies over the family of all compact, balanced and convex subsets of X. This topology is called the *topology of compact convergence* on E, and is denoted by $\gamma(E, X)$. Because $U_\epsilon(A) = U_1(\epsilon^{-1}A)$, we can say that $\gamma(E, X)$ has a local base consisting of the open sets

$$U_1(A) = \left\{ g \in E : \max_{x\in A} |\langle g, x\rangle| < 1 \right\},$$

where A varies over the family of all compact, balanced and convex subsets of X.

In conclusion, given a topological vector space X, the *algebraic dual* of X has one locally convex TVS structure, namely $\sigma(E, X)$, the weak* topology, while the *dual* of X two locally convex TVS structures, namely $\sigma(E, X)$, the weak* topology, and $\gamma(E, X)$, the topology of compact convergence. It is clear from the definitions that $\gamma(E, X)$ is *finer* than $\sigma(E, X)$.

Proposition 4.1. *Let X be a TVS and E be the dual of X, i.e. the set of all linear continuous maps from X to $\mathbb{R}$. Let F be the algebraic dual of E, i.e. the set of all linear maps from E to $\mathbb{R}$. Equip E with the weak* topology $\sigma(E, X)$ and F with the weak* topology $\sigma(F, E)$. Then the map $J : X \to (F, \sigma(F, E))$, $J_x(g) = \langle g, x\rangle$ for all $g \in E$, is linear and continuous.*

Proof. We know that $(F, \sigma(F, E))$ has a local base consisting of the open sets

$$V_\epsilon(g_1, \ldots, g_m) = \{v \in F : \; |\langle v, g_i\rangle| < \epsilon \quad \forall 1 \leq i \leq m\},$$

where $\epsilon > 0$ and the set $\{g_1, \ldots, g_m\}$ varies over the family of all finite subsets of E. For a fixed set $V_\epsilon(g_1, \ldots, g_m)$, we put $U = \bigcap_{i=1}^{m} g_i^{-1}((-\epsilon, \epsilon))$.

Since $g_i \in E$, the set $g_i^{-1}((-\epsilon, \epsilon))$ is open in X. Thus, U is open in X. For each $x \in U$, we have $|\langle J_x, g_i\rangle| = |\langle g_i, x\rangle| < \epsilon$ for all $1 \leq i \leq m$. Thus, $J_x \in V_\epsilon(g_1, \ldots, g_m)$. This means $J(U) \subset V_\epsilon(g_1, \ldots, g_m)$. Therefore, J is continuous from X to $(F, \sigma(F, E))$. $\square$

Proposition 4.2. *Let E be a vector space and F be the algebraic dual of E, i.e. the set of all linear maps from E to $\mathbb{R}$. Equip F with the weak* topology $\sigma(F,E)$. Let $\Lambda : (F, \sigma(F,E)) \to \mathbb{R}$ be a linear continuous map. Then there exists $g \in E$ such that*

$$\langle \Lambda, h \rangle = \langle h, g \rangle \quad \forall h \in F.$$

Proof. By the definition of the weak* topology, F has a local base consisting of the sets

$$U_\epsilon(x_1, \ldots, x_m) = \{h \in F : |\langle h, x_i \rangle| < \epsilon \quad \forall 1 \le i \le m\},$$

where $\epsilon > 0$ and the set $\{x_1, \ldots, x_m\}$ varies over the family of all finite subsets of E. Because Λ is continuous, there exist $\epsilon > 0$ and $x_1, \ldots, x_m \in E$ such that

$$\Lambda(U_\epsilon(x_1, \ldots, x_m)) \subset (-1, 1).$$

If $h \in F$ satisfies $\langle h, x_1 \rangle = \ldots \langle h, x_m \rangle = 0$, then $h \in U_\epsilon(x_1, \ldots, x_m)$; and thus $|\langle \Lambda, h \rangle| < 1$. Consider a linear map $T : F \to \mathbb{R}^{m+1}$,

$$T(h) = (\langle \Lambda, h \rangle, \langle h, x_1 \rangle, \ldots, \langle h, x_m \rangle) \quad \forall h \in F.$$

The point (1,0,. . . ,0) does not belong to $T(F)$. Let S_1 be the orthogonal component of $T(F)$ in $\mathbb{R}^{m+1}$ with respect to the usual inner product.

Put $S_2 = \{(0, y_1, \ldots, y_m) : \ y_i \in \mathbb{R}\} \subset \mathbb{R}^{m+1}$. Suppose by contradiction that $S_1 \subset S_2$. Then $S_2^\perp \subset S_1^\perp$. Thus, $\{(y_0, 0, \ldots, 0) : \ y_0 \in \mathbb{R}\} \subset T(F)$. This implies $(1, 0, \ldots, 0) \in T(F)$, which is a contradiction. Therefore, $S_1 \not\subset S_2$. This means there exists a vector $\alpha = (\alpha_0, \ldots, \alpha_m) \in S_1$ with $\alpha_0 \neq 0$. By replacing α by $\alpha_0^{-1}\alpha$, we can assume $\alpha = (1, \alpha_1, \ldots, \alpha_m)$. We have $\alpha \perp T(F)$. Thus, $\alpha \cdot T(h) = 0$ for all $h \in F$. Hence,

$$\langle \Lambda, h \rangle + \alpha_1 \langle h, x_1 \rangle + \ldots + \alpha_m \langle h, x_m \rangle = 0 \quad \forall h \in F.$$

Put $g = -\alpha_1 x_1 - \ldots - \alpha_m x_m \in E$. We have $\langle \Lambda, h \rangle - \langle h, g \rangle = 0$ for all $h \in F$. Therefore, $\langle \Lambda, h \rangle = \langle h, g \rangle$ for all $h \in F$. □

Proposition 4.3. *Let X be a TVS, E be its dual, i.e. the set of all linear continuous maps from X to $\mathbb{R}$, and $f : E \to \mathbb{R}$ be a linear map. Then f is continuous on $(E, \gamma(E,X))$ if and only if it is continuous on $(E, \sigma(E,X))$.*

Proof. We know that $(E, \gamma(E,X))$ is finer than $(E, \sigma(E,X))$. Thus, if f is continuous on $(E, \sigma(E,X))$, it is also continuous on $(E, \gamma(E,X))$. Now suppose that f is continuous on $(E, \gamma(E,X))$. We show that it is continuous on $(E, \sigma(E,X))$. Let F be the algebraic dual of E, i.e. the set of all linear maps from E to $\mathbb{R}$. Equip F with the weak* topology $\sigma(F,E)$. By Proposition 4.1, the map $J : X \to F$, $J_x(g) = \langle g, x \rangle$ for all $g \in E$, is linear continuous from X to $(F, \sigma(F,X))$.

Since $f : E \to \mathbb{R}$ is linear, $f \in F$. We know that $(E, \gamma(E,X))$ has a local base consisting of the open sets

$$U_1(A) = \{g \in E : \ |\langle g, x \rangle| < 1 \quad \forall x \in A\},$$

where A varies over the family of all compact, balanced, convex subsets of X. Because f is continuous on $(E, \gamma(E, X))$, there exists a compact balanced convex subset A of X such that

$$f(U_1(A)) \subset (-1, 1). \tag{4.1}$$

Put $K = J(A) \subset F$. Since A is compact and J is continuous, K is also compact. Because A is balanced convex and J is linear, K is also balanced convex. We want to show $f \in K$. Suppose by contradiction that $f \in F \backslash K$. Applying Lemma 4.6 with x_0 therein being replaced by f, we conclude that there is a linear continuous map $\Lambda : (F, \sigma(F, E)) \to \mathbb{R}$ such that $\langle \Lambda, f \rangle = 1$ and $|\langle \Lambda, h \rangle| < 1$ for all $h \in K$. By Proposition 4.2, there exists $g \in E$ such that $\langle \Lambda, h \rangle = \langle h, g \rangle$ for all $h \in F$. Thus, $\langle \Lambda, f \rangle = \langle f, g \rangle$ and $\langle \Lambda, h \rangle = \langle h, g \rangle$ for all $h \in K$. Therefore,

$$\begin{aligned} \langle \Lambda, f \rangle &= \langle f, g \rangle, \\ |\langle h, g \rangle| &< 1 \quad \forall h \in K = J(A). \end{aligned} \tag{4.2}$$

For every $x \in A$, we have

$$|\langle g, x \rangle| = |\langle J_x, g \rangle| < 1.$$

Thus, $g \in U_1(A)$. By (4.1), $\langle f, g \rangle < 1$. This contradicts (4.2). Therefore, $f \in J(A)$. Then there exists $x \in A$ such that $f = J_x$. Thus, $\langle f, g \rangle = \langle g, x \rangle$ for all $g \in E$. Then it is straightforward from the definition of the weak* topology $(E, \sigma(E, X))$ that f is continuous on $(E, \sigma(E, X))$. □

Proposition 4.4. *Let X be a TVS, E be its dual, i.e. the set of all linear continuous maps from X to $\mathbb{R}$, and S be a convex subset of E. Then S is closed in $(E, \sigma(E, X))$ if and only if it is closed in $(E, \gamma(E, X))$.*

Proof. ($\Leftarrow$) Suppose that S is closed in $(E, \gamma(E, X))$. Because $\gamma(E, X)$ is finer than $\sigma(E, X)$, S is also closed in $(E, \sigma(E, X))$.

($\Rightarrow$) Suppose that S is closed in $(E, \sigma(E, X))$ Take any y in the closure of S with respect to $(E, \gamma(E, X))$. By Hahn-Banach theorem, to show that $y \in S$, it suffices to show that $\langle \Lambda, y \rangle = 0$ for all linear continuous map $\Lambda : (E, \sigma(E, X)) \to \mathbb{R}$ which vanishes on S. Let Λ be such a map. Then by Proposition 4.3, Λ is continuous on $(E, \sigma(E, X))$. Applying Proposition 4.2 with E therein being replaced by X and F therein being replaced by E, we conclude that there exists $x \in X$ such that $\langle \Lambda, z \rangle = \langle z, x \rangle$ for all $z \in E$. Thus, $\langle \Lambda, y \rangle = \langle y, x \rangle$. Put

$$A = \{tx : \ -1 \le t \le 1\}.$$

Then A is a compact, balanced, convex subset of X. For each $\epsilon > 0$, we know from the definition of the topology $\gamma(E, X)$ that the set

$$U_\epsilon(A) = \{g \in E : \ |\langle g, z \rangle| < \epsilon \quad \forall z \in A\}$$

is a neighborhood of 0 in $(E, \gamma(E, X))$. Thus, $y + U_\epsilon(A)$ is a neighborhood of y in $(E, \gamma(E, X))$. Because y lies in the closure of S with respect to $(E, \gamma(E, X))$, there exists $y_\epsilon \in S$ such that $y_\epsilon \in y + U_\epsilon(A)$. Thus,

$$|\langle y_\epsilon - y, z \rangle| < \epsilon \quad \forall z \in A, \forall \epsilon > 0.$$

In particular, $|\langle y_\epsilon - y, x\rangle| < \epsilon$ for all $\epsilon > 0$. We have

$$\langle y_\epsilon, x\rangle = \langle \Lambda, y_\epsilon\rangle = 0.$$

Thus, $|\langle y, x\rangle| < \epsilon$ for all $\epsilon > 0$. Therefore, $\langle y, x\rangle = 0$ and thus $\langle \Lambda, y\rangle = 0$. $\square$

Let X be a TVS and E be its dual, i.e. the set of all linear continuous maps from X to $\mathbb{R}$. For each subset U of X, we put

$$U^o = \{g \in E : \ |\langle g, x\rangle| \le 1 \quad \forall x \in U\},$$

which is called the *polar* of U. It is easy to see that U^o is balanced and convex. We know that the weak* topology $\sigma(E, X)$ is the topology generated be the separating family of seminorms $\{p_x\}_{x\in X}$. Thus, each map p_x is continuous on $(E, \sigma(E, X))$. Since $U^o = \bigcap_{x\in U} p_x^{-1}([-1, 1])$, U^o is closed in $(E, \sigma(E, X))$.

If U is a neighborhood of 0 in X then by Banach-Alaoglu theorem [Rud73, p.67], U^o is compact in $(E, \sigma(E, X))$. In the sequel, we write $(U^o, \sigma(E, X))$ to indicate the topology which U^o inherits from $(E, \sigma(E, X))$.

Proposition 4.5. *Let X be a TVS and E be its dual, i.e. the set of all linear continuous maps from X to $\mathbb{R}$. Let S be a convex subset of E. Then S is closed in $(E, \sigma(E, X))$ if and only if $S \cap U^o$ is closed in $(U^o, \sigma(E, X))$ for every neighborhood U of 0 in X.*

Proof. If S is closed in $(E, \sigma(E, X))$ then it is obvious that $S \cap \tilde{U}$ is closed in $(\tilde{U}, \sigma(E, X))$ for every subset $\tilde{U}$ of E.

Now suppose that $S \cap U^o$ is closed in $(U^o, \sigma(E, X))$ for every neighborhood U of 0 in X. Since S and U^o are convex, so is $S \cap U^o$. By Proposition 4.4, $S \cap U^o$ is closed in $(U^o, \gamma(E, X))$. We need to show that S is closed in $(E, \sigma(E, X))$. By Proposition 4.4 again, this is equivalent to showing that S is closed in $(E, \gamma(E, X))$. Put $\mathscr{O} = E \backslash S$. Then

$$\mathscr{O} \cap U^o = (E\backslash S) \cap U^o = U^o \backslash (S \cap U^o),$$

which is open in $(U^o, \gamma(E, X))$ for every open neighborhood U of 0 in X. We need to show that $\mathscr{O}$ is open in $(E, \gamma(E, X))$.

Take any $y \in \mathscr{O}$, and put $W = \mathscr{O} - \{y\} = \{g - y : \ g \in \mathscr{O}\}$. We show that W is a neighborhood of 0 in $(E, \gamma(E, X))$. Since $y \in E$, there exists an open neighborhood $\tilde{U}$ of 0 in X such that $|\langle y, z\rangle| < 1$ for all $z \in \tilde{U}$. Because X is metrizable, there exists a countable local base $U_0 \supset U_1 \supset U_2 \supset \ldots$ We can assume $U_0 = U_1 = X$ and $U_2 = \tilde{U}$. Then $U_{m+1} \subset \tilde{U}$ for all $m \ge 1$. Thus,

$$|\langle y, z\rangle| < 1 \quad \forall z \in U_{m+1},\ \forall m \ge 1. \tag{4.3}$$

We will show that there is a sequence of finite sets $B_0, B_1, B_2, \ldots$ such that $B_m \subset U_m$ for all $m \ge 0$ and $U_m^o \cap A_m^o \subset W$ for all $m \ge 1$, where

$$A_m = \bigcap_{k=0}^{m-1} B_k.$$

Choose $B_0 = \{0\}$. Then $A_1 = B_0 = \{0\}$. We have

$$U_1^o = X^o = \{g \in E : \ |\langle g, x\rangle| \leq 1 \ \ \forall x \in X\} = \{0\}.$$

Thus, $U_1^o \cap A_1^o = \{0\} \cap A_1^o = \{0\} \subset W$. Now suppose that for some $m \geq 1$, we have found $B_0, B_1, \ldots, B_{m-1}$ such that $U_m^o \cap A_m^o \subset W$. We need to find a finite set $B_m \subset U_m$ such that $U_{m+1}^o \cap (A_m \cup B_m)^o \subset W$.

Suppose by contradiction that there is no such B_m. Then for every finite set $B \subset U_n$, $U_{m+1}^o \cap (A_m \cup B)^o \not\subset W$. Put $K_m = U_{m+1}^o \cap (E\backslash W)$. Then for every finite set $B \subset U_m$,

$$K_m \cap (A_m \cup B)^o = U_{m+1}^o \cap (A_m \cup B)^o \cap (E\backslash W) \neq \emptyset.$$

Denote by $\mathscr{P}$ the family of all sets $K_m \cap (A_m \cup B)^o$ where B varies over the family of all finite subsets of U_m. Then every member of $\mathscr{P}$ is nonempty. If B and $\tilde{B}$ are two finite subsets of U_n then

$$\begin{aligned}
[K_m \cap (A_m \cup B)^o] \cap [K_m \cap (A_m \cup \tilde{B})^o] &= K_m \cap [(A_m \cup B)^o \cap (A_m \cup \tilde{B})^o] \\
&= K_m \cap [(A_m \cup B) \cup (A_m \cup \tilde{B})]^o \\
&= K_m \cap (A_m \cup \hat{B})^o,
\end{aligned}$$

where $\hat{B} = B \cup \tilde{B}$, which is a finite subset of U_n. Thus, $\mathscr{P}$ is closed under finite intersection. Since every member of $\mathscr{P}$ is nonempty, $\mathscr{P}$ has the finite intersection property. We have

$$\begin{aligned}
K_m &= U_{m+1}^o \cap (E\backslash W) \\
&= U_{m+1}^o \cap [(E\backslash \mathscr{O}) - \{y\}] \\
&= U_{m+1}^o \cap (S - \{y\}) \\
&= [(U_{m+1}^o + \{y\}) \cap S] - \{y\}.
\end{aligned}$$

Put $\hat{U} = \frac{1}{2}U_{m+1}$, which is a neighborhood of 0 in X. For every $y \in U_{m+1}^o$ and $w \in U$, we write $w = \frac{1}{2}z$ for some $z \in U_{m+1}$. Then

$$\begin{aligned}
|\langle g + y, w\rangle| &= \frac{1}{2}|\langle g + y, z\rangle| \\
&\leq \frac{1}{2}|\langle g, z\rangle| + \frac{1}{2}|\langle y, z\rangle| \\
&< \frac{1}{2} + \frac{1}{2} = 1 \ \ \text{(by (4.3))}.
\end{aligned}$$

Thus, $g + y \in (\hat{U})^o$. Thus, $U_{m+1}^o + \{y\} \subset (\hat{U})^o$. By the hypothesis, $(\hat{U})^o \cap S$ is closed in $((\hat{U})^o, \sigma(E, X))$. Thus, $(U_{m+1}^o + \{y\}) \cap S$ is closed in $U_{m+1}^o + \{y\}$. Then $[(U_{m+1}^o + \{y\}) \cap S] - \{y\}$ is closed in U_{m+1}^o. Thus, $U_{m+1}^o \cap (S - \{y\})$ is closed in U_{m+1}^o. Therefore, K_m is closed in $(U_{m+1}^o, \sigma(E, X))$. According to the remark before the statement of Proposition 4.5, U_{m+1}^o is compact in $(E, \sigma(E, X))$. Thus, K_m is compact in $(E, \sigma(E, X))$.

Each member of $\mathscr{P}$ is of the form $K_m \cap (A_m \cup B)^o$ where B is a finite subset of U_m. According to a remark before the statement of Proposition 4.5, $(A_m \cup B)^o$ is closed in $(E, \sigma(E, X))$. Thus, $\mathscr{P}$ is a family of closed subsets of $(K_m, \sigma(E, X))$ that has the finite intersection property. Thus, the intersection of all members of $\mathscr{P}$ is nonempty. In particular,

$$\begin{aligned}
\emptyset \neq \bigcap\nolimits_{z \in U_m} [K_m \cap (A_n \cup \{z\})^o] &= K_m \cap \left[\bigcup_{z \in U_m} (A_m \cup \{z\})\right]^o \\
&= K_m \cap (A_m \cup U_m)^o \\
&= K_m \cap A_m^o \cap U_m^o \\
&\subset (E \backslash W) \cap (A_m^o \cap U_m^o).
\end{aligned}$$

However, $(E\backslash W) \cap (A_m^o \cap U_m^o) = \emptyset$ because $A_m^o \cap U_m^o \subset W$. This is a contradiction. Therefore, we have finished proving the existence of the sequence of finite sets $B_0, B_1, B_2, \ldots$ such that $B_m \subset U_m$ for all $m \geq 0$ and $U_m^o \cap A_m^o \subset W$ for all $m \geq 1$. Put

$$A = \bigcup\nolimits_{n=1}^{\infty} A_n = \{0\} \cup \left(\bigcup\nolimits_{m=1}^{\infty} B_m\right).$$

Then A is a countable and compact subset of X. We have $A^o = \bigcap_{m=1}^{\infty} A_m^o$. Thus, for each $m \in \mathbb{N}$, $U_m^o \cap A^o \subset U_m^o \cap A_m^o \subset W$. This implies

$$\left(\bigcup\nolimits_{m=1}^{\infty} U_m^o\right) \cap A^o = \bigcup\nolimits_{m=1}^{\infty} (U_m^o \cap A^o) \subset W.$$

Moreover,

$$\bigcup\nolimits_{m=1}^{\infty} U_m^o = \left(\bigcap\nolimits_{m=1}^{\infty} U_m\right)^o = \{0\}^o = E.$$

Hence,

$$A^o = E \cap A^o = \left(\bigcup\nolimits_{m=1}^{\infty} U_m^o\right) \cap A^o \subset W.$$

By the definition of the topology of compact convergence $\gamma(E, X)$, the set $U_{\frac{1}{2}}(A) = \left\{g \in E : |\langle g, x\rangle| < \frac{1}{2}\ \forall x \in A\right\}$ is an open neighborhood of 0 in E. Because $U_{\frac{1}{2}}(A) \subset A^o \subset W$, W is an open neighborhood of 0 in $(E, \sigma(E, X))$. $\square$

Lemma 4.6. *Let F be a locally convex TVS, K be a closed balanced convex subset of F, and $x_0 \in F\backslash K$. Then there exists a linear continuous map $\Lambda : F \to \mathbb{R}$ such that $\langle \Lambda, x_0\rangle = 1$ and $|\langle \Lambda, x\rangle| < 1$ for all $x \in K$.*

Proof. Because K is closed in F and $x_0 \in F\backslash K$, there exists a neighborhood U of 0 in F such that $(x_0 + U) \cap K = \emptyset$. Since F is locally convex, we can assume that U is convex. By Part (ii) of Proposition 6.1, we can even assume that U is balanced. Put $L = K - U = \{x - y : x \in K, y \in U\}$. Then L is a neighborhood of 0 in F and $x_0 \notin L$. Because K and U are balanced and convex, so is L. By Proposition 6.4, the Minkowski functional $\mu_L : F \to \mathbb{R}$ is a seminorm. Since $0 \in K$ and $x_0 \in F\backslash K$, $x_0 \neq 0$. Put $F_0 = \{\lambda x_0 : \lambda \in \mathbb{R}\}$, which is a vector subspace of F.

Because the scalar multiplication on F is continuous, the family $\{(-\epsilon, \epsilon)x_0\}_{\epsilon>0}$ is a local base of F_0. Thus, the map $\Lambda_0 : F_0 \to \mathbb{R}$, $\Lambda_0(\lambda x_0) = \lambda$, is linear and continuous on F_0. Since $x_0 \notin L$, $\mu_L(x_0) > 1$. Hence,

$$|\Lambda(\lambda x_0)| = |\lambda| \leq |\lambda|\mu_L(x_0) = \mu_L(\lambda x_0) \quad \forall \lambda \in \mathbb{R}.$$

By Hahn-Banach theorem [Rud73, p.57], Λ_0 has a linear continuous extension $\Lambda : F \to \mathbb{R}$ such that $|\Lambda(x)| \leq \mu_L(x)$ for all $x \in F$. We have $\Lambda(x_0) = \Lambda_0(x_0) = 1$. For every $x \in K$, x lies in the interior of L. Thus, $|\Lambda(x)| \leq \mu_L(x) < 1$. □

4.2 Some topological properties of the dual of $C^\infty(\mathbb{R}^n)$

In this section, we denote by X the metrizable TVS $C^\infty(\mathbb{R}^n)$ as described in Section 6.2. We also denote by E the dual of X, i.e. the set of all linear continuous maps from $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}$. By Proposition 6.9, X is a Fréchet space.

Proposition 4.7. *For each $m \in \mathbb{N}$ and $\phi \in C^\infty(\mathbb{R}^n)$, we put*

$$p_m(\phi) = \max\left\{|D^\alpha \phi(x)| : \ |\alpha| \leq m, x \in \bar{B}_m\right\},$$

where $\bar{B}_m$ is the closed ball in $\mathbb{R}^n$ which is centered at the origin and with radius m. Let U be a neighborhood of 0 in X. Then there exists $N \in \mathbb{N}$ such that

$$|T\psi| \leq 2Np_N(\psi) \quad \forall \psi \in X, \ T \in U^o.$$

An immediate consequence of Proposition 4.7 is that supp$T \subset \bar{B}_N$ for all $T \in U^o$.

Proof of Proposition 4.7. For each $m \in \mathbb{N}$, we put

$$V_m = \left\{\phi \in C^\infty(\mathbb{R}^n) : \ |D^\alpha \phi(x)| < \frac{1}{m} \ \ \forall |\alpha| \leq m, \ \forall x \in \bar{B}_m\right\}.$$

By the definition of the topology on $C^\infty(\mathbb{R}^n)$ in Section 6.2, the family $\{V_m\}_{m\in\mathbb{N}}$ is a local base of $C^\infty(\mathbb{R}^n)$. Thus, there exists $N \in \mathbb{N}$ such that $V_N \subset U$. Thus, for every $T \in U^o$,

$$|T(\phi)| \leq 1 \quad \forall \phi \in V_N.$$

For each $\psi \in C^\infty(\mathbb{R}^n)$ and $\epsilon > 0$, we put

$$\phi_\epsilon = \frac{\psi}{2Np_N(\psi) + \epsilon} \in C^\infty(\mathbb{R}^n).$$

Then

$$p_N(\phi_\epsilon) = \frac{p_N(\psi)}{2Np_N(\psi) + \epsilon} < \frac{1}{2N}.$$

Thus, $\phi_\epsilon \in V_N$. Hence, $|T(\phi_\epsilon)| \leq 1$. Then

$$|T\psi| = (2Np_N(\psi) + \epsilon)|T\phi_\epsilon| \leq 2Np_N(\psi) + \epsilon.$$

Because this inequality is true for every $\epsilon > 0$, we must have $|T\psi| \leq 2Np_N(\psi)$ for all $\psi \in C^\infty(\mathbb{R}^n)$. □

Proposition 4.8. *Let $\mathscr{D}(\mathbb{R}^n)$ be the test-function space on $\mathbb{R}^n$, i.e. the TVS defined in Proposition 6.10. Then the embedding $\mathscr{D}(\mathbb{R}^n) \hookrightarrow X$ is linear and continuous.*

Proof. It is clear that the identity map $\mathscr{D}(\mathbb{R}^n) \hookrightarrow C^\infty(\mathbb{R}^n)$ is linear. Proposition 6.14 gives us a method to show that this map is continuous. Let (ϕ_n) be any sequence in $\mathscr{D}(\mathbb{R}^n)$ that converges to 0. We show that (ϕ_n) also converges to 0 in $C^\infty(\mathbb{R}^n)$. By Proposition 6.8, we need to show that for every multi-index α, $(D^\alpha\phi_n)$ converges to 0 uniformly on every compact subset of $\mathbb{R}^n$. By Proposition 6.13, there exists a compact set $K \subset \mathbb{R}^n$ such that $\phi_n \in \mathscr{D}_K$ for all $n \in \mathbb{N}$, and that $(D^\alpha\phi_n)$ converges to 0 uniformly on K for every multi-index α. Let L be any compact set in $\mathbb{R}^n$. We have $\phi_n|_L$ is supported in $K \cap L$. Because $(D^\alpha\phi_n)$ converges to 0 uniformly on $K \cap L$, it converges to 0 uniformly on L. □

Recall that $\mathscr{E}'(\mathbb{R}^n)$ denotes that set of all compactly supported distributions on $\mathbb{R}^n$. By Part (ii) of Proposition 6.17, an element in $\mathscr{E}'(\mathbb{R}^n)$ can be considered as an element in E thanks to the unique linear continuous extension from $\mathscr{D}(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$. Now let us consider $T \in E$. Thanks to Proposition 4.8, the restriction of T on $\mathscr{D}(\mathbb{R}^n)$ is a distribution on $\mathbb{R}^n$. Since T is continuous, there exists a neighborhood U of 0 in X such that $|T(\phi)| < 1$ for all $\phi \in U$. Thus, $T \in U^o$. Then by Proposition 4.7, T is compactly supported. Hence, $T|_{\mathscr{D}(\mathbb{R}^n)} \in \mathscr{E}'(\mathbb{R}^n)$. Therefore, E can be identified with the set $\mathscr{E}'(\mathbb{R}^n)$ in a natural way.

Proposition 4.9. *Let $P \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$. Then the differential operator $P(D) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is linear and continuous.*

Proof. It suffices to show that for every multi-index α, the differential operator $D^\alpha : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is continuous. Proposition 6.8 gives a necessary and sufficient condition for the convergence of a sequence in $C^\infty(\mathbb{R}^n)$. Let (f_n) be any sequence in $C^\infty(\mathbb{R}^n)$ that converges to $f \in C^\infty(\mathbb{R}^n)$. Then for every multi-index β, the sequence $(D^\beta D^\alpha f_n)$ converges to $D^\beta D^\alpha f$ uniformly on every compact subset of $\mathbb{R}^n$. Thus, $(D^\alpha f_n)$ converges to $D^\alpha f$ in $C^\infty(\mathbb{R}^n)$. Therefore, D^α is a continuous map. □

Proposition 4.10. *Let $P \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$. Define a map $f : X \to X$, $f(\phi) = P(D)\phi$ for all $\phi \in X$. Note that by Proposition 4.9, f is linear and continuous. Let $f^* : E \to E$ be the dual map of f, i.e. $f^*(T) = T \circ f$ for all $T \in E$. Let U be a neighborhood of 0 in X. Then $f^*(E) \cap U^o$ is closed in $(U^o, \sigma(E, X))$.*

Proof. Consider a net $\{T_i\}_{i\in I}$ in $f^*(E) \cap U^o$ that converges to $T_0 \in U^o$ in $(U^o, \sigma(E, X))$. We need to show that $T_0 \in f^*(E)$. By Proposition 4.7, there exists $N \in \mathbb{N}$ such that

$$|\langle T, \phi\rangle| \leq 2Np_N(\phi) \quad \forall T \in U^o,\ \forall \phi \in X,$$

where $p_N(\phi) = \max\left\{|D^\alpha\phi(x)| : |\alpha| \leq N, x \in \bar{B}_N\right\}$ and $\bar{B}_N$ is the closed ball in $\mathbb{R}^n$ which is centered at the origin and with radius N. Thus,

$$|\langle T_i, \phi\rangle| \leq 2Np_N(\phi) \quad \forall i \in I,\ \forall \phi \in X. \tag{4.4}$$

If $\phi \in \mathscr{D}(\mathbb{R}^n\backslash\bar{B}_N)$ then $|\langle T_i, \phi\rangle| \le 2Np_N(\phi) = 0$. Thus, T_i vanishes in $\mathbb{R}^n\backslash\bar{B}_N$. This implies supp$T_i \subset \bar{B}_N$ for all $i \in I$. Because $T_i \in f^*(E)$, there exists $S_i \in E$ such that $T_i = f^*(S_i) = S_i \circ f$. For every $\phi \in X$ we have

$$\langle T_i, \phi\rangle = \langle S_i \circ f, \phi\rangle = \langle S_i, f(\phi)\rangle = \langle S_i, P(D)\phi\rangle = \langle P(-D)S_i, \phi\rangle . \tag{4.5}$$

Hence, $T_i = P(-D)S_i$ for all $i \in I$. Because $S_i \in E$, $S_i|_{\mathscr{D}(\mathbb{R}^n)} \in \mathscr{E}'(\mathbb{R}^n)$ by the remark before Proposition 4.9. By Proposition 3.4, we have

$$[\mathrm{supp}S_i] = [\mathrm{supp}T_i] \subset \bar{B}_N \quad \forall i \in I,$$

where $[K]$ denotes the convex hull of a set $K \subset \mathbb{R}^n$. Take $\psi_0 \in C^\infty(\mathbb{R}^n)$ arbitrarily. We show that the net $\{\langle S_i, \psi_0\rangle\}_{i\in I}$ is bounded and convergent in $\mathbb{R}$.

Let χ be a function in $\mathscr{D}(\mathbb{R}^n)$ such that $\chi = 1$ in $\bar{B}_{N+1}$. Then $\chi\psi_0 = \psi_0$ in $\bar{B}_{N+1}$, which implies supp$(\chi\psi_0 - \psi_0) \subset \mathbb{R}^n\backslash B_{N+1}$. Thus,

$$\mathrm{supp}S_i \cap \mathrm{supp}(\chi\psi_0 - \psi_0) \subset B_N \cap (\mathbb{R}^n\backslash B_{N+1}) = \emptyset.$$

By Part (i) of Proposition 6.17, $\langle S_i, \chi\psi_0 - \psi_0\rangle = 0$. Hence,

$$\langle S_i, \chi\psi_0\rangle = \langle S_i, \psi_0\rangle. \tag{4.6}$$

Since $\chi\psi_0 \in \mathscr{D}(\mathbb{R}^n)$, by Proposition 4.13, there exists $\phi_0 \in X$ such that $P(D)\phi_0 = \chi\psi_0$. Then from (4.5) we have

$$\langle S_i, \chi\psi_0\rangle = \langle S_i, P(D)\psi_0\rangle = \langle T_i, \phi_0\rangle .$$

Together with (4.6), this identity yields

$$\langle S_i, \psi_0\rangle = \langle T_i, \phi_0\rangle \quad \forall i \in I. \tag{4.7}$$

Because the net $\{T_i\}_{i\in I}$ converges to T_0 in $(U^o, \sigma(E, X))$, the net $\{\langle T_i, \phi_0\rangle\}_{i\in I}$ converges in $\mathbb{R}$. Thus, the net $\{\langle S_i, \psi_0\rangle\}_{i\in I}$ converges in $\mathbb{R}$. Denote by $\langle S, \psi_0\rangle$ the limit. Then we get a linear map $S : X \to \mathbb{R}$ such that $\lim \langle S_i, \psi\rangle = \langle S, \psi\rangle$ for all $\psi \in X$. By (4.4) and (4.7), we have

$$|\,\langle S_i, \psi_0\rangle\,| \le 2Np_N(\phi_0) \quad \forall i \in I.$$

Thus, the set $\{\langle S_i, \psi_0\rangle :\ i \in I\}$ is bounded in $\mathbb{R}$. Put $\Gamma = \{S_i :\ i \in I\}$. Then Γ is a family of linear continuous maps from X to $\mathbb{R}$ which is pointwise bounded. Because X is a Fréchet space (Proposition 6.9), X is a complete metric space. According to Banach-Steinhaus theorem (see [Rud73, p.43]), Γ is equicontinuous. In other words, for each $\epsilon > 0$, there exists a neighborhood U of 0 in X such that

$$|\,\langle S_i, \psi\rangle\,| < \epsilon \quad \forall \psi \in U,\ \forall i \in I.$$

Thus, $|\langle S, \psi\rangle| \le \epsilon$ for all $\psi \in U$. Thus, S is a continuous map from X to $\mathbb{R}$. This means $S \in E$. For every $\phi \in X$, we have

$$\langle T_i, \phi\rangle = \langle S_i, f(\phi)\rangle \to \langle S, f(\phi)\rangle = \langle S \circ f, \phi\rangle = \langle f^*(S), \phi\rangle .$$

On the other hand, we know that $\langle T_i, \phi\rangle \to \langle T_0, \phi\rangle$. Thus,

$$\langle f^*(S), \phi\rangle = \langle T_0, \phi\rangle \quad \forall \phi \in X.$$

Therefore, $T_0 = f^*(S) \in f^*(E)$. $\square$

4.3 Existence of smooth solutions to $P(D)u = f$

With the background in the dual space of $C^\infty(\mathbb{R}^n)$ discussed in the previous section, we are now able to make concrete the ideas mentioned at the beginning of Section 4. It is necessary to approximate (in C^k-norm) a function v satisfying $P(D)v = 0$ in a finite ball in $\mathbb{R}^n$ by functions satisfying the same equation in the whole space $\mathbb{R}^n$. If $P(D)$ is the Laplacian, this can be done by applying, first, Walsh's theorem (quoted in Proposition 2.1) to get an approximation for the function v itself, and, secondly, the estimation of the derivatives of v by the function v itself (quoted in Proposition 2.2). Proposition 4.12 is a generalization of Walsh's theorem for a general linear differential operator $P(D)$.

Let us introduce some notations. For each $s > 0$, we denote by B_s the open ball in $\mathbb{R}^n$ which is centered at the origin and with radius s. Recall that for each nonnegative integer k and nonempty open bounded set $\Omega \subset \mathbb{R}^n$, $C^k(\bar{\Omega})$ denotes the vector space of all functions $f : \Omega \to \mathbb{R}$ such that $D^\alpha f$ exists and is uniformly continuous in Ω for all multi-indices $|\alpha| \le k$. It is well-known that $C^k(\bar{\Omega})$ is a normed vector space with

$$||u||_{C^k(\bar{\Omega})} = \max\left\{|D^\alpha u(x)| : \ |\alpha| \le k, \ x \in \bar{\Omega}\right\} \quad \forall u \in C^k(\bar{\Omega}).$$

Proposition 4.11. *Let k be a nonnegative integer and Ω be a nonempty open bounded subset of $\mathbb{R}^n$. Suppose that $L : C^k(\bar{\Omega}) \to \mathbb{R}$ is a linear continuous map. Then there exists a distribution $T \in \mathscr{E}'(\mathbb{R}^n)$ supported in $\bar{\Omega}$ such that*

$$L(u|_{\bar{\Omega}}) = \langle T, u\rangle \quad \forall u \in C^\infty(\mathbb{R}^n).$$

Proof. Because L is continuous, there exists a constant $C > 0$ such that

$$|L\phi| \le C\|\phi\|_{C^k(\bar{\Omega})} \quad \forall \phi \in C^k(\bar{\Omega}).$$

Define a linear map $T : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$, $\langle T, \phi\rangle = L(\phi|_\Omega)$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$. For every $\phi \in \mathscr{D}(\mathbb{R}^n)$,

$$\begin{aligned} |\langle T, \phi\rangle| = |L(\phi|_\Omega)| &\le C\|\phi|_\Omega\|_{C^k(\bar{\Omega})} \\ &= C\max\left\{|D^\alpha\phi(x)| : \ x \in \bar{\Omega}, \ |\alpha| \le k\right\} \\ &= C\max\left\{|D^\alpha\phi(x)| : \ x \in \mathbb{R}^n, \ |\alpha| \le k\right\}. \end{aligned} \tag{4.8}$$

According to the notation in Proposition 6.15, we have

$$|\langle T, \phi\rangle| \le C||\phi||_k \quad \forall \phi \in \mathscr{D}(\mathbb{R}^n)$$

and thus $T \in \mathscr{D}'(\mathbb{R}^n)$. By (4.8), T vanishes in $\mathbb{R}^n \backslash \bar{\Omega}$. Thus, $T \in \mathscr{E}'(\mathbb{R}^n)$. Define a linear map $\tilde{T} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, $\tilde{T}(u) = L(u|_\Omega)$ for all $u \in C^\infty(\mathbb{R}^n)$. Let (u_m) be any sequence in $C^\infty(\mathbb{R}^n)$ which converges to $u_0 \in C^\infty(\mathbb{R}^n)$. By Proposition 6.8, $D^\beta u_m \to D^\beta u_0$ uniformly on $\bar{\Omega}$ as $m \to \infty$ for every multi-index β. In particular, $\|u_m - u_0\|_{C^k(\bar{\Omega})} \to 0$ as $m \to \infty$. Because L is continuous, $L(u_m|_\Omega) \to L(u_0|_\Omega)$ in $\mathbb{R}$. Then $\tilde{T}(u_m) \to \tilde{T}(u_0)$. Therefore, $\tilde{T}$ is continuous. Because $\tilde{T}(u) = \langle T, u\rangle =$

$L(u|_\Omega)$ for all $u \in \mathscr{D}(\mathbb{R}^n)$, $\tilde{T}$ is the linear continuous extension of T to $C^\infty(\mathbb{R}^n)$. This allows us to write $\langle T, u\rangle = L(u|_\Omega)$ for all $u \in C^\infty(\mathbb{R}^n)$.

Next, we show that T is supported in $\bar{\Omega}$. For every $u \in \mathscr{D}(\mathbb{R}^n \backslash \bar{\Omega})$,

$$\langle T, u\rangle = L(u|_\Omega) = L(0) = 0.$$

Thus, T vanishes in $\mathbb{R}^n \backslash \bar{\Omega}$. Therefore, $\mathrm{supp} T \subset \bar{\Omega}$. □

Proposition 4.12. *Let k be a nonnegative integer and $0 < s < r$. Let $P \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$ and $v \in C^\infty(\mathbb{R}^n)$ such that $P(D)v = 0$ in B_r. Then there exists a sequence (v_m) in $C^\infty(\mathbb{R}^n)$ such that $P(D)v_m = 0$ in $\mathbb{R}^n$ and*

$$\lim_{m\to\infty} ||v_m - v||_{C^k(\bar{B}_s)} = 0.$$

Proof. Denote by X the metrizable TVS $C^\infty(\mathbb{R}^n)$ as defined in Section 6.2, and E its dual, i.e. the set of all linear continuous maps from X to $\mathbb{R}$. Equip E with the weak* topology $\sigma(E, X)$. Define a linear map $f : X \to X$, $f(\phi) = P(D)\phi$ for all $\phi \in X$. By Proposition 4.9, f is continuous. Let $f^* : E \to E$ be the dual map of f, i.e. $f^*(T) = T \circ f$ for all $T \in E$. Put

$$X_1 = \{u \in C^k(\bar{B}_s) : \exists \text{extension } \tilde{u} \in C^\infty(\mathbb{R}^n) \text{ such that } P(D)\tilde{u} = 0 \text{ in } \mathbb{R}^n\}.$$

Then X_1 is a vector subspace of X. We want to show that $v|_{B_s} \in \bar{X}_1$. Let L be any linear continuous functional on $C^k(\bar{B}_s)$ which vanishes on X_1. We want to show that $L(v|_{B_s}) = 0$. By Proposition 4.11, there exists a distribution $T \in \mathscr{E}'(\mathbb{R}^n)$ supported in $\bar{B}_s$ such that

$$L(u|_{\bar{B}_s}) = \langle T, u\rangle \quad \forall u \in C^\infty(R^n). \tag{4.9}$$

By Part (ii) of Proposition 6.17, we can regard $T \in E$. We will show that $T \in f^*(E)$. For every $\phi \in \ker f$, $\phi_{\bar{B}_s} \in X_1$. Thus, $\langle T, \phi\rangle = L(\phi_{\bar{B}_s}) = 0$. This implies that T vanishes on $\ker f$. First, we show that T belongs to the closure of $f^*(E)$ in $(E, \sigma(E, X))$. Let Γ be any linear continuous map from $(E, \sigma(E, X))$ to $\mathbb{R}$ such that

$$\langle \Gamma, h\rangle = 0 \quad \forall h \in f^*(E). \tag{4.10}$$

We need to show that $\langle \Gamma, T\rangle = 0$. According to Proposition 4.2, where E therein is replaced by X, F therein is replaced by E, and Λ therein by Γ, we conclude that there exists $\phi_0 \in X$ such that

$$\langle \Gamma, h\rangle = \langle h, \phi_0\rangle \quad \forall h \in E. \tag{4.11}$$

By (4.10), we have $\langle h, \phi_0\rangle = 0$ for all $h \in f^*(E)$. Thus, for every $S \in E$,

$$\langle S, f(\phi_0)\rangle = \langle S \circ f, \phi_0\rangle = \langle f^*(S), \phi_0\rangle = 0.$$

Hence, $f(\phi_0) = 0$ by Hahn-Banach theorem. Thus, $\phi_0 \in \ker f$. By (4.11), $\langle \Gamma, T\rangle = \langle T, \phi_0\rangle$, which is zero because T vanishes on $\ker f$. We have proved that T lies in the closure of $f^*(E)$ in $(E, \sigma(E, X))$.

Now we show that $f^*(E)$ is a closed set in $(E, \sigma(E, X))$. Since f^* is a linear map, $f^*(E)$ is a vector subspace of E. In particular, $f^*(E)$ is convex. According to Proposition 4.5 where S therein is replaced by $f^*(E)$, it suffices to show that $f^*(E) \cap U^o$ is closed in $(U^o, \sigma(E, X))$ for every neighborhood U of 0 in X. This was proved in Proposition 4.10. Therefore, we have proved that $T \in f^*(E)$. Write $T = f^*(\Lambda) = \Lambda \circ f$. Then

$$\langle T, \phi \rangle = \langle \Lambda \circ f, \phi \rangle = \langle \Lambda, f(\phi) \rangle = \langle \Lambda, P(D)\phi \rangle \quad \forall \phi \in X. \tag{4.12}$$

Thus, $\langle T, \phi \rangle = \langle P(-D)\Lambda, \phi \rangle$ for all $\phi \in X$. Hence, $P(-D)\Lambda = T$. Because $\Lambda \in E$, $\Lambda|_{\mathscr{D}(\mathbb{R}^n)} \in \mathscr{E}'(\mathbb{R}^n)$ by Proposition 4.8. Then by Proposition 3.4,

$$[\mathrm{supp}\Lambda] = [\mathrm{supp}T] \subset \bar{B}_s.$$

We have $\mathrm{supp}\Lambda \cap \sup\mathrm{p}(P(D)v) \subset \bar{B}_s \cap (R^n \backslash B_r) = \emptyset$. By Part (i) of Proposition 6.17, $\langle \Lambda, P(D)v \rangle = 0$. By (4.12), we get

$$\langle T, v \rangle = \langle \Lambda, P(D)v \rangle = 0.$$

Therefore, from (4.9) we get $L(v|_{\bar{B}_s}) = \langle T, v \rangle = 0$. □

Proposition 4.13. *Let $P \in \mathbb{R}[x_1, \ldots, x_n] \backslash \{0\}$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$. Then there exists $u \in C^\infty(\mathbb{R}^n)$ such that $P(D)u = \phi$ in $\mathbb{R}^n$.*

Proof. By Proposition 3.8 (the Malgrange-Ehrenpreis theorem), the differential operator $P(D)$ has a fundamental solution $\Gamma \in \mathscr{D}'(\mathbb{R}^n)$. By Proposition 6.20, the function $u = \Gamma * \phi$ belongs to $C^\infty(\mathbb{R}^n)$. We also have

$$P(D)u = P(D)(\Gamma * \phi) = (P(D)\Gamma) * \phi.$$

Thus, $P(D)u = \delta_0 * \phi = \phi$ by Proposition 6.22. □

Proposition 4.14. *Let $P \in \mathbb{R}[x_1, \ldots, x_n] \backslash \{0\}$ and $f \in C^\infty(\mathbb{R}^n)$. Then there exists $u \in C^\infty(\mathbb{R}^n)$ such that $P(D)u = f$ in $\mathbb{R}^n$.*

Proof. Put

$$\begin{aligned} A_0 &= \{x \in \mathbb{R}^n : |x| < 2\}, \\ A_k &= \{x \in \mathbb{R}^n : k < |x| < k+2\} \quad \forall k \in \mathbb{N}. \end{aligned}$$

Then the collection $\{A_k : k = 0, 1, 2, \ldots\}$ is an open cover of $\mathbb{R}^n$. Let $\{\phi_k : k = 0, 1, \ldots\}$ be a smooth partition of unity subordinate to this cover. Put

$$f_k(x) = f(x)\phi_k(x) \quad \forall x \in \mathbb{R}^n, \ \forall k \geq 0.$$

Then $f = \sum_{k=0}^\infty f_k$. Because $f_k \in \mathscr{D}(\mathbb{R}^n)$, by Proposition 4.13, there exists $\eta_k \in C^\infty(\mathbb{R}^n)$ such that $P(D)\eta_k = f_k$. Then $P(D)\eta_k = 0$ in B_k. Applying Proposition 4.12 for $s = k - \frac{1}{2}$, $r = k$ and $v = \eta_k$, we conclude that there exists $\tilde{\eta}_k \in C^\infty(\mathbb{R}^n)$ such that $P(D)\tilde{\eta}_k = 0$ in $\mathbb{R}^n$ and

$$\|\tilde{\eta}_k - \eta_k\|_{C^k(\bar{B}_{k-1/2})} < \frac{1}{k^2}.$$

Put $u_k = \eta_k - \tilde{\eta}_k \in C^\infty(\mathbb{R}^n)$. Then $P(D)u_k = P(D)\eta_k - P(D)\tilde{\eta}_k = P(D)\eta_k = f_k$ and

$$\|u_k\|_{C^k(\bar{B}_{k-1/2})} < \frac{1}{k^2}.$$

For each compact set $A \subset \mathbb{R}^n$, there exists $k_0 \in \mathbb{N}$ such that $A \subset B_{k-1/2}$ for all $k > k_0$. Then

$$|u_k(x)| \leq \|u_k\|_{C^k(\bar{B}_{k-1/2})} < \frac{1}{k^2} \quad \forall x \in A,\ \forall k > k_0.$$

Thus, the series $\sum_{k=0}^\infty u_k$ converges uniformly on every compact set in $\mathbb{R}^n$. Put $u = \sum_{k=0}^\infty u_k$. Then for every multi-index α, we have

$$|D^\alpha u_k(x)| \leq \|u_k\|_{C^k(\bar{B}_{k-1/2})} < \frac{1}{k^2} \quad \forall x \in A,\ \forall k > \max\{k_0, \alpha\}.$$

Hence, the series $\sum_{k=0}^\infty D^\alpha u_k$ converges uniformly on every compact subset of $\mathbb{R}^n$. Therefore, $u \in C^\infty(\mathbb{R}^n)$ and $D^\alpha u = \sum_{k=0}^\infty D^\alpha u_k$. Consequently,

$$P(D)u = \sum\nolimits_{k=0}^\infty P(D)u_k = \sum\nolimits_{k=0}^\infty f_k = f.$$

□

5 Some applications

In this section, we present some applications of Proposition 4.14. An immediate application is the existence of solutions of a system of linear partial differential equations with constant coefficients. This was first pointed out in [Ehr54, Theorem 15] and [Mal56, Prop. 8, p.318].

Proposition 5.1. *Consider a system of differential equations in $\mathbb{R}^n$ of the form*

$$\sum_{k=1}^N P_{jk}(D)u_k = f_j \quad \forall 1 \leq j \leq N, \tag{5.1}$$

where $P_{jk} \in \mathbb{R}[x_1, \ldots, x_n]$ and $f_j \in C^\infty(\mathbb{R}^n)$ are given. Suppose that the determinant $\det(P_{jk})$ *is not the zero-polynomial. Then the system* (5.1) *has solutions $u_1, u_2, \ldots, u_N \in C^\infty(\mathbb{R}^n)$.*

According to the terminology in [Ehr54], when $\det(P_{jk}) \equiv 0$, the matrix differential operator $(P_{jk}(D))$ is called *degenerate*. Otherwise, it is called *non-degenerate*.

Proof of Proposition 5.1. Put $Q = \det(P_{jk}) \in \mathbb{R}[x_1, \ldots, x_n]\backslash\{0\}$. By Proposition 4.14, for each $1 \leq j \leq N$, the equation $Q(D)v_j = f_j$ in $\mathbb{R}^n$ has a solution $v_j \in C^\infty(\mathbb{R}^n)$. For each $x \in \mathbb{R}^n$, we denote by $(P^{jk}(x))$ the adjugate matrix of $(P_{jk}(x))$. Note that $P^{jk}(x) \in \mathbb{R}[x_1, \ldots, x_n]$. Moreover,

$$\sum\nolimits_{k=1}^N P^{ik}(x)P_{kj}(x) = \det(P_{jk}(x))\delta_{ij} = Q(x)\delta_{ij} \quad \forall x \in \mathbb{R}^n,$$

where δ_{ij} is the Kronecker delta. For each $1 \leq k \leq N$, we put

$$u_k = \sum\nolimits_{k=1}^{N} P^{ik}(D) v_i \in C^\infty(\mathbb{R}^n).$$

Then

$$\begin{aligned}
\sum\nolimits_{k=1}^{N} P_{jk}(D) u_k &= \sum\nolimits_{k=1}^{N} \sum\nolimits_{i=1}^{N} P_{jk}(D) P^{ik}(D) v_i \\
&= \sum\nolimits_{i=1}^{N} \left(\sum\nolimits_{k=1}^{N} P_{jk}(D) P^{ik}(D) \right) v_i \\
&= \sum\nolimits_{k=1}^{N} Q(D)(\delta_{ij} v_i) \\
&= Q(D) v_j = f_j.
\end{aligned}$$

Therefore, $(u_1, u_2, \ldots, u_N)$ is a solution to the system (5.1). $\square$

Proposition 5.2. *Consider the linear Stokes equations in the whole space* $\mathbb{R}^3$ *without the initial condition.*

$$\left\{ \begin{array}{c} \partial_t \vec{u} - \Delta \vec{u} + \nabla p = \vec{f}, \\ \nabla \cdot \vec{u} = 0. \end{array} \right. \tag{5.2}$$

where

$$\begin{aligned}
&\vec{u} = (u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t)), \\
&p = p(x_1, x_2, x_3, t), \\
&\vec{f} = (f_1(x_1, x_2, x_3, t), f_2(x_1, x_2, x_3, t), f_3(x_1, x_2, x_3, t)).
\end{aligned}$$

Suppose that $f_1, f_2, f_3 \in C^\infty(\mathbb{R}^4)$. *Then the system* (5.2) *has solutions* $\vec{u} \in [C^\infty(\mathbb{R}^4)]^3$ *and* $p \in C^\infty(\mathbb{R}^4)$.

Proof. Put

$$\begin{aligned}
&u_4(x_1, x_2, x_3, t) = p(x_1, x_2, x_3, t), \\
&x_4 = t, \\
&f_4(x_1, x_2, x_3, t) = 0.
\end{aligned}$$

Then the system (5.2) becomes a system of linear differential equations in $\mathbb{R}^4$,

$$\left\{ \begin{array}{r} D_4 u_1 - D_{11} u_1 - D_{22} u_1 - D_{33} u_1 + D_1 u_4 = f_1, \\ D_4 u_2 - D_{11} u_2 - D_{22} u_2 - D_{33} u_2 + D_2 u_4 = f_2, \\ D_4 u_3 - D_{11} u_3 - D_{22} u_3 - D_{33} u_3 + D_3 u_4 = f_3, \\ D_1 u_1 + D_2 u_2 + D_3 u_3 = f_4. \end{array} \right. \tag{5.3}$$

These equations can be rewritten as

$$\sum_{k=1}^{4} P_{jk}(D) u_k = f_j \quad \forall 1 \leq j \leq 4,$$

where

$$\begin{aligned}
&P_{11} = P_{22} = P_{33} = x_4 - x_1^2 - x_2^2 - x_3^2, \\
&P_{14} = x_1, \; P_{24} = x_2, \; P_{34} = x_3, \\
&P_{41} = x_1, \; P_{42} = x_2, \; P_{43} = x_3, \\
&P_{jk} = 0 \; \text{ for other pairs } (j, k).
\end{aligned}$$

Thus,

$$
(P_{jk}(x_1, x_2, x_3, x_4)) = \begin{pmatrix} x_4 - x_1^2 - x_2^2 - x_3^2 & 0 & 0 & x_1 \\ 0 & x_4 - x_1^2 - x_2^2 - x_3^2 & 0 & x_2 \\ 0 & 0 & x_4 - x_1^2 - x_2^2 - x_3^2 & x_3 \\ x_1 & x_2 & x_3 & 0 \end{pmatrix}
$$

Then $\det(P_{jk}) = (x_4 - x_1^2 - x_2^2 - x_3^2)^2(x_1^2 + x_2^2 + x_3^2)$, which is not a zero polynomial. By Proposition 5.1, the system (5.3) has solutions $u_1, u_2, u_3, u_4 \in C^\infty(\mathbb{R}^4)$. $\square$

6 Some background in test functions and distributions

This section collects some basic properties of topological vector spaces, test functions and distributions in $\mathbb{R}^n$ that are needed to study the existence of smooth solutions to the problem $P(D)u = f$. Most of these properties are taken from Chapters 1 and 6 in [Rud73]. All vector spaces of our concern are over the field $\mathbb{R}$.

6.1 Topological vector spaces

Below is a list of terminologies that are used in the paper.

1. A vector space X equipped with a topology τ is called a *topological vector space*, abbreviated by TVS, if it is a T_1 space (i.e. every singleton is closed) on which the addtition and scalar multiplication maps are continuous.

2. A family $\mathscr{B}$ of neighborhoods of 0 in a TVS X is called a *local base* if every neighborhood of 0 in X contains a member of $\mathscr{B}$. A local base $\mathscr{B}$ of X is said to be *balanced* (respectively *convex*) if every member of its is balanced (respectively convex).

3. A TVS (X, τ) is said to be *metrizable* if τ is induced by a metric on X. A metric d on X is called *translation-invariant* if $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$.

4. A TVS X is said to be *locally convex* if it has a convex local base.

5. A TVS (X, τ) is said to be a *Fréchet space* if

   (i) (X, τ) is locally convex,

   (ii) (X, τ) is metrized by a translation-invariant metric d,

   (iii) (X, d) is a complete metric space.

6. A subset A of a TVS X is said to be *topologically bounded* if for every open neighborhood V of 0, there exists a number $s > 0$ such that $A \subset tV$ for all $t > s$.

7. A map $p : X \to \mathbb{R}$ is called a *seminorm* on a TVS X if

   (i) $p(x) \geq 0 \;\; \forall x \in X$,

   (ii) $p(x+y) \leq p(x) + p(y) \;\; \forall x, y \in X$,

   (iii) $p(\alpha x) = |\alpha| p(x) \;\; \forall \alpha \in \mathbb{R}, x \in X$.

   A family $\mathscr{P}$ of seminorms on X is said to be *separating* if for every $x \in X \backslash \{0\}$, there exists $p \in \mathscr{P}$ such that $p(x) \neq 0$.

8. Let A be a subset of a TVS X. We say that A is *balanced* if $A \neq \emptyset$ and $tA \subset A$ for all $-1 \leq t \leq 1$. We say that A is *absorbing* if $X = \bigcup_{t>0} tA$.

9. Let A be an absorbing subset of a TVS X. The map $\mu_A : X \to \mathbb{R}$, $\mu_A(x) = \inf\{t > 0 : t^{-1}x \in X\}$, is called the *Minkowski functional* of A.

Proposition 6.1. *Let X be a TVS and U be an neighborhood of 0 in X. We have the following statements.*

(i) There is an open neighborhood V_1 of 0 in X such that V_1 is balanced, $V_1 + V_1 \subset U$, and $\bar{V}_1 \subset U$.

(ii) If U is convex, it contains an open neighborhood V_2 of 0 in X such that V_2 is balanced and convex.

Proof. (i) Because the addition map $X \times X \to X$, $(x, y) \mapsto x + y$ is continuous, there exist open neighborhoods $\mathscr{O}_1, \mathscr{O}_2$ of 0 in X such that $\mathscr{O}_1 + \mathscr{O}_2 \subset U$. Put $\mathscr{O}_3 = \mathscr{O}_1 \cap \mathscr{O}_2$. Because the scalar multiplication $\mathbb{R} \times X \to X$, $(\lambda, x) \mapsto \lambda x$ is continuous, there exist a number $\delta > 0$ and an open neighborhood $\mathscr{O}_4$ of 0 in X such that $(-\delta, \delta) \times \mathscr{O}_4 \subset \mathscr{O}_3$. Put

$$V_1 = \bigcup_{|t|<\delta} t\mathscr{O}_4.$$

Then V_1 is also an open neighborhood of 0 in X. Moreover, V_1 is balanced and $V_1 \subset \mathscr{O}_3$. We have $V_1 + V_1 \subset \mathscr{O}_3 + \mathscr{O}_3 \subset \mathscr{O}_1 + \mathscr{O}_2 \subset U$. Next, we show that $\bar{V}_1 \subset U$. Take any $x \in \bar{V}_1$. Then $(x + V_1) \cap V_1 \neq \emptyset$. Thus, there exists $y, z \in V_1$ such that $x = y - z$. By the definition of V_1, we have $-V_1 = V_1$. In particular, $-z \in V_1$. Then $x = y + (-z) \in V_1 + V_1 \subset U$. Therefore, $\bar{V}_1 \subset U$.

(ii) Put $V_0 = U \cap (-U)$. Because U and $-U$ are convex neighborhoods of 0, V_0 is also a convex neighborhood of 0. Now we show that V_0 is balanced. Take any $x \in V_0$ and $t \in [-1, 1]$. By replacing x by $-x$ if necessary, we can assume $t \in [0, 1]$. Then tx lies on the line segment from 0 to x. This segment lies entirely in U because U is convex. Similarly, this segment lies entirely in $-U$ because $-U$ is convex. Therefore, $tx \in U \cap (-U) = V_0$. Let V_2 be the interior of V_0. Then $V_2 \subset U$ and V_2 is an open neighborhood of 0.

We show that V_2 is balanced. Take any $x \in V_2$ and $t \in [-1, 1]$. If $t = 0$ then $tx = 0 \in V_2$. Consider ther case $t \neq 0$. Because x lies in the interior of V_0, there exists an open neighborhood $\mathscr{O}$ of x contained in V_0. Since V_0 is balanced, $t\mathscr{O} \subset tV_0 \subset V_0$. Note that $t\mathscr{O}$ is an open neighborhood of tx. Thus, $tx \in V_2$.

We show that V_2 is convex. Take any $x, y \in V_2$ and $t \in [-1, 1]$. Because x and y lie in the interior of V_0, there exist open neighborhoods $\mathscr{O}_1$ of x and $\mathscr{O}_2$ of y such that $\mathscr{O}_1, \mathscr{O}_2 \subset V_0$. Since V_0 is convex, $t\mathscr{O}_1 + (1-t)\mathscr{O}_2 \subset V_0$. Note that $t\mathscr{O}_1 + (1-t)\mathscr{O}_2$ is an open neighborhood of $tx + (1-t)y$. Thus, $tx + (1-t)y \in V_2$. □

Proposition 6.2. *Let X be a TVS. We have the following statements.*

(i) Every singleton is topologically bounded.

(ii) Every convergent sequence of X is topologically bounded.

Proof. (i) Take $a \in X$ and let V be any neighborhood of 0 in X. Because the scalar multiplication is continuous, there exists $\epsilon > 0$ such that $\delta a \in V$ for all $0 < \delta < \epsilon$. Thus, $a \in \delta^{-1}V$ for all $0 < \delta < \epsilon$. Hence, $a \in tV$ for all $t > \epsilon^{-1}$. Therefore, the singleton $\{a\}$ is topologically bounded in X.

(ii) Let (x_n) be a convergent sequence in X. Denote $a = \lim x_n$. Let V be any neighborhood of 0 in X. By Part (i) of Proposition 6.1, there exists a balanced neighborhood W of 0 in X such that $W + W \subset V$. Because $x_n \to a$, there exists $n_0 \in \mathbb{N}$ such that $x_n - a \in W$ for all $n > n_0$. Because the singleton $\{a\}$ is topologically bounded, there exists a number $s > 1$ such that $a \in tW$ for all $t > s$. Then

$$x_n \in a + W \subset tW + W \subset tW + tW = t(W + W) \subset tV \quad \forall n > n_0, \ \forall t > s.$$

Because each singleton $\{x_i\}$, $1 \leq i \leq n_0$, is topologically bounded in X, there exists $s_0 > 0$ such that $x_i \in tV$ for all $t > s_0$ and $1 \leq i \leq n_0$. Take $s_1 = \max\{s, s_0\} > 1$. Then $x_n \in tV$ for all $t > s_1$ and $n \in \mathbb{N}$. Therefore, the sequence $\{x_n\}$ is topologically bounded. □

Proposition 6.3. *Let X and Y be two TVS and $\Lambda : X \to Y$ be a linear map. Consider the following statements.*

(i) Λ is continuous.

(ii) Λ is bounded, i.e. Λ maps topologically bounded sets into topologically bounded sets.

(iii) $\Lambda(x_n) \to 0$ if the sequence (x_n) converges to 0.

Then (i) implies (ii). If X is metrizable then (i),(ii),(iii) are equivalent.

Proof. (i)⇒(ii). Suppose that Λ is continuous. Let A be any topologically bounded subset of X and V be any neighborhood of 0 in Y. We need to find $s > 0$ such that $\Lambda(A) \subset tV$ for all $t > s$. Since Λ is continuous, there exists a neighborhood U of 0 in X such that $\Lambda(U) \subset V$. Since A is topologically bounded in X, there is $s_0 > 0$ such that $A \subset tU$ for all $t > s_0$. Thus, $\Lambda(A) \subset \Lambda(tU) = t\Lambda(U) \subset tV$ for all $t > s_0$. We can choose $s = s_0$.

(ii)⇒(iii) (assuming that X is metrizable). Suppose by contradiction that there exists a sequence (x_n) in X such that $x_n \to 0$ and $\Lambda(x_n) \not\to 0$. Then there exists an open neighborhood $\mathscr{O}$ of 0 in Y and a subsequence (x_{n_k}) such that $\Lambda(x_{n_k}) \notin \mathscr{O}$ for all $k \in \mathbb{N}$. By replacing (x_n) with this subsequence, we can assume that $\Lambda(x_n) \notin \mathscr{O}$ for all $n \in \mathbb{N}$. Let d be a metric on X that is compatible with the topology on X. Since $x_n \to 0$ in sense of topology, we have $d(x_n, 0) \to 0$. For each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $d(x_{n_k}, 0) < \frac{1}{k^2}$. Moreover, the sequence (n_k) can be chosen to be increasing in $\mathbb{N}$. Then (x_{n_k}) is a subsequence of (x_n). By replacing (x_n) with this subsequence, we can assume that $d(x_n, 0) < \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Thus,

$$d(nx_n, 0) \leq d(x_n, 0) + \ldots + d(x_n, 0) < \frac{1}{n^2} + \ldots + \frac{1}{n^2} = \frac{1}{n}.$$

This means the sequence (nx_n) converges to 0 in X. By Part (ii) of Proposition 6.2, both (x_n) and (nx_n) are topologically bounded sequences in X. Because Λ is a bounded map, (Λx_n) and $(n\Lambda x_n)$ are topologically bounded in Y. Thus, there exists $s > 0$ such that $n\Lambda x_n \in t\mathscr{O}$ for all $t > s$ and $n \in \mathbb{N}$. Consider $n > s$ and $t = n$. Then $n\Lambda x_n \in n\mathscr{O}$ for all $n > s$. Thus, $\Lambda x_n \in \mathscr{O}$ for all $n > s$. This is a contradiction.

(iii)⇒(i) (assuming that X is metrizable). It suffices to show that Λ is continuous at the origin of X. Let U be a neighborhood of 0 in Y. We need to find a neighborhood V of 0 in X such that $\Lambda(V) \subset U$. For each $n \in \mathbb{N}$, denote by $B_{1/n}$ the open ball in X centered at 0 with radius $\frac{1}{n}$. Suppose by contradiction that $\Lambda(B_{1/n}) \not\subset U$ for all $n \in \mathbb{N}$. Then there exists $y_n \in \Lambda(B_{1/n}) \backslash U$ for all $n \in \mathbb{N}$. Write $y_n = \Lambda(x_n)$ for some $x_n \in B_{1/n}$. Then $x_n \to 0$ in X and $y_n = \Lambda(x_n) \notin U$ for all $n \in \mathbb{N}$. This contradicts the fact that $\Lambda(x_n) \to 0$ in Y. Thus, there exists $n_0 \in \mathbb{N}$ such that $\Lambda(B_{1/n_0}) \subset U$. We now can choose $V = B_{1/n_0}$. □

Proposition 6.4. *Let A be a balanced, convex, absorbing subset of a TVS X. Then the Minkowski functional μ_A is a seminorm on X.*

Proof. By the definition of μ_A, it is clear that $\mu_A(x) \geq 0$ for all $x \in X$. Take $x \in X$ and $\alpha \in \mathbb{R}$. We show that $\mu_A(\alpha x) = |\alpha|\mu_A(x)$. Because A is balanced, $0 \in A$. Thus, $\mu_A(0) = 0$. This implies that our claim is true for the case $\alpha = 0$. Consider $\alpha \neq 0$. We have $\mu_A(\alpha x) = \inf A_1$ and $\mu_A(x) = \inf A_2$, where

$$A_1 = \left\{t > 0 : t^{-1}\alpha x \in A\right\}, \quad A_2 = \left\{s > 0 : s^{-1}x \in A\right\}.$$

We want to show that $\inf A_1 = |\alpha| \inf A_2$. It suffices to show $A_1 = |\alpha| A_2$. Because A is balanced, $A_1 = \{t > 0 : \ t^{-1}|\alpha|x \in A\}$. Take $t \in A_1$. To show that $t \in |\alpha|A_2$, we will show that $|\alpha|^{-1}t \in A_2$. We have $t^{-1}|\alpha|x \in A$. Thus, $(t|\alpha|^{-1})^{-1}x \in A$, which implies $t|\alpha|^{-1} \in A_2$. Therefore, $A_1 \subset |\alpha|A_2$. Now take $s \in A_2$. We will show that $|\alpha|s \in A_1$. We have $s^{-1}x \in A$. Thus, $s^{-1}|\alpha|^{-1}|\alpha|x \in A$. Thus, $(s|\alpha|)^{-1}|\alpha|x \in A$. This implies $s|\alpha| \in A_1$. Therefore, $|\alpha|A_2 \subset A_1$.

Next, we show that $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$ for all $x, y \in X$. We have

$$\begin{aligned} \mu_A(x) &= \inf\left\{t > 0 : t^{-1}x \in A\right\}, \\ \mu_A(y) &= \inf\left\{s > 0 : s^{-1}y \in A\right\}, \\ \mu_A(x+y) &= \inf\left\{r > 0 : r^{-1}(x+y) \in A\right\}. \end{aligned}$$

Take any $t, s > 0$ such that $t^{-1}x \in A$ and $s^{-1}y \in A$. Since A is convex,

$$(t^{-1}x)\frac{t}{t+s} + (s^{-1}y)\frac{s}{t+s} \in A.$$

In other words, $(t+s)^{-1}(x+y) \in A$. Thus, $t+s \geq \mu_A(x+y)$. Because this is true for all $t, s > 0$ satisfying $t^{-1}x,\ s^{-1}y \in A$, we have $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$. □

Proposition 6.5. *Let X be a vector space and $\mathscr{P}$ be a separating family of seminorms on X. For each $p \in \mathscr{P}$ and $n \in \mathbb{N}$, we put*

$$V(p,n) = \left\{x \in X :\ p(x) < \frac{1}{n}\right\}.$$

Let $\mathscr{B}$ be the family of all finite intersections of these sets. Then we have the following statements.

(i) There is a topology τ on X such that (X, τ) is a TVS and $\mathscr{B}$ is a convex balanced local base consisting of open sets.

(ii) Each $p \in \mathscr{P}$ is continuous.

(iii) A subset E of X is topologically bounded if and only if $p(E)$ is bounded in $\mathbb{R}$ for all $p \in \mathscr{P}$.

Proof. (i) Define a collection

$$\tau = \left\{\bigcup\nolimits_{i\in I}(x_i + U_i) :\ x_i \in X, U_i \in B, \text{ and } I \text{ is some index set}\right\}.$$

We show that τ is a topology on X. It is clear that $\emptyset, X \in \tau$, and that τ is closed under arbitrary union. It remains to show that τ is closed under finite intersection. It suffices to show that the intersection of two elements in τ also belongs to τ. Let $\bigcup_{i\in I}(x_i + U_i)$ and $\bigcup_{j\in J}(y_j + V_j)$ be two elements of τ. The intersection is

$$\bigcup\nolimits_{i\in I, j\in J}[(x_i + U_i) \cap (y_j + V_j)].$$

Therefore, it suffices to show that each $(x_i + U_i) \cap (y_j + V_j)$ belongs to τ. Because $U_i \in \mathscr{B}$, there are $p_{i_1}, \ldots, p_{i_m} \in \mathscr{P}$ and $n_{i_1}, \ldots, n_{i_m} \in \mathbb{N}$ such that

$$U_i = \left\{x \in X : p_{i_k}(x) < \frac{1}{n_{i_k}} \quad \forall 1 \leq k \leq m\right\}.$$

Because $V_j \in \mathscr{B}$, there are $p_{j_1}, \ldots, p_{j_l} \in \mathscr{P}$ and $n_{j_1} \ldots, n_{j_l} \in \mathbb{N}$ such that

$$V_j = \left\{x \in X : p_{j_s}(x) < \frac{1}{n_{j_s}} \quad \forall 1 \leq s \leq l\right\}.$$

For each $z \in (x_i + U_i) \cap (y_j + V_j)$,

$$\begin{cases} p_{i_k}(x_i - z) < \frac{1}{n_{i_k}} & \forall 1 \leq k \leq m, \\ p_{j_s}(y_j - z) < \frac{1}{n_{j_s}} & \forall 1 \leq s \leq l. \end{cases}$$

There exists $N_z \in \mathbb{N}$ such that

$$\begin{cases} p_{i_k}(x_i - z) + \frac{1}{N_z} < \frac{1}{n_{i_k}} & \forall 1 \le k \le m, \\ p_{j_s}(y_j - z) + \frac{1}{N_z} < \frac{1}{n_{j_s}} & \forall 1 \le s \le l. \end{cases}$$

Put

$$W_z = \left\{ w \in X : \ p_{i_k}(w) < \frac{1}{N_z} \ \forall 1 \le k \le m, \text{ and } p_{j_s}(w) < \frac{1}{N_z} \ \forall 1 \le s \le l \right\}.$$

Then $W_z \in \mathscr{B}$. We will show that $z + W_z \subset (x_i + U_i) \cap (y_j + V_j)$. Take any $w \in W_z$, we need to show $z + w \in (x_i + U_i) \cap (y_j + V_j)$. For any $1 \le k \le m$ and $1 \le s \le l$, we need to show $p_{i_k}(x_i - z - w) < \frac{1}{n_{i_k}}$ and $p_{j_s}(y_j - z - w) < \frac{1}{n_{j_s}}$. We have

$$p_{i_k}(x_i - z - w) \le p_{i_k}(x_i - z) + p_{i_k}(w) < p_{i_k}(x_i - z) + \frac{1}{N_z} < \frac{1}{n_{i_k}},$$

$$p_{j_s}(x_i - z - w) \le p_{j_s}(x_i - z) + p_{j_s}(w) < p_{j_s}(x_i - z) + \frac{1}{N_z} < \frac{1}{n_{j_s}}.$$

Therefore, we have proved that τ is a topology on X. Moreover, τ is translation-invariant in sense that a set $\mathscr{O}$ is open in X if and only if every translation of $\mathscr{O}$ in X is also open.

Next, we will show that (X, τ) is a TVS. We first show that every singleton is closed in X. Because τ is translation-invariant, it suffices to show that $\{0\}$ is closed in X. For each $p \in \mathscr{P}$ and $n \in \mathbb{N}$, we will show that the set

$$\tilde{V}(p, n) = \left\{ x \in X : \ p(x) > \frac{1}{n} \right\}$$

is open in X. Take any $x \in \tilde{V}(p, n)$. We want to show that there is an open neighborhood of x contained in $\tilde{V}(p, n)$. Because $p(x) > \frac{1}{n}$, there exists $N \in \mathbb{N}$ such that $p(x) - \frac{1}{N} > \frac{1}{n}$. We will show that $x + V(p, N) \subset \tilde{V}(p, n)$. Take any $y \in x + V(p, N)$. Then $y - x \in V(p, N)$. Thus, $p(y - x) < \frac{1}{N}$. Then

$$p(y) \ge p(x) - p(x - y) > p(x) - \frac{1}{N} > \frac{1}{n}.$$

Hence, $y \in \tilde{V}(p, n)$. Therefore, $x + V(p, N) \subset \tilde{V}(p, n)$. We have proved that the set $\tilde{V}(p, n)$ is open in X. Thus, the set

$$A = \bigcap_{\substack{p \in \mathscr{P} \\ n \in \mathbb{N}}} (X \backslash \tilde{V}(p, n))$$

is closed in X. We have

$$\begin{aligned} A &= \left\{ x \in X : p(x) \le \frac{1}{n} \forall p \in P, \forall n \in N \right\} \\ &= \{ x \in X : p(x) = 0 \}. \end{aligned}$$

Because $\mathscr{P}$ is a separating family of seminorms, $A = \{0\}$. Therefore, $\{0\}$ is closed in X.

Next, we will show that the addition map is continuous. Take $x, y \in X$ and put $z = x + y$. We will show that the addition map $X \times X \to X$ is continuous at (x, y). Any neighborhood of z contains an open set of the form $z + V$ for some $V \in \mathscr{B}$. By the definition of $\mathscr{B}$, there are $p_1, \dots, p_m \in \mathscr{P}$ and $n_1, \dots, n_m \in \mathbb{N}$ such that

$$V = \left\{ u \in X : \ p_k(u) < \frac{1}{n_k} \quad \forall 1 \le k \le m \right\}.$$

Put $N = 2 \max\{n_1, \dots, n_m\}$ and

$$V_1 = \left\{ u \in X : \ p_k(u) < \frac{1}{N} \quad \forall 1 \le k \le m \right\}.$$

We will show that $(x + V_1) \cap (y + V_1) \subset (z + V)$. We have

$$\begin{aligned} x + V_1 &= \left\{ v \in X : \ p_k(x - v) < \tfrac{1}{N} \ \forall 1 \le k \le m \right\}, \\ y + V_1 &= \left\{ w \in X : \ p_k(y - w) < \tfrac{1}{N} \ \forall 1 \le k \le m \right\}, \\ z + V &= \left\{ u \in X : \ p_k(z - u) < \tfrac{1}{n_k} \ \forall 1 \le k \le m \right\}. \end{aligned}$$

Take $v \in x + V_1$ and $w \in y + V_1$. We will show that $v + w \in z + V$. We have

$$p_k(z - v - w) = p_k(x - v + y - w) \le p_k(x - v) + p_k(y - w) < \frac{1}{N} + \frac{1}{N} = \frac{2}{N} \le \frac{1}{n_k}.$$

Therefore, $v + w \in z + V$.

Next, we show that the scalar multiplication is continuous from $\mathbb{R} \times X$ to X. Take $\lambda \in \mathbb{R}$ and $x \in X$. Put $y = \lambda x$. Every neighborhood of y in X contains an open set of the form $y + V$ with $V \in \mathscr{B}$. We want to show that the scalar multiplication map is continuous at (λ, x). Write

$$V = \left\{ u \in X : \ p_k(u) < \frac{1}{n_k} \quad \forall 1 \le k \le m \right\}.$$

Let $N \in \mathbb{N}$ be any number such that

$$N > \max\{n_1, \dots, n_m\} \left(|\lambda| + 1 + \max\{p_1(x), \dots, p_m(x)\} \right).$$

Put

$$V_2 = \left\{ v \in X : \ p_k(x) < \frac{1}{N} \ \forall 1 \le k \le m \right\}.$$

We will show that

$$\left(\lambda - \frac{1}{N}, \lambda + \frac{1}{N} \right) (x + V_2) \subset y + V.$$

We have

$$\begin{aligned} x + V_2 &= \left\{ v \in X : \ p_k(x - v) < \tfrac{1}{N} \ \forall 1 \le k \le m \right\}, \\ y + V &= \left\{ u \in X : \ p_k(y - u) < \tfrac{1}{n_k} \ \forall 1 \le k \le m \right\}. \end{aligned}$$

For $-\frac{1}{N} < t < \frac{1}{N}$ and $v \in x + V_2$, we will show that $(\lambda + t)v \in y + V$.

$$\begin{aligned} p_k((\lambda + t)v - y) &= p_k(\lambda(v - x) + tv) \\ &\leq p_k(\lambda(v - x)) + p_k(tv) \\ &\leq |\lambda| p_k(v - x) + |t| p_k(v) \\ &\leq |\lambda| \frac{1}{N} + \frac{1}{N}(p_k(v - x) + p_k(x)) \\ &< \frac{|\lambda|}{N} + \frac{1}{N}\left(\frac{1}{N} + p_k(x)\right) \\ &\leq \frac{|\lambda| + 1 + p_k(x)}{N} \\ &< \frac{1}{n_k} \quad \forall 1 \leq k \leq m. \end{aligned}$$

Thus, $(\lambda + t)v \in y + V$.

(ii) Take any seminorm $p \in \mathscr{P}$. We will show that $p : X \to \mathbb{R}$ is continuous. Take $x \in X$ arbitrarily and put $\alpha = p(x) \in \mathbb{R}$. For every $\epsilon > 0$, we will find a set $V \in \mathscr{B}$ such that $p(x + V) \subset (\alpha - \epsilon, \alpha + \epsilon)$. Let $n \in \mathbb{N}$ be any number such that $\frac{1}{n} < \epsilon$. Put $V = V(p, n) \in \mathscr{B}$. For each $y \in x + V$, we have $y - x \in V$. Thus, $p(y - x) < \frac{1}{n} < \epsilon$. Then

$$|p(y) - \alpha| = |p(y) - p(x)| \leq p(y - x) < \epsilon.$$

Thus, $p(y) \in (\alpha - \epsilon, \alpha + \epsilon)$. Therefore, $p(x + V) \subset (\alpha - \epsilon, \alpha + \epsilon)$.

(iii) Let E be a topologically bounded subset of X. We will show that $p(E)$ is bounded in $\mathbb{R}$ for every $p \in \mathscr{P}$. Fix $p \in \mathscr{P}$. Because E is topologically bounded, there exists a number $s > 0$ such that $E \subset sV(p, 1)$. Then $s^{-1}E \subset V(p, 1)$. Thus $s^{-1}x \in V(p, 1)$ for every $x \in E$. Thus, $p(s^{-1}x) < 1$. Thus, $p(x) < s$. This means $p(E) \subset [0, s)$. Therefore, $p(E)$ is bounded in $\mathbb{R}$.

Next, suppose that $p(E)$ is bounded in $\mathbb{R}$ for every $p \in \mathscr{P}$. We will show that E is topologically bounded in X. Take any open neighborhood V of 0 in X. We will find $s > 0$ such that $E \subset tV$ for all $t > s$. Because V contains an element of $\mathscr{B}$, we can shrink V if necessary to be able to assume $V \in \mathscr{B}$. Then there are $p_1, \ldots, p_m \in V$ and $n_1, \ldots, n_m \in \mathbb{N}$ such that

$$V = \left\{ x \in X : \; p_k(x) < \frac{1}{n_k} \quad \forall 1 \leq k \leq m \right\}.$$

Because $p_1(E), \ldots, p_m(E)$ are bounded in $\mathbb{R}$, there exists a number $M > 0$ such that $p_k(E) \subset [0, M]$ for all $1 \leq k \leq m$. Choose $s = (n_1 + \ldots + n_m)M > 0$. For $t > s$, $x \in E$, $1 \leq k \leq m$, we have

$$p_k(t^{-1}x) = t^{-1}p_k(x) \leq t^{-1}M < s^{-1}M < \frac{1}{n_1 + \ldots + n_m} \leq \frac{1}{n_k}.$$

Thus, $t^{-1}x \in V$. This means $x \in tV$. Therefore, $E \subset tV$ for all $t > s$. □

Proposition 6.6. *Let X be a vector space and $\mathscr{P} = \{p_n : n \in \mathbb{N}\}$ be a separating family of seminorms on X. Let τ be the topology on X generated by $\mathscr{P}$ in the manner of Proposition 6.5. Define a map $d : X \times X \to \mathbb{R}$,*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{2^{-n} p_n(x - y)}{1 + p_n(x - y)}.$$

Then d is a translation-invariant metric on X and (X, τ) is metrized by d.

Proof. First, we will show that d is a translation-invariant metric on X. It is clear that d is a well-defined map and $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$. Also, we have $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. It remains to check the triangle inequality. It suffices to show that

$$\frac{p_n(x - y)}{1 + p_n(x - y)} \leq \frac{p_n(x - z)}{1 + p_n(x - z)} + \frac{p_n(z - y)}{1 + p_n(z - y)} \quad \forall n \in N,\ \forall x, y, z \in X.$$

Put $a = p_n(x - y)$, $b = p_n(x - z)$, $c = p_n(z - y)$. Then $a, b, c \geq 0$ and $a \leq b + c$. We want to show that

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c}.$$

This inequality is equivalent to $a(1 + b)(1 + c) \leq (1 + a)[b(1 + c) + c(1 + b)]$. It is true because RHS-LHS$=abc + 2bc + b + c - a \geq 0$.

Next, we will show that d is compatible with τ. Let $\mathscr{B}$ be the local base of τ as in Proposition 6.5. Because d is translation-invariant, it suffices to show that each ball $B_r = \{x \in X : d(x, 0) < r\}$ contains a member of $\mathscr{B}$ and each member of $\mathscr{B}$ contains such a ball. First, we will find $V \in \mathscr{B}$ such that $V \subset B_r$. There exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} 2^{-n} < r/2$. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{r}{2}$. Put

$$V = \left\{ x \in X : p_k(x) < \frac{1}{n_0} \ \ \forall 1 \leq k \leq N \right\}.$$

For each $x \in V$, we have

$$\begin{aligned} d(x, 0) &= \sum_{n=1}^{N} \frac{2^{-n} p_n(x)}{1 + p_n(x)} + \sum_{n=N+1}^{\infty} \frac{2^{-n} p_n(x)}{1 + p_n(x)} \\ &\leq \sum_{n=1}^{N} 2^{-n} p_n(x) + \sum_{n=N+1}^{\infty} 2^{-n} \\ &< \sum_{n=1}^{N} 2^{-n} \frac{1}{n_0} + \frac{r}{2} < \sum_{n=1}^{\infty} 2^{-n} \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Thus, $x \in B_r$. Therefore, $V \subset B_r$.

Next, for each $V \in \mathscr{B}$, we will find $r > 0$ such that $B_r \subset V$. There are $p_{i_1}, \ldots, p_{i_m} \in \mathscr{P}$ and $n_1, \ldots, n_m \in \mathbb{N}$ such that

$$V = \left\{ x \in X : p_{i_k}(x) < \frac{1}{n_k} \ \ \forall 1 \leq k \leq m \right\}.$$

Take any $r > 0$ such that

$$0 < \frac{r2^{n_k}}{1 - r2^{n_k}} < \frac{1}{n_k} \quad \forall 1 \leq k \leq m.$$

For each $x \in B_r$, we have

$$\frac{2^{-n_k} p_{i_k}(x)}{1 + p_{i_k}(x)} \leq d(x, 0) < r.$$

Multiplying both sides by (-1) and then adding 2^{-n_k}, we get

$$\frac{2^{-n_k}}{1 + p_{i_k}(x)} > 2^{-n_k} - r > 0.$$

Hence,

$$1 + p_{i_k}(x) < \frac{1}{1 - r2^{n_k}}.$$

Therefore,

$$p_{i_k}(x) < \frac{r2^{n_k}}{1 - r2^{n_k}} < \frac{1}{n_k}.$$

Thus, $x \in V$, Hence, $B_r \subset V$. $\square$

6.2 The spaces $C^\infty(\Omega)$ and $\mathscr{D}_K$

Let Ω be a nonempty open subset of a Euclidean space. This section gives a construction for a topology on $C^\infty(\Omega)$ which turns it into a Fréchet space. For the purpose of studying the existence of smooth solutions to the problem $P(D)u = f$ where f is a given smooth function on $\mathbb{R}^n$, we only consider the case $\Omega = \mathbb{R}^n$. However, we will still deal with a generic open set Ω in this section because the method below still works in such a case.

Let (K_n) be a sequence of compact subsets of Ω such that each K_n is contained in the interior of K_{n+1} and that $\Omega = \bigcup_{n=1}^\infty K_n$. For example, we can choose

$$K_n = \left\{ x \in \Omega : \ \operatorname{dist}(x, \Omega^c) \geq \frac{1}{n}, \ |x| \leq n \right\},$$

where n may start from some index. For each $n \in \mathbb{N}$, we define a map $p_n : C^\infty(\Omega) \to \mathbb{R}$,

$$p_n(f) = \max \left\{ |D^\alpha f(x)| : \ x \in K_n, \ |\alpha| \leq n \right\}.$$

This is a seminorm on $C^\infty(\Omega)$. If $f \in C^\infty(\Omega) \backslash \{0\}$, there exists $x_0 \in \Omega$ such that $f(x_0) \neq 0$; then for each $n \in \mathbb{N}$ such that $x_0 \in K_0$ we have $p_n(f) \neq 0$. Thus, the family of seminorms p_n is separating.

By Proposition 6.5, the family $\{p_n\}$ generates a locally convex TVS structure on $C^\infty(\Omega)$. Moreover, $C^\infty(\Omega)$ has a local base $\mathscr{B}$ consisting of open sets

$$V(n, k) = \left\{ f \in C^\infty(\Omega) : p_n(f) < \frac{1}{k} \right\} \quad \forall n, k \in N.$$

Because $V(k,k) \subset V(n,k)$ for all $k \geq n$, the family $\{V(k,k) : \ k \in \mathbb{N}\}$ is a local base of $C^\infty(\Omega)$. Moreover, by Proposition 6.6, the topology on $C^\infty(\Omega)$ is metrized by a translation-invariant metric

$$d(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} p_n(f-g)}{1+p_n(f-g)} \quad \forall f,g \in C^\infty(\Omega). \tag{6.1}$$

For a compact set $K \subset \Omega$, we put

$$\mathscr{D}_K = \{f \in C^\infty(\Omega) : \mathrm{supp} f \subset K\}.$$

Then $\mathscr{D}_K$ is a vector subspace of $C^\infty(\Omega)$ and hence inherits the TVS structure for $C^\infty(\Omega)$.

Proposition 6.7. *The topology which we have defined on $C^\infty(\Omega)$ does not depend on the specific choice of a sequence (K_n).*

Proof. Let τ be the topology corresponding to a sequence (K_n) of compact subsets of Ω such that each K_n is contained in the interior of K_{n+1} and that $\Omega = \bigcup_{n=1}^\infty K_n$. Then τ has a local base $\mathscr{B}$ consisting of the sets

$$V(n,k) = \left\{ f \in C^\infty(\Omega) : \ p_n(f) < \frac{1}{k} \right\} \quad \forall n,k \in \mathbb{N}.$$

Let $(\tilde{K}_n)$ be another sequence of compact subsets of Ω such that each $\tilde{K}_n$ is contained in the interior of $\tilde{K}_{n+1}$ and that $\Omega = \bigcup_{n=1}^\infty \tilde{K}_n$. For each $n \in \mathbb{N}$, we define a map $\tilde{p}_n : C^\infty(\Omega) \to \mathbb{R}$,

$$\tilde{p}_n(f) = \max\left\{ |D^\alpha f(x)| : \ x \in \tilde{K}_n, \ |\alpha| \leq n \right\}.$$

Let $\tilde{\tau}$ be the topology on $C^\infty(\Omega)$ generated by the sequence of seminorm $(\tilde{p}_n)$. Then $\tilde{\tau}$ has a local base $\tilde{\mathscr{B}}$ consisting of the sets

$$\tilde{V}(n,k) = \left\{ f \in C^\infty(\Omega) : \ \tilde{p}_n(f) < \frac{1}{k} \right\} \quad \forall n,k \in \mathbb{N}.$$

To show that $\tau = \tilde{\tau}$, it suffices to show that each member of $\mathscr{B}$ contains a member of $\tilde{\mathscr{B}}$ and vice versa. To do so, it suffices to show each set $V(n,k)$ contains a set $\tilde{V}(m,l)$ and vice versa.

Consider a set $V(n,k)$. Because $\bigcup_{m=1}^\infty \tilde{K}_m = \Omega$ and $\tilde{K}_1 \subset \tilde{K}_2 \ldots$, there exists $m \geq n$ such that $K_n \subset \tilde{K}_m$. For each $f \in \tilde{V}(m,k)$, we have

$$\frac{1}{k} > \tilde{p}_m(f) = \sup_{\tilde{K}_m} \{|D^\alpha f(x)| : \ |\alpha| \leq m\} \geq \sup_{K_n} \{|D^\alpha f(x)| : \ |\alpha| \leq n\} = p_n(f).$$

Thus, $f \in V(n,k)$. This means $\tilde{V}(m,k) \subset V(n,k)$.

Consider a set $\tilde{V}(m,l)$. Because $\bigcup_{n=1}^\infty K_n = \Omega$ and $K_1 \subset K_2 \ldots$, there exists $n \geq m$ such that $\tilde{K}_m \subset K_n$. For each $f \in V(n,l)$, we have

$$\frac{1}{l} > p_n(f) = \sup_{K_n} \{|D^\alpha f(x)| : \ |\alpha| \leq n\} \geq \sup_{\tilde{K}_m} \{|D^\alpha f(x)| : \ |\alpha| \leq m\} = p_m(f).$$

Thus, $f \in \tilde{V}(m,l)$. This means $V(n,l) \subset \tilde{V}(m,l)$. □

Proposition 6.8. *Let (f_n) be a sequence in $C^\infty(\Omega)$ and $f \in C^\infty(\Omega)$. Then (f_n) converges to f if and only if for every multi-index β, the sequence $(D^\beta f_n)$ converges to $D^\beta f$ uniformly on every compact subset of Ω.*

Proof. We know that the sequential convergence in the topological space $C^\infty(\Omega)$ is the same as that in the metric space $(C^\infty(\Omega), d)$ where d is given by (6.1). Suppose that $d(f_n, f) \to 0$ as $n \to \infty$. For $m \in \mathbb{N}$,

$$\frac{2^{-m} p_m(f_n - f)}{1 + p_m(f_n - f)} \leq d(f_n, f)$$

for all n sufficiently large. Thus, $p_m(f_n - f) \to 0$ as $n \to \infty$. Thus, for each $|\alpha| \leq m$, the sequence $(D^\alpha f_n)$ converges to $D^\alpha f$ uniformly on K_m as $n \to \infty$. Because m can be chosen arbitrarily large, the sequence $(D^\alpha f_n)$ converges to $D^\alpha f$ uniformly on every compact subset of Ω.

Now suppose that for every multi-index β, the sequence $(D^\beta f_n)$ converges to $D^\beta f$ uniformly on every compact subset of Ω. For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{m=N+1}^\infty 2^{-m} < \epsilon/2$. There exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$|D^\beta f_n(x) - D^\beta f(x)| < \frac{\epsilon}{2} \quad \forall |\beta| \leq N, \ \forall x \in K_N.$$

Thus,

$$p_m(f_n - f) < \frac{\epsilon}{2} \quad \forall 1 \leq m \leq N, \ \forall n > n_0.$$

For $n > n_0$,

$$\begin{aligned} d(f_n, f) &= \sum_{m=1}^N \frac{2^{-m} p_m(f_n - f)}{1 + p_m(f_n - f)} + \sum_{m=N+1}^\infty \frac{2^{-m} p_m(f_n - f)}{1 + p_m(f_n - f)} \\ &\leq \sum_{m=1}^N 2^{-m} \underbrace{p_m(f_n - f)}_{<\epsilon/2} + \underbrace{\sum_{m=N+1}^\infty 2^{-m}}_{<\epsilon/2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, $d(f_n, f) \to 0$ as $n \to \infty$. □

Proposition 6.9. *$C^\infty(\Omega)$ is a Fréchet space.*

Proof. From the construction of the topology on $C^\infty(\Omega)$, $C^\infty(\Omega)$ is a locally convex TVS and is metrized by a translation-invariant metric d. It remains to show that d is a complete metric. Let $(f_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in $(C^\infty(\Omega), d)$. Then for each $n \in \mathbb{N}$,

$$\frac{2^{-n} p_n(f_i - f_j)}{1 + p_n(f_i - f_j)} \leq d(f_i, f_j).$$

Thus,

$$0 \leq p_n(f_i - f_j) \leq \frac{2^n d(f_i, f_j)}{1 - 2^n d(f_i, f_j)}$$

for all $i, j \in \mathbb{N}$ that are sufficiently large. Put $p_n(f_i - f_j) \to 0$ as $i, j \to \infty$. Thus, for $|\alpha| \leq n$, $\max\{|D^\alpha f_i(x) - D^\alpha f_j(x)| : x \in K_n\} \to 0$ as $i, j \to \infty$. This means for every multi-index α, $\{(D^\alpha f_i)|_{K_n}\}$ is a Cauchy sequence in $C(K_n)$. Thus, $\{(D^\alpha f_i)|_{K_n}\}$ converges to a function in $C(K_n)$. Because this is true for all $n \in \mathbb{N}$ and that K_n is contained in the interior of K_{n+1}, there exists a function $f_\alpha \in C(\Omega)$ such that $(D^\alpha f_i)|_{K_n} \to f_\alpha|_{K_n}$ in $C(K_n)$.

For $|\alpha| = 0$, we denote $f = f_\alpha$. Because the sequence $(D^\alpha f_i)$ converges to f_α uniformly on every compact subset of Ω for all α, we have $f \in C^\infty(\Omega)$ and $D^\alpha f = f_\alpha$. Thus, the sequence $(D^\alpha f_i)$ converges to $D^\alpha f$ uniformly on every compact subset of Ω. By Proposition 6.8, the sequence (f_i) converges to f in $C^\infty(\Omega)$. $\square$

6.3 The test-function space $\mathscr{D}(\Omega)$

Let Ω be a nonempty open subset of $\mathbb{R}^n$. Denote by $\mathscr{D}(\Omega)$ the set of all functions $\phi \in C^\infty(\Omega)$ which are compactly supported in Ω. It is clear that $\mathscr{D}(\Omega)$ is a vector subspace of $C^\infty(\Omega)$. Hence, $\mathscr{D}(\Omega)$ inherits a topology from that of $C^\infty(\Omega)$ which we defined in Section 6.2. However, in this section, we will introduce a new topology on $\mathscr{D}(\Omega)$ in which the convergence of sequences becomes more demanding. With the new topology, $\mathscr{D}(\Omega)$ is called a *test-function space*. For the purpose of studying the existence of smooth solutions to the problem $P(D)u = f$ where f is a given smooth function on $\mathbb{R}^n$, we only consider the case $\Omega = \mathbb{R}^n$. However, we will still deal with a generic open set Ω in this section because the method below still works in such a case.

For each compact set $K \subset \Omega$, we denote by τ_K the topology on $\mathscr{D}_K$ as described in Section 6.2. We know that $(\mathscr{D}_K, \tau_K)$ is a locally convex TVS.

Proposition 6.10. *Let $\tilde{\mathscr{B}}$ be the collection of all balanced convex sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}_K \cap W \in \tau_K$ for every compact set $K \subset \Omega$. Let τ be the collection of all unions of sets of the form $\phi + W$ where $\phi \in \mathscr{D}(\Omega)$ and $W \in \tilde{\mathscr{B}}$. Then we have the following statements.*

(i) τ is a topology on $\mathscr{D}(\Omega)$.

(ii) Every neighborhood of $\phi \in \mathscr{D}(\Omega)$ contains an open neighborhood $\phi + W$ for some $W \in \tilde{\mathscr{B}}$.

(iii) $(\mathscr{D}(\Omega), \tau)$ is a locally convex TVS, and $\tilde{\mathscr{B}}$ is a local base consisting of balanced convex open sets.

According to [Rud73, Remark 6.9, p.141], $\mathscr{D}(\Omega)$ is not metrizable. Consequently, the topology on $\mathscr{D}(\Omega)$ defined in Proposition 6.10 is different from the topology which $\mathscr{D}(\Omega)$ inherits from $C^\infty(\mathbb{R}^n)$.

Proof of Proposition 6.10. (i) It is clear that $\emptyset, \mathscr{D}(\Omega) \in \tau$ and that τ is closed under unions. Now we will show that τ is closed under finite intersections. To do so, we take $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$ and $W_1, W_2 \in \tilde{\mathscr{B}}$ arbitrarily and show that $(\phi_1 + W_1) \cap$

$(\phi_2 + W_2) \in \tau$. Take any $\phi \in (\phi_1 + W_1) \cap (\phi_2 + W_2)$. We will find a set $W \in \tilde{\mathscr{B}}$ such that $\phi + W \subset (\phi_1 + W_1) \cap (\phi_2 + W_2)$. Put

$$K_0 = (\text{supp}\phi) \cup (\text{supp}\phi_1) \cup (\text{supp}\phi_2).$$

Then K_0 is a compact subset of Ω. Because $\phi - \phi_1 \in W_1$ and $\text{supp}(\phi - \phi_1) \subset K_0$, we have $\phi - \phi_1 \in W_1 \cap \mathscr{D}_{K_0}$ which is a balanced convex open subset of $(\mathscr{D}_{K_0}, \tau_{K_0})$. Thus, there exists a number $\delta_1 \in (0,1)$ such that $\phi - \phi_1 \in (1-\delta_1)(W_1 \cap \mathscr{D}_{K_0})$. We write

$$\phi - \phi_1 = (1-\delta_1)\psi_1,$$

for some $\psi_1 \in W_1$. Similarly, there exists $\delta_2 \in (0,1)$ and $\psi_2 \in W_2$ such that $\phi - \phi_2 = (1-\delta_2)\psi_2$. Put $W = (\delta_1 W_1) \cap (\delta_2 W_2)$. Then W is a convex balanced subset of $\mathscr{D}(\Omega)$ because it is the intersection of two convex balanced subsets of $\mathscr{D}(\Omega)$. For each compact set $K \subset \Omega$,

$$\begin{aligned} D_K \cap W &= (D_K \cap \delta_1 W_1) \cap (D_K \cap \delta_2 W_2) \\ &= [\delta_1(\underbrace{D_K \cap W_1}_{\in \tau_K})] \cap [\delta_2(\underbrace{D_K \cap W_2}_{\in \tau_K})] \\ &\in \tau_K. \end{aligned}$$

Therefore, $W \in \tilde{\mathscr{B}}$. We will show that $\phi + W \subset (\phi_1 + W_1) \cap (\phi_2 + W_2)$. For each $\psi \in W$, we will show that $\phi + \psi \in (\phi_1 + W_1) \cap (\phi_2 + W_2)$. Write $\psi = \delta_1\psi_3 = \delta_2\psi_4$ for some $\psi_3 \in W_1$ and $\psi_4 \in W_2$. Then

$$\begin{aligned} &\phi + \psi - \phi_1 = (1-\delta_1)\psi_1 + \delta_1\psi_3 \in W_1 \quad \text{(since } W_1 \text{ is convex)}, \\ &\phi + \psi - \phi_2 = (1-\delta_2)\psi_2 + \delta_2\psi_4 \in W_2 \quad \text{(since } W_2 \text{ is convex)}. \end{aligned}$$

Thus, $\phi + \psi \in \phi_1 + W_1$ and $\phi + \psi \in \phi_2 + W_2$. We have proved that τ is a topology on $\mathscr{D}(\Omega)$.

(ii) Take $\phi \in \mathscr{D}(\Omega)$ arbitrarily. By the definition of the topology τ on $\mathscr{D}(\Omega)$, every neighborhood of ϕ in $\mathscr{D}(\Omega)$ contains an open neighborhood of the form $\phi_0 + W_0$ for some $\phi_0 \in \mathscr{D}(\Omega)$ and $W_0 \in \tilde{\mathscr{B}}$. Put $\phi_1 = \phi$, $\phi_2 = \phi$, $W_1 = W_0$ and $W_2 = W_0$. Then $\phi \in (\phi_1 + W_1) \cap (\phi_2 + W_2)$. In the proof of Part (i), we showed that there exists $W \in \tilde{\mathscr{B}}$ such that $\phi + W \subset (\phi_1 + W_1) \cap (\phi_2 + W_2)$. Thus,

$$\phi + W \subset (\phi + W_0) \cap (\phi_0 + W_0) \subset \phi_0 + W_0.$$

Therefore, every neighborhood of ϕ in $\mathscr{D}(\Omega)$ contains an open neighborhood of the form $\phi + W$ for some $W \in \tilde{\mathscr{B}}$.

(iii) Each element in $\tilde{\mathscr{B}}$ is balanced and convex. By part (ii), $\tilde{\mathscr{B}}$ is a basis of open neighborhood of 0. We will show that $(\mathscr{D}, \tau)$ is a TVS. First, we will show that $(\mathscr{D}, \tau)$ is a T_1 space. For $\phi_1, \phi_2 \in$, $\phi_1 \neq \phi_2$, we put

$$W = \left\{ \phi \in \mathscr{D}(\Omega) : \ \max_{\Omega} |\phi| < \max_{\Omega} |\phi_2 - \phi_1| \right\}.$$

Then $\phi_1 \in \phi_1 + W$ but $\phi_2 \notin \phi_1 + W$. It remains to show that $W \in \tilde{\mathscr{B}}$. For each $s \in [-1,1]$ and $\phi \in W$,

$$\max_{\Omega} |s\phi| = |s| \max_{\Omega} |\phi| \leq \max_{\Omega} |\phi| < \max_{\Omega} |\phi_2 - \phi_1|.$$

Thus, $s\phi \in W$. This implies that W is balanced in $\mathscr{D}(\Omega)$. For $\psi_1, \psi_2 \in W$, $s \in [0,1]$ and $x \in \Omega$, we have

$$\begin{aligned} |s\psi_1(x) + (1-s)\psi_2(x)| &\leq s|\psi_1(x)| + (1-s)|\psi_2(x)| \\ &\leq s \max_\Omega |\phi_1| + (1-s) \max_\Omega |\phi_2| \\ &< s \max_\Omega |\phi_2 - \phi_1| + (1-s) \max_\Omega |\phi_2 - \phi_1| \\ &= \max_\Omega |\phi_2 - \phi_1|. \end{aligned}$$

Thus, $s\psi_1 + (1-s)\psi_2 \in W$. This implies that W is a convex subset of $\mathscr{D}(\Omega)$. For every compact set $K \subset \Omega$, we will show that $\mathscr{D}_K \cap W \in \tau_K$. We have

$$\mathscr{D}_K \cap W = \left\{ \phi \in \mathscr{D}_K : \ \max_\Omega |\phi| < \max_\Omega |\phi_2 - \phi_1| \right\}.$$

We know from Section 6.2 that $(\mathscr{D}_K, \tau_K)$ is a metric space with

$$d(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} p_n(f-g)}{1 + p_n(f-g)} \quad \forall f, g \in \mathscr{D}_K,$$

where $p_n(h) = \max\{|D^\alpha h(x)| : \ |\alpha| \leq n, \ x \in K_n\}$ and (K_n) is a sequence of compact subsets of Ω such that each K_n is contained in the interior of K_{n+1} and $\Omega = \sum_{n=1}^{\infty} K_n$. Take any $\phi \in \mathscr{D}_K \cap W$. Choose a number $\epsilon \in (0,2)$ such that

$$\frac{\epsilon}{2-\epsilon} < \max_\Omega |\phi_2 - \phi_1| - \max_K |\phi|.$$

There exists $m \in \mathbb{N}$ such that $K \subset K_m$. We will show that the set

$$\mathscr{O} = \left\{ \psi \in \mathscr{D}_K : \ d(\psi, \phi) < \epsilon 2^{-m-1} \right\}$$

is contained in $\mathscr{D}_K \cap W$. For each $\psi \in \mathscr{O}$,

$$\frac{2^{-m} p_m(\psi, \phi)}{1 + p_m(\psi, \phi)} \leq d(\psi, \phi) < \epsilon 2^{-m-1}.$$

Thus,

$$\frac{p_m(\psi, \phi)}{1 + p_m(\psi, \phi)} < \frac{\epsilon}{2},$$

which implies

$$p_m(\psi, \phi) < \frac{\epsilon}{2-\epsilon} < \max_\Omega |\phi_2 - \phi_1| - \max_K |\phi|.$$

Thus,

$$\max_K |\psi - \phi| \leq \max_{K_m} |\psi - \phi| \leq p_m(\psi, \phi) < \max_\Omega |\phi_1 - \phi_2| - \max_K |\phi|.$$

Thus,

$$\max_K |\psi| \leq \max_K |\psi - \phi| + \max_K |\phi| < \max_\Omega |\phi_1 - \phi_2|.$$

This implies $\psi \in W$. Therefore, $\phi \in \mathscr{O} \subset \mathscr{D} \cap W$. Note that $\mathscr{O}$ is an open subset in $\mathscr{D}_K$ depending on the choice of ϕ in $\mathscr{D}_K \cap W$. If we denote this dependence by writing $\mathscr{O}_\phi$ instead of $\mathscr{O}$, then

$$\mathscr{D}_K \cap W = \bigcup\nolimits_{\phi \in \mathscr{D}_K \cap W} O_\phi.$$

Thus, $\mathscr{D}_K \cap W$ is open in $\mathscr{D}_K$. We have proved that $W \in \tilde{\mathscr{B}}$.

Next, we will show that the addition map on $\mathscr{D}(\Omega)$ is continuous. Take any $\phi_1, \phi_2 \in \mathscr{D}(\Omega)$ and put $\phi_3 = \phi_1 + \phi_2$. We will show that the addition map is continuous at (ϕ_1, ϕ_2). By Part (ii), every neighborhood of ϕ_3 contains an open neighborhood of the form $\phi_3 + W$ for some $W \in \tilde{\mathscr{B}}$. Because W is balanced and convex, so is the set $\frac{1}{2}W$. For every compact set $K \subset \Omega$,

$$\mathscr{D}_K \cap \left(\frac{1}{2}W\right) = \frac{1}{2}(\underbrace{\mathscr{D}_K \cap W}_{\in \tau_K}) \in \tau_K.$$

Thus, $\frac{1}{2}W \in \tilde{\mathscr{B}}$. We will show that

$$\left(\phi_1 + \frac{1}{2}W\right) + \left(\phi_2 + \frac{1}{2}W\right) \subset \phi_3 + W. \tag{6.2}$$

For $\psi_1, \psi_2 \in W$, we have

$$\left(\phi_1 + \frac{1}{2}W\right) + \left(\phi_2 + \frac{1}{2}W\right) = (\phi_1 + \phi_2) + \left(\frac{1}{2}\psi_1 + \frac{1}{2}\psi_2\right) = \phi_3 + \left(\frac{1}{2}\psi_1 + \frac{1}{2}\psi_2\right).$$

Since W is convex, $\frac{1}{2}\psi_1 + \frac{1}{2}\psi_2 \in W$. Therefore, $\phi_3 + \left(\frac{1}{2}\psi_1 + \frac{1}{2}\psi_2\right) \in \phi_3 + W$. Thus, we have proved (6.2).

Next, we will show that the scalar multiplication map $\mathbb{R} \times \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$, $(\lambda, f) \mapsto \lambda f$ is continuous. Take any $\lambda \in \mathbb{R}$ and $\phi \in \mathscr{D}(\Omega)$. We will show that the scalar multiplication map is continuous at (λ, ϕ). By Part (ii), every neighborhood of $\lambda\phi$ contains an open neighborhood of the form $\lambda\phi + W$ for some $W \in \tilde{\mathscr{B}}$. Since $\phi \in \mathscr{D}(\Omega)$, $\phi \in \mathscr{D}_K$ where $K = \mathrm{supp}\phi$. Because $W \in \tilde{\mathscr{B}}$,$\mathscr{D}_K \cap W$ is open in $\mathscr{D}_K$. Thus, there exists a number $\epsilon \in \left(0, \frac{1}{2(|\lambda|+1)}\right)$ such that $2\epsilon\phi \in (\mathscr{D}_K \cap W)$. Note that $\epsilon W \in \tilde{\mathscr{B}}$. We will show that

$$(\lambda - \epsilon, \lambda + \epsilon)(\phi + \epsilon W) \subset \lambda\phi + W. \tag{6.3}$$

For any $t \in (-\epsilon, \epsilon)$ and $\psi \in W$, we have

$$\begin{aligned}(\lambda + t)(\phi + \epsilon\psi) - \lambda\phi &= t\phi + \epsilon(\lambda + t)\psi \\ &= \frac{1}{2}(2t)\phi + \frac{1}{2}(2\epsilon(\lambda + t))\psi. \end{aligned} \tag{6.4}$$

Because $\frac{|t|}{\epsilon} \leq 1$, $2t\phi = \frac{|t|}{\epsilon}(2\epsilon\phi)$, $2\epsilon\phi \in W$, and W is balanced, we conclude that $2t\phi \in W$. Because

$$|2\epsilon(\lambda + t)| \leq 2\epsilon(|\lambda| + \epsilon) < 2\epsilon(|\lambda| + 1) < 1,$$

$\psi \in W$, and W is balanced, we conclude that $2\epsilon(\lambda+t)\psi \in W$. Because $2t\phi \in W$, $2\epsilon(\lambda+t)\psi \in W$ and W is convex, we conclude that

$$\frac{1}{2}(2t)\phi + \frac{1}{2}(2\epsilon(\lambda+t))\psi \in W.$$

Then (6.4) implies that $(\lambda+t)(\phi+\epsilon\psi) - \lambda\phi \in W$. Thus,

$$(\lambda+t)(\phi+\epsilon\psi) \in \lambda\phi \quad \forall t \in (-\epsilon,\epsilon),\ \forall \psi \in W.$$

Thus, (6.3) is proved. $\square$

Proposition 6.11. *Denote by $(\mathscr{D}(\Omega),\tau)$ the TVS as defined in Proposition 6.10. For each compact set $K \subset \Omega$, we denote by τ_K the topology which $\mathscr{D}_K$ inherits from $C^\infty(\Omega)$. Then*

$$\tau_K = \{\mathscr{D}(\Omega) \cap W : \ W \in \tau\}.$$

In other words, τ_K coincides with the topology that $\mathscr{D}_K$ inherits from $(\mathscr{D}(\Omega),\tau)$.

Proof. First, we will show that $\{\mathscr{D}(\Omega) \cap W : \ W \in \tau\} \subset \tau_K$. Take any $W \in \tau$. We want to show that $\mathscr{D}(\Omega) \cap W \in \tau_K$. Take any $\phi \in \mathscr{D}(\Omega) \cap W$. By the definition of τ in Proposition 6.10, there exist $\phi_0 \in \mathscr{D}(\Omega)$ and $W_0 \in \tilde{\mathscr{B}}$ such that $\phi \in \mathscr{D}_K \cap (\phi_0 + W_0)$. Thus, $0 \in \mathscr{D}_K \cap (\phi_0 - \phi + W_0)$. Thus, $\phi_0 - \phi + W_0$ is a neighborhood of 0 in $\mathscr{D}(\Omega)$. Because $\tilde{\mathscr{B}}$ is a local base of τ, there exists $W_1 \in \tilde{\mathscr{B}}$ such that $W_1 \subset \phi_0 - \phi + W_0$. Thus,

$$\mathscr{D}_K \cap W_1 \subset \mathscr{D}_K \cap (\phi_0 - \phi + W_0).$$

We have

$$\phi + (\mathscr{D}_K \cap W_1) \subset \mathscr{D}_K \cap (\phi_0 + W_0) \subset \mathscr{D}_K \cap W.$$

Since $W_1 \in \tilde{\mathscr{B}}$, $\mathscr{D}_K \cap W_1 \in \tau_K$. Thus, $\phi + (\mathscr{D}_K \cap W_1)$ is an open neighborhood of ϕ in $(\mathscr{D}_K, \tau_K)$ that is contained in $\mathscr{D}_K \cap W$. Because ϕ was chosen arbitrarily in $\mathscr{D}_K \cap W$, we conclude that $\mathscr{D}_K \cap W \in \tau_K$.

Next, we will show that $\tau_K \subset \{\mathscr{D}(\Omega) \cap W : \ W \in \tau\}$. We know that τ_K is the topology which $\mathscr{D}_K$ inherits from the metric space $(C^\infty(\Omega), d)$,

$$d(f,g) = \sum_{n=1}^{\infty} \frac{2^{-n} p_n(f-g)}{1+p_n(f-g)} \quad \forall f,g \in C^\infty(\Omega),$$

where $p_n : C^\infty(\Omega) \to \mathbb{R}$, $p_n(f) = \max\{|D^\alpha f(x)| : \ |\alpha| \le n,\ x \in K_n\}$ and (K_n) is a sequence of compact subsets of Ω such that each K_n is contained in the interior of K_{n+1} and $\bigcup_{n=1}^{\infty} K_n = \Omega$. Take any $\mathscr{O} \in \tau_K$. We will show that there exists $V \in \tau$ such that $\mathscr{O} = \mathscr{D}_K \cap V$. Take any $\phi \in \mathscr{O}$. Because $(\mathscr{D}_K, d)$ is a metric space, there exists $r > 0$ such that $\phi + B_r \subset \mathscr{O}$ where

$$B_r = \{\psi \in \mathscr{D}_K : \ d(\psi,0) < r\} = \left\{\psi \in \mathscr{D}_K : \ \sum_{n=1}^{\infty} \frac{2^{-n} p_n(\psi)}{1+p_n(\psi)} < r\right\}.$$

Because K is compact and (K_n) is an increasing sequence with $\bigcup_{n=1}^{\infty} K_n = \Omega$, there exists $N \in \mathbb{N}$ such that $K \subset K_N$. We can assume $\sum_{n=N+1}^{\infty} 2^{-n} < r/2$. Put $W = \{\psi \in \mathscr{D}(\Omega) :\ p_N(\psi) < r/2\}$. For every $\psi \in W$, we have

$$0 \le p_1(\psi) \le p_2(\psi) \le \ldots \le p_N(\psi) < \frac{r}{2}.$$

Thus,

$$d(\psi, 0) = \sum_{n=1}^{N} \frac{2^{-n} p_n(\psi)}{1 + p_n(\psi)} + \sum_{n=N+1}^{\infty} \frac{2^{-n} p_n(\psi)}{1 + p_n(\psi)} \le \sum_{n=1}^{N} 2^{-n} \underbrace{p_n(\psi)}_{<r/2} + \underbrace{\sum_{n=N+1}^{\infty} 2^{-n}}_{<r/2} < \frac{r}{2} + \frac{r}{2} = r.$$

Thus, $W \subset \{\psi \in \mathscr{D}(\Omega) :\ d(\psi, 0) < r\}$. Thus,

$$\mathscr{D}_K \cap W \subset \{\psi \in \mathscr{D}_K :\ d(\psi, 0) < r\} = B_r.$$

Hence,

$$\mathscr{D}_K \cap (\phi + W) = \phi + (\mathscr{D}_K \cap W) \subset \phi + B_r \subset \mathscr{O}. \tag{6.5}$$

Because p_N is a seminorm on $\mathscr{D}(\Omega)$, W is a balanced and convex subset of $\mathscr{D}(\Omega)$. For every compact set $L \subset \Omega$, we have

$$\mathscr{D}_L \cap W = \left\{\psi \in \mathscr{D}_L :\ p_N(\psi) < \frac{r}{2}\right\} = \mathscr{D}_L \cap U,$$

with $U = \{\psi \in C^\infty(\Omega) :\ p_N(\psi) < r/2\}$. By Proposition 6.5, the map $p_N : C^\infty(\Omega) \to \mathbb{R}$ is continuous. Thus, the set $U = p_N^{-1}((-\infty, r/2))$ is open in $C^\infty(\Omega)$. Hence, $\mathscr{D}_L \cap U$ is open in $\mathscr{D}_L$. This means $\mathscr{D}_L \cap W$ is open in $\mathscr{D}_L$. Thus, $W \in \tilde{\mathscr{B}}$. Then the set $W_\phi = \phi + W$ is an open neighborhood of ϕ in $\mathscr{D}(\Omega)$. By (6.5), we have $\phi \in \mathscr{D}_K \cap W_\phi \subset \mathscr{O}$. Therefore,

$$\mathscr{O} = \bigcup_{\phi \in \mathscr{O}} (W_\phi \cap \mathscr{D}_K) = \left(\bigcup_{\phi \in \mathscr{O}} W_\phi \right) \cap \mathscr{D}_K = V \cap \mathscr{D}_K,$$

where $V = \bigcup_{\phi \in \mathscr{O}} W_\phi$ is an open subset of $\mathscr{D}(\Omega)$. $\square$

Proposition 6.12. *A set E is topologically bounded in $\mathscr{D}(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ such that $E \subset \mathscr{D}_K$ and E is topologically bounded in $\mathscr{D}_K$.*

Proof. ($\Leftarrow$) Suppose that there exists a compact set $K \subset \Omega$ such that $E \subset \mathscr{D}_K$ and E is topologically bounded in $\mathscr{D}_K$. For each neighborhood V of 0 in $\mathscr{D}(\Omega)$, we find $s > 0$ such that $E \subset tV$ for all $t > s$. By Proposition 6.11, the $V \cap \mathscr{D}_K$ is a neighborhood of 0 in $\mathscr{D}_K$. Because E is topologically bounded in $\mathscr{D}_K$, there exists $s > 0$ such that $E \subset t(V \cap \mathscr{D}_K)$ for all $t > s$. Thus, $E \subset tV$ for all $t > s$.

($\Rightarrow$) Consider a topologically bounded subset E of $\mathscr{D}(\Omega)$. Suppose by contradiction that there is no compact set $K \subset \Omega$ such that $E \subset \mathscr{D}_K$. Let (K_n) be a sequence of compact subsets of Ω such that each K_n is contained in the interior of K_{n+1} and $\Omega = \sum_{n=1}^{\infty} K_n$. Because $E \not\subset \mathscr{D}_{K_n}$, there exists $\phi_n \in E\backslash\mathscr{D}_{K_n}$. Thus there is $x_n \in \Omega\backslash K_n$ such that $\phi_n(x_n) \neq 0$. We claim that the sequence (x_n) has no accumulation point in Ω. Suppose otherwise. Then there exists a subsequence (x_{n_k}) of (x_n) and $x_0 \in \Omega$ such that $x_{n_k} \to x_0$. Since $x_0 \in \Omega = \sum_{n=1}^{\infty} K_n$, there exists $N \in \mathbb{N}$ such that $x_0 \in K_N$. Because K_N is contained in the interior of K_{N+1}, there exists $k_0 \in \mathbb{N}$ such that $x_{n_k} \in K_{N+1}$ for all $k > k_0$. For $m > \max\{N, k_0\}$, we have

$$x_{n_m} \in \Omega\backslash K_{n_m} \subset \Omega\backslash K_m \subset \Omega\backslash K_{N+1}.$$

This is a contradiction. Therefore, our claim is proved. Put

$$W = \left\{ \phi \in \mathscr{D}(\Omega) : \ |\phi(x_n)| < \frac{1}{n}|\phi_n(x_n)| \ \ \forall n \in \mathbb{N} \right\}.$$

It is clear that W is a balanced convex subset of $\mathscr{D}(\Omega)$. We will show that $W \in \tilde{\mathscr{B}}$, where $\tilde{\mathscr{B}}$ is the local base of $\mathscr{D}(\Omega)$ as defined in Proposition 6.10. For each compact set $L \subset \Omega$, we will show that $\mathscr{D}_L \cap W$ is open in $\mathscr{D}_L$. Because the sequence (x_n) has no limit accumulation point in L and L is compact, there are only finitely many terms of the sequence (x_n) lying in L. Let us call them $x_{n_1}, x_{n_2}, \ldots, x_{n_m}$ for $m \geq 0$. Then

$$\mathscr{D}_L \cap W = \left\{ \phi \in \mathscr{D}_L : \ |\phi(x_{n_i})| < \frac{1}{n_i}|\phi_{n_i}(x_{n_i})| \ \ \forall 1 \leq i \leq m \right\}.$$

For each $1 \leq i \leq m$, we put

$$\alpha_i = \frac{1}{n_i}|\phi_{n_i}(x_{n_i})| > 0$$

and define a map $J_i : \mathscr{D}_L \to \mathbb{R}$, $J_i(f) = f(x_{n_i})$. Since L is compact, there exists $N \in \mathbb{N}$ such that $L \subset K_N$. By Part (ii) of Proposition 6.5, the map $p_N : C^{\infty}(\Omega) \to \mathbb{R}$ is continuous. For every sequence (f_j) in $\mathscr{D}_L$ which converges to some $f \in \mathscr{D}_L$, we have

$$|J_i(f_j) - J_i(f)| = |f_j(x_{n_i}) - f(x_{n_i})| \leq |p_N(f_j) - p_N(f)| \to 0 \text{ as } j \to \infty.$$

Hence, J_i is continuous. We have

$$D_L \cap W = \{\phi \in \mathscr{D}_L : |J_i(\phi)| < \alpha_j \ \ \forall 1 \leq i \leq m\} = \bigcup_{i=1}^{m} J_i^{-1}((-\alpha_i, \alpha_i)),$$

which is an open subset of $\mathscr{D}_L$. We have proved that $W \in \tilde{\mathscr{B}}$. Therefore, W is an open neighborhood of the origin in $\mathscr{D}(\Omega)$. Because E is topologically bounded in $\mathscr{D}(\Omega)$, there exists $s > 0$ such that $E \subset tW$ for all $t > s$. In particular, $\phi_n \in (s+1)W$ for all $n \in \mathbb{N}$. Thus,

$$\frac{1}{s+1}\phi_n \in W \quad \forall n \in \mathbb{N}.$$

This means
$$\frac{1}{s+1}|\phi_n(x_n)| < \frac{1}{n}|\phi_n(x_n)| \quad \forall n \in N.$$
Since $|\phi_n(x_n)| > 0$, the above inequality is equivalent to $n < s+1$ for all $n \in \mathbb{N}$. This is a contradiction. $\square$

Proposition 6.13. *Denote by $(\mathscr{D}(\Omega), \tau)$ the TVS as defined in Proposition 6.10. Let (ϕ_n) be a sequence in $(\mathscr{D}(\Omega), \tau)$. Then (ϕ_n) converges to 0 if and only if there is a compact set $K \subset \Omega$ such that $\phi_n \in \mathscr{D}_K$ for all $n \in \mathbb{N}$ and $D^\alpha \phi_n \to 0$ uniformly on K for every multi-index α.*

Proof. ($\Leftarrow$) Suppose that there is a compact set $K \subset \Omega$ such that $\phi_n \in \mathscr{D}_K$ for all $n \in \mathbb{N}$ and $D^\alpha \phi_n \to 0$ uniformly on K for every multi-index α. By Proposition 6.8, the sequence (ϕ_n) converges to 0 in $C^\infty(\Omega)$. Thus, (ϕ_n) converges to 0 in the topology $(\mathscr{D}_K, \tau_K)$. By Proposition 6.11, τ_K is also the topology that $\mathscr{D}_K$ inherits from $(\mathscr{D}(\Omega), \tau)$. We conclude that (ϕ_n) converges to 0 in $(\mathscr{D}(\Omega), \tau)$.

($\Rightarrow$) Consider a sequence (ϕ_n) in $(\mathscr{D}(\Omega), \tau)$ which converges to 0. By Part (ii) of Proposition 6.2, (ϕ_n) is topologically bounded. Then by Proposition 6.12, there exists a compact set $K \subset \Omega$ such that $\phi_n \in \mathscr{D}_K$ for all $n \in \mathbb{N}$. Because $\phi_n \to 0$ in $(\mathscr{D}(\Omega), \tau)$, $\phi_n \to 0$ in $(\mathscr{D}_K, \tau_K)$. Thus, $\phi_n \to 0$ in $C^\infty(\Omega)$. By Proposition 6.8, for every multi-index α and for every compact subset L of Ω, $D^\alpha \phi_n \to 0$ uniformly on L as $n \to \infty$. Taking $L = K$, we have proved the claim. $\square$

Proposition 6.14. *Denote by $(\mathscr{D}(\Omega), \tau)$ the TVS as defined in Proposition 6.10. Let Y be a locally convex TVS and $\Lambda : \mathscr{D}(\Omega) \to Y$ be a linear map. Then Λ is continuous if and only if for every sequence (ϕ_n) converging to 0 in $\mathscr{D}(\Omega)$, the sequence $(\Lambda\phi_n)$ converges to 0 in Y.*

Proof. ($\Rightarrow$) Suppose that Λ is continuous. Let (ϕ_n) be a sequence converging to 0 in $\mathscr{D}(\Omega)$ and V be a neighborhood of 0 in Y. Because Λ is continuous, there exists a neighborhood U of 0 in $\mathscr{D}(\Omega)$ such that $\Lambda(U) \subset V$. Since $\phi_n \to 0$ in $\mathscr{D}(\Omega)$, there exists $N \in \mathbb{N}$ such that $\phi_n \in U$ for all $n > N$. Thus, $\Lambda(\phi_n) \in \Lambda(U) \subset V$ for all $n > N$. Hence, $\Lambda(\phi_n) \to 0$ in Y.

($\Leftarrow$) For each compact set $K \subset \Omega$, we know from Section 6.2 that $\mathscr{D}_K$ is metrizable. Every sequence (ϕ_n) in $\mathscr{D}_K$ that converges to 0 also converges to 0 as a sequence in $\mathscr{D}(\Omega)$. Then we have $\Lambda(\phi_n) \to 0$. Thus, the restriction of Λ on $\mathscr{D}_K$ is continuous. Now we will show that Λ is continuous on $\mathscr{D}(\Omega)$. Because Y is locally convex, it has a balanced convex local base consisting of open sets, namely $\mathscr{B}_1$, according to Part (ii) of Proposition 6.1. Take any $V \in \mathscr{B}_1$. We will show that $\Lambda^{-1}(V) \subset U$ is open in $\mathscr{D}(\Omega)$. Since V is balanced and convex, so is U. For each compact set $K \subset \Omega$, we have
$$\mathscr{D}_K \cap U = \mathscr{D}_K \cap \Lambda^{-1}(V) = (\Lambda|_{\mathscr{D}_K})^{-1}(V),$$
which is open in $\mathscr{D}_K$ because $\Lambda|_{\mathscr{D}_K}$ is continuous. Thus, $U \in \tilde{\mathscr{B}}$, the local base of $\mathscr{D}(\Omega)$ as defined in Proposition 6.10. Therefore, U is open in $\mathscr{D}(\Omega)$. $\square$

6.4 The distribution spaces $\mathscr{D}'(\Omega)$ and $\mathscr{E}'(\Omega)$

Let Ω be a nonempty open subset of a Euclidean space. The test-function space $\mathscr{D}(\Omega)$ is a TVS as defined in Proposition 6.10. For each $\phi \in \mathscr{D}(\Omega)$ and $N \in \mathbb{N}$, we denote $||\phi||_N = \max\{|D^\alpha \phi(x)| : \ |\alpha| \leq N, x \in \Omega\}$. A map $\Lambda : \mathscr{D}(\Omega) \to \mathbb{R}$ is called a *distribution* in Ω if Λ is linear and continuous. The set of all distributions in Ω is denoted by $\mathscr{D}'(\Omega)$. This is clearly a vector space (over $\mathbb{R}$).

Proposition 6.15. *Let $\Lambda : \mathscr{D}(\Omega) \to \mathbb{R}$ be a linear map. Then $\Lambda \in \mathscr{D}'(\Omega)$ if and only if for every compact set $K \subset \Omega$, there exist a nonnegative integer $N = N(K)$ and a number $C = C(K) > 0$ such that*

$$|\Lambda\phi| \leq C||\phi||_N \quad \forall \phi \in \mathscr{D}_K.$$

Proof. ($\Rightarrow$) Suppose that $\Lambda \in \mathscr{D}'(\Omega)$. Take any compact set $K \subset \Omega$. The map $\Lambda|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathbb{R}$ is linear and continuous. By Proposition 6.3, $\Lambda|_{\mathscr{D}_K}$ is a bounded map. Let (K_n) be a sequence of compact subset of Ω such that K_n lies in the interior of K_{n+1} and $\bigcup_{n=1}^\infty K_n = \Omega$. Then we have a family of seminorms $\{p_n : \ n \in \mathbb{N}\}$ with $p_n : C^\infty(\Omega) \to \mathbb{R}$, $p_n(f) = \max\{|D^\alpha f(x)| : \ |\alpha| \leq n, \ x \in K_n\}$. By Part (iii) of Proposition 6.5, a set E is topologically bounded in $C^\infty(\Omega)$ if and only if $p_n(E)$ is bounded in $\mathbb{R}$ for every $n \in \mathbb{N}$.

Because K is compact, there exists $N \in \mathbb{N}$ such that $K \subset K_N$. Put $E = \{\phi \in \mathscr{D}_K : \ p_N(\phi) = 1\}$. We will show that E is topologically bounded in $\mathscr{D}_K$. Because $\mathscr{D}_K$ is a topological subspace of $C^\infty(\Omega)$, it suffices to show that E is topologically bounded in $C^\infty(\Omega)$. We have

$$0 \leq p_1(f) \leq p_2(f) \leq \ldots \leq p_N(f) = p_{N+1}(f) = \ldots = 1 \quad \forall f \in E.$$

Therefore, $p_n(E) \subset [0, 1]$ for all $n \in \mathbb{N}$. Thus, E is topologically bounded in $C^\infty(\Omega)$.

Because $\Lambda|_{\mathscr{D}_K}$ is a bounded map, $\Lambda(E)$ is bounded in $\mathbb{R}$. Thus, there exists $C \in \mathbb{R}$ such that $|\Lambda(\phi)| \leq C$ for all $\phi \in E$. For every $\psi \in \mathscr{D}\backslash\{0\}$, we put

$$\phi = \frac{\psi}{||\psi||_N} \in E.$$

Then $|\Lambda(\psi)| = ||\psi||_N |\Lambda(\phi)| \leq C||\psi||_N$.

($\Leftarrow$) Suppose that for every compact set $K \subset \Omega$, there exists a nonnegative integer $N = N(K)$ and a number $C = C(K) > 0$ such that $|\Lambda(\phi)| \leq C||\phi||_N$ for all $\phi \in \mathscr{D}_K$. We will show that for every compact set $K \subset \Omega$, the map $\Lambda|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathbb{R}$ is continuous. For such a set K, there exists $N_0 \in \mathbb{N}$ such that $K \subset K_{N_0}$. We know that $\mathscr{D}_K$ is a topological subspace of $C^\infty(\Omega)$. Consider a sequence (ϕ_m) in $\mathscr{D}_K$ which converges to $0 \in \mathscr{D}_K$. Then $\phi_m \to 0$ in $C^\infty(\Omega)$. Put $N_1 = \max\{N(K), N_0\}$. Then

$$||\phi_m||_N \leq p_{N_1}(\phi_m) \to 0 \text{ as } m \to \infty.$$

Since $|\Lambda(\phi_m)| \leq ||\phi_m||_N$, $\Lambda(\phi_m) \to 0$ as $m \to \infty$. Because the topology on $\mathscr{D}_K$ is metrizable, the map $\Lambda|_{\mathscr{D}_K}$ is continuous according to Proposition 6.3.

We have showed that the map $\Lambda|_{\mathscr{D}_K} : \mathscr{D}_K \to \mathbb{R}$ is continuous for every compact set $K \subset \Omega$. Now we show that $\Lambda : \mathscr{D}(\Omega) \to \mathbb{R}$ is continuous. It suffices to show that Λ is continuous at 0. Consider an open neighborhood of 0 in $\mathbb{R}$ of the form $(-\epsilon, \epsilon)$, $\epsilon > 0$. We need to show that the set $V = \Lambda^{-1}((-\epsilon, \epsilon))$ is open in $\mathscr{D}(\Omega)$. By the linearity of Λ, V is balanced and convex. We will show that $V \in \tilde{\mathscr{B}}$, where $\tilde{\mathscr{B}}$ is the local base of $\mathscr{D}(\Omega)$ as defined in Proposition 6.10. To do so, it remains to show that $\mathscr{D}_K \cap V$ is open in $\mathscr{D}_K$ for every compact set $K \subset \Omega$. We have

$$\mathscr{D}_K \cap V = \mathscr{D}_K \cap \Lambda^{-1}((-\epsilon, \epsilon)) = (\Lambda|_{\mathscr{D}_K})^{-1}((-\epsilon, \epsilon)).$$

This is an open set in $\mathscr{D}_K$ because the map $\Lambda|_{\mathscr{D}_K}$ is continuous. □

Let $\Lambda \in \mathscr{D}'(\Omega)$. If ω is an open subset of Ω such that $\Lambda\phi = 0$ for all $\phi \in \mathscr{D}(\omega)$, we say that Λ *vanishes in* ω. Let W be the union of all open sets $\omega \subset \Omega$ in which Λ vanishes. Then $\Omega \backslash W$ is called the *support* of Λ, denoted by $\mathrm{supp}\Lambda$. The set of all distributions whose supports are compact subsets of Ω is denoted by $\mathscr{E}'(\Omega)$. This is clearly a vector subspace of $\mathscr{D}'(\Omega)$.

Proposition 6.16. *Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Define a map $\Lambda_u : \mathscr{D} \to \mathbb{R}$,*

$$\Lambda_u(\phi) = \int_\Omega u\phi dx.$$

Then $\Lambda_u \in \mathscr{D}'(\Omega)$. For this reason, we usually view a locally integrable function as a distribution by identifying u with Λ_u.

Proof. It is clear from the definition of Λ_u that Λ_u is a linear map. For any compact set $K \subset \Omega$, we put

$$C = C(K) = \int_K |u| dx < \infty.$$

Then for every $\phi \in \mathscr{D}_K$,

$$|\Lambda_u(\phi)| = \left| \int_\Omega u\phi dx \right| = \left| \int_K u\phi dx \right| \le \int_K |u||\phi| dx \le C \max\{|\phi(x)| : x \in K\} = C||\phi||_0.$$

By Proposition 6.15, $\Lambda_u \in \mathscr{D}'(\Omega)$. □

Proposition 6.17. *Let $\Lambda \in \mathscr{D}'(\Omega)$. Then we have the following statements.*

(i) If $\phi \in \mathscr{D}(\Omega)$ and $\mathrm{supp}\phi \cap \mathrm{supp}\Lambda = \emptyset$, then $\Lambda\phi = 0$.

(ii) If $\Lambda \in \mathscr{E}'(\Omega)$ then there are a number $C > 0$ and a nonnegative integer N such that

$$|\Lambda\phi| \le C||\phi||_N \quad \forall \phi \in \mathscr{D}(\Omega).$$

Furthermore, Λ extends in a unique way to a linear continuous functional on $C^\infty(\Omega)$.

Proof. (i) Let $\phi \in \mathscr{D}(\Omega)$. Suppose that $\operatorname{supp}\phi \cap \operatorname{supp}\Lambda = \emptyset$. Then $\operatorname{supp}\phi \subset \Omega\backslash\operatorname{supp}\Lambda$. Denote by $\mathscr{S}$ the family of all open subsets ω in Ω in which Λ vanishes. Put $W = \bigcup_{\omega\in\mathscr{S}}\omega$. Then $\operatorname{supp}\Lambda = \Omega\backslash W$ by the definition of supports. Thus, $\operatorname{supp}\phi \subset W$. For each $x \in \operatorname{supp}\phi$, $x \in W$. Thus, there exists $\omega_x \in \mathscr{S}$ such that $x \in \omega_x$. Thus, the family $\{\omega_x : \ x \in \operatorname{supp}\phi\}$ is an open cover of $\operatorname{supp}\phi$. Because $\operatorname{supp}\phi$ is compact, we can extract a finite subcover and rename it as $\{U_1, U_2, \ldots, U_m\}$.

Put $U = \Omega\backslash\operatorname{supp}\phi$. Then $\{U, U_1, \ldots, U_m\}$ is an open cover of Ω. Consider a smooth partition of unity subordinate to this cover, namely $\{\psi, \psi_1, \ldots, \psi_m\}$. We have $\operatorname{supp}\psi \subset U$ and $\operatorname{supp}\psi_i \subset U_i$ for all $1 \le i \le m$. Then $\phi = \phi\psi+\phi\psi_1+\ldots+\phi\psi_m$. Since $\operatorname{supp}\psi \subset U = \Omega\backslash\operatorname{supp}\phi$, $\phi\psi = 0$ in Ω. Put $\phi_i = \phi\psi_i \in \mathscr{D}(U_i)$. Then $\phi = \phi_1+\phi_2+\ldots+\phi_m$. Because Λ is linear, $\Lambda\phi = \Lambda\phi_1+\Lambda\phi_2+\ldots+\Lambda\phi_m$. Because $U_i \in \mathscr{S}$ and $\phi_i \in \mathscr{D}(U_i)$, we have $\Lambda\psi_i = 0$ for all $1 \le i \le m$. Therefore, $\Lambda\phi = 0$.

(ii) Consider a distribution $\Lambda \in \mathscr{D}'(\Omega)$ whose support is a compact subset of Ω. Then there exists a umber $\epsilon > 0$ such that $\operatorname{dist}(\operatorname{supp}\Lambda, \Omega^c) > \epsilon$. Put

$$\begin{aligned} K_1 &= \{x \in \Omega : \operatorname{dist}(x, \operatorname{supp}\Lambda) \le \epsilon/2\}, \\ K_2 &= \{x \in \Omega : \operatorname{dist}(x, \operatorname{supp}\Lambda) \le \epsilon\}. \end{aligned}$$

Then K_1 and K_2 are compact subsets of Ω. Moreover, $\operatorname{supp}\Lambda$ is contained in the interior of K_1. Aslo, K_1 is contained in the interior of K_2 and $\operatorname{supp}\Lambda \subset K_1 \subset K_2 \subset \Omega$. Let χ be a function in $\mathscr{D}(\Omega)$ such that $\chi = 1$ in K_1 and $\chi = 0$ in $\Omega\backslash K_2$. Because $\Lambda \in \mathscr{D}'(\Omega)$, by Proposition 6.15 there exists a nonnegative number N and a number $C_0 > 0$ such that

$$|\Lambda(\phi)| \le C_0||\phi||_N \quad \forall\phi \in \mathscr{D}_{K_2}.$$

Consider any function $\psi \in \mathscr{D}(\Omega)$. Then $\psi\chi \in \mathscr{D}_{K_2}$ and $\psi - \psi\chi = 0$ in K_1. Thus, $\operatorname{supp}(\psi - \psi\chi) \subset \Omega\backslash\operatorname{supp}\Lambda$. Thus,

$$\operatorname{supp}(\psi - \psi\chi) \cap \operatorname{supp}\Lambda = \emptyset.$$

By Part (i), $\Lambda(\psi - \psi\chi) = 0$. Hence, $\Lambda(\psi) = \Lambda(\psi\chi)$. Therefore,

$$|\Lambda(\psi)| = |\Lambda(\psi\chi)| \le C_0||\psi\chi||_N. \tag{6.6}$$

By (3.5), we have

$$D^\alpha(\psi\chi) = \sum_{\{\beta:\ \beta\le\alpha\}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (D^\beta\psi)(D^{\alpha-\beta}\chi).$$

For each $|\alpha| \le N$, we have

$$|D^\alpha(\psi\chi)(x)| \le M||\psi||_N \quad \forall x \in \Omega,$$

where

$$M = \max\left\{ \sum_{\{\beta:\ \beta\le\alpha\}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} |D^{\alpha-\beta}\chi(x)| : \ |\alpha| \le N,\ x \in \Omega \right\}.$$

Thus, $||\psi\chi||_N \le M||\psi||_N$. Then (6.6) implies

$$|\Lambda\psi| \le C_0||\psi\chi||_N \le C_0 M||\psi||_N \quad \forall\psi \in \mathscr{D}(\Omega).$$

Therefore, we can choose $C = C_0 M > 0$.

Next, we show that Λ extends in a unique way to a linear and continuous functional on $C^\infty(\Omega)$. Suppose that $\tilde{\Lambda} : C^\infty(\Omega) \to \mathbb{R}$ is such an extension. For each function $f \in C^\infty(\Omega)$, we have $f\chi \in \mathscr{D}_{K_2}$ and $f = f\chi$ in K_1. Thus, $\text{supp}(f - f\chi) \subset \Omega\backslash\text{supp}\Lambda$. Thus,

$$\text{supp}(f - f\chi) \cap \text{supp}\Lambda = \emptyset.$$

By Part (i), $\Lambda(f - f\chi) = 0$. Thus,

$$\tilde{\Lambda}(f) = \tilde{\Lambda}(f\chi) + \tilde{\Lambda}(f - f\chi) = \Lambda(f\chi) + \Lambda(f - f\chi) = \Lambda(f\chi).$$

This means $\tilde{\Lambda}$ is uniquely determined. Now we show that the functional $\tilde{\Lambda} : C^\infty(\Omega) \to \mathbb{R}$, $\tilde{\Lambda}(f) = \Lambda(f\chi)$ is actually a linear continuous extension of Λ. If $f \in \mathscr{D}(\Omega)$ then

$$\tilde{\Lambda}(f) = \Lambda(f\chi) = \Lambda(f) - \Lambda(f - f\chi) = \Lambda(f).$$

Thus, $\tilde{\Lambda} = \Lambda$ on $\mathscr{D}(\Omega)$. By the definition of $\tilde{\Lambda}$, it is clear that $\tilde{\Lambda}$ is linear. Let (f_n) be a sequence in $C^\infty(\Omega)$ which converges to $f \in C^\infty(\Omega)$. By Proposition 6.8, for each multi-index α, $D^\alpha f_n \to D^\alpha f$ uniformly on every compact subset of Ω. By (3.5), we have

$$D^\alpha(f_n\chi) = \sum_{\{\beta:\beta\le\alpha\}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (D^\beta f_n)(D^{\alpha-\beta}\chi),$$

which converges to

$$\sum_{\{\beta:\beta\le\alpha\}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (D^\beta f)(D^{\alpha-\beta}\chi) = D^\alpha(f\chi)$$

on every compact subset of Ω. Note that $(f_n\chi)$ is a sequence in $\mathscr{D}_{K_2}$. Thus, $f_n\chi \to f\chi$ in $\mathscr{D}(\Omega)$ according to Proposition 6.13. Since Λ is continuous on $\mathscr{D}(\Omega)$, $\Lambda(f_n\chi) \to \Lambda(f\chi)$. Thus, $\tilde{\Lambda}(f_n) \to \tilde{\Lambda}(f)$. Therefore, $\tilde{\Lambda}$ is continuous on $C^\infty(\Omega)$. □

Proposition 6.18. *For each multi-index α and $\Lambda \in \mathscr{D}'(\Omega)$, we define a map $D^\alpha\Lambda : \mathscr{D}(\Omega) \to \mathbb{R}$,*

$$(D^\alpha\Lambda)(\phi) = (-1)^\alpha\Lambda(D^\alpha\phi) \quad \forall\phi \in \mathscr{D}(\Omega).$$

Then $D^\alpha\Lambda \in \mathscr{D}'(\Omega)$.

Proof. It is clear that $D^\alpha\Lambda$ is a linear map. Because $\Lambda \in \mathscr{D}'(\Omega)$, for any compact set $K \subset \Omega$, according to Proposition 6.15 there exists a nonnegative integer $N = N(K)$ and a number $C = C(K) > 0$ such that

$$|\Lambda(\phi)| \le C||\phi||_N \quad \forall\phi \in \mathscr{D}_K.$$

For every $\phi \in \mathscr{D}_K$, we have $D^\alpha \phi \in \mathscr{D}_K$. Thus,

$$|(D^\alpha \Lambda)(\phi)| = |\Lambda(D^\alpha \phi)| \leq C||D^\alpha \phi||_N \leq C||\phi||_{N+|\alpha|}.$$

Therefore, $D^\alpha \Lambda \in \mathscr{D}'(\Omega)$ according to Proposition 6.15. $\square$

Proposition 6.19. *Let $\Lambda \in \mathscr{D}'(\Omega)$ and α, β be two multi-indices. Then*

$$(D^\alpha D^\beta \Lambda)(\phi) = (D^{\alpha+\beta}\Lambda)(\phi) = (D^\beta D^\alpha \Lambda)(\phi) \quad \forall \phi \in \mathscr{D}(\Omega).$$

Proof. By the definition of partial derivatives of a distribution, we have

$$\begin{aligned} (D^\alpha D^\beta \Lambda)(\phi) &= (-1)^{|\alpha|}(D^\beta \Lambda)(D^\alpha \phi) \\ &= (-1)^{|\alpha|}(-1)^{|\beta|}\Lambda(D^\beta D^\alpha \phi) \\ &= (-1)^{|\alpha|+|\beta|}\Lambda(D^{\alpha+\beta}\phi) \\ &= (D^{\alpha+\beta}\Lambda)(\phi) \end{aligned}$$

Switching α with β, we get $(D^\beta D^\alpha \Lambda)(\phi) = (D^{\beta+\alpha}\Lambda)(\phi)$. Because $\alpha + \beta = \beta + \alpha$, we obtain $(D^\beta D^\alpha \Lambda)(\phi) = (D^\alpha D^\beta \Lambda)(\phi)$. $\square$

If we have a map $v : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ then $\tau_x v$ and $\check{v}$ are the functions on $\mathbb{R}^n$ defined by

$$(\tau_x v)(y) = v(y - x), \quad \check{v}(y) = v(-y).$$

If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$ then their *convolution* $u * \phi$ is defined as the function

$$(u * \phi)(x) = u(\tau_x \check{\phi}) \quad \forall x \in \mathbb{R}^n. \tag{6.7}$$

By Part (ii) of Proposition 6.17, each $u \in \mathscr{E}'(\mathbb{R}^n)$ can extend in a unique way to a linear continuous functional on $C^\infty(\mathbb{R}^n)$. This allows us to define the convolution $u * \phi$ for $u \in \mathscr{E}'(\mathbb{R}^n)$ and $\phi \in C^\infty(\mathbb{R}^n)$ as follows.

$$(u * \phi)(x) = u(\tau_x \check{\phi}) \quad \forall x \in \mathbb{R}^n. \tag{6.8}$$

Of course, the definitions (6.7) and (6.8) agree whenever two ways of defining $u * \phi$ are possible. When $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, u can be viewed as a distribution in $\mathbb{R}^n$ by Proposition 6.16. Then

$$(u * \phi)(x) = u(\tau_x \check{\phi}) = \int_{\mathbb{R}^n} u(y)\tau_x \check{\phi}(y)dy = \int_{\mathbb{R}^n} u(y)\phi(x - y)dy,$$

which coincides with the usual convolution of two functions in $\mathbb{R}^n$.

Proposition 6.20. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$. Then $u * \phi \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha \phi)$ for every multi-index α.*

Proof. Note that $D^\alpha u \in \mathscr{D}'(\mathbb{R}^n)$ according to Proposition 6.18. Thus, the convolution $(D^\alpha u) * \phi$ is well-defined. First, we show that $(D^\alpha u) * \phi = u * (D^\alpha \phi)$. For each $x \in \mathbb{R}^n$, we have

$$((D^\alpha u) * \phi)(x) = (D^\alpha u)(\tau_x \check{\phi}) = (-1)^{|\alpha|} u(D^\alpha(\tau_x \check{\phi})). \tag{6.9}$$

For each $y \in \mathbb{R}^n$, $\tau_x \check{\phi}(y) = \check{\phi}(y-x) = \phi(x-y)$. Thus, $D^\alpha(\tau_x \check{\phi}) = (-1)^{|\alpha|} D^\alpha \phi(x-y)$. Then (6.9) becomes

$$\begin{aligned}
((D^\alpha u) * \phi)(x) &= (-1)^{|\alpha|} u((-1)^{|\alpha|} D^\alpha \phi(x-y)) \\
&= (-1)^{|\alpha|} (-1)^{|\alpha|} u((D^\alpha \phi)(x-y)) \\
&= u((D^\alpha \phi)(x-y)) = u((D^\alpha \phi)\check{}\,(y-x)) \\
&= u(\tau_x((D^\alpha \phi)\check{}\,)) = (u * (D^\alpha \phi))(x).
\end{aligned}$$

Therefore, $(D^\alpha u) * \phi = u * (D^\alpha \phi)$.

Next, we show that $D^\alpha(u * \phi) = u * (D^\alpha \phi)$ for every multi-index α. By induction on the length $|\alpha|$, it suffices for us to show that $D_a(u * \phi) = u * (D_a \phi)$ for every unit vector $a \in \mathbb{R}^n$, where D_a denotes the directional derivative in a-direction. For $x \in \mathbb{R}^n$ and $h \in (-1, 1) \backslash \{0\}$, we have

$$\begin{aligned}
\frac{(u * \phi)(x + ha) - (u * \phi)(x)}{h} &= \frac{u(\tau_{x+ha} \check{\phi}) - u(\tau_x \check{\phi})}{h} \\
= u\left(\frac{\tau_{x+ha}\check{\phi} - \tau_x \check{\phi}}{h}\right) &= u\left(\tau_x\left(\frac{\tau_{ha}\check{\phi} - \check{\phi}}{h}\right)\right).
\end{aligned}$$

Also, $(u * (D^\alpha \phi))(x) = u(\tau_x((D^\alpha \phi)\check{}\,))$. Thus, showing that

$$\lim_{h \to 0} \frac{(u * \phi)(x + ha) - (u * \phi)(x)}{h} = (u * (D_a \phi))(x)$$

is equivalent to showing that

$$\lim_{h \to 0} u\left(\tau_x\left(\frac{\tau_{ha}\check{\phi} - \check{\phi}}{h}\right)\right) = u(\tau_x((D_a \phi)\check{}\,)).$$

Because u is continuous from $\mathscr{D}(\mathbb{R}^n)$ to $\mathbb{R}^n$, it suffices to show that

$$\lim_{h \to 0} \tau_x\left(\frac{\tau_{ha}\check{\phi} - \check{\phi}}{h}\right) = \tau_x((D_a \phi)\check{}\,).$$

By the definition of the operator τ_x, this will be proved if we can show that

$$\lim_{h \to 0} \frac{\tau_{ha}\check{\phi} - \check{\phi}}{h} = (D_a \phi)\check{}\,. \quad \text{(convergence in } \mathscr{D}(\mathbb{R}^n))$$

More explicitly, we want to show that

$$\left(y \mapsto \frac{\phi(ha - y) - \phi(-y)}{h}\right) \to (y \mapsto D_a \phi(-y))$$

in $\mathscr{D}(\mathbb{R}^n)$ as $h \to \infty$. Applying the check operator on both sides, we are supposed to show that

$$\left(y \mapsto \frac{\phi(ha+y) - \phi(y)}{h}\right) \to (y \mapsto D_a\phi(y))$$

in $\mathscr{D}(\mathbb{R}^n)$ as $h \to \infty$. Thus, we want to show that

$$\lim_{h\to 0} \frac{\tau_{-ha}\phi - \phi}{h} = D_a\phi. \tag{6.10}$$

For each $\psi \in \mathscr{D}(\mathbb{R}^n)$, we have

$$\begin{aligned} \frac{(\tau_{-ha}\psi)(y) - \psi(y)}{h} - D_a\psi(y) &= \frac{\psi(y+ha) - \psi(y)}{h} - D_a\psi(y) \\ &= D_a\psi(y + \theta_h a) - D_a\psi(y) \ \ \text{(where } |\theta_h| < |h|\text{)} \\ &= \theta_h(D_aD_a)\psi(y + \tilde{\theta}_h a) \ \ \text{(where } |\tilde{\theta}_h| \le |\theta_h| < |h|\text{)}. \end{aligned}$$

Put $M = \max\{|D_aD_a\psi(x)| : \ x \in \mathbb{R}^n\}$. Then for every $y \in \mathbb{R}^n$,

$$\left|\frac{(\tau_{-ha}\psi)(y) - \psi(y)}{h} - D_a\psi(y)\right| \le |\theta_h|M \le |h|M.$$

Thus, $\frac{\tau_{-ha}\psi - \psi}{h}$ converges to $D_a\psi$ uniformly in $\mathbb{R}^n$ as $h \to 0$. Applying this result for $\psi = D^\beta\phi$, where β is any multi-index, we conclude that

$$D^\beta\left(\frac{\tau_{-ha}\phi - \phi}{h}\right) = \frac{\tau_{-ha}(D^\beta\phi) - D^\beta\phi}{h}$$

converges to $D_a(D^\beta\phi) = D^\beta(D_a\phi)$ uniformly in $\mathbb{R}^n$ as $h \to 0$. Put $K = \bar{B}_1 + \mathrm{supp}\phi$, where $\bar{B}_1$ is the closed unit ball in $\mathbb{R}^n$. Then $\frac{\tau_{-ha}\phi - \phi}{h} \in \mathscr{D}_K$ for all $h \in (-1,1)\backslash\{0\}$. By Proposition 6.8, $\frac{\tau_{-ha}\phi - \phi}{h}$ converges to $D_a\phi$ in $\mathscr{D}_K$. Therefore, we obtain the convergence in $\mathscr{D}(\mathbb{R}^n)$ according to Proposition 6.13. □

Proposition 6.21. *Let $u \in \mathscr{E}'(\mathbb{R}^n)$ and $\phi \in C^\infty(\mathbb{R}^n)$. We have the following statements.*

(i) $u * \phi \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(u * \phi) = (D^\alpha u) * \phi = u * (D^\alpha\phi)$.

(ii) If $\phi \in \mathscr{D}(\mathbb{R}^n)$ then $u * \phi \in \mathscr{D}(\mathbb{R}^n)$ and $\mathrm{supp}(u * \phi) \subset (\mathrm{supp}u) + (\mathrm{supp}\phi)$.

Proof. (i) The proof of this part is almost a repetition of that of Proposition 6.20 with $\mathscr{D}(\mathbb{R}^n)$ being replaced by $C^\infty(\mathbb{R}^n)$. We only need to adjust the proof of (6.10), namely to show that $\frac{\tau_{-ha}\phi - \phi}{h}$ converges to $D_a\phi$ in $C^\infty(\mathbb{R}^n)$ as $h \to 0$. Consider $h \in (-1,1)\backslash\{0\}$. For any $\psi \in C^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \frac{(\tau_{-ha}\psi)(y) - \psi(y)}{h} - D_a\psi(y) &= \frac{\psi(y+ha) - \psi(y)}{h} - D_a\psi(y) \\ &= D_a\psi(y + \theta_h a) - D_a\psi(y) \ \ \text{(where } |\theta_h| < |h|\text{)} \\ &= \theta_h(D_aD_a)\psi(y + \tilde{\theta}_h a) \ \ \text{(where } |\tilde{\theta}_h| \le |\theta_h| < |h|\text{)}. \end{aligned}$$

For every compact set $K \subset \mathbb{R}^n$, we put

$$M_K = \max\left\{|D_a D_a \psi(x)| : x \in K + \bar{B}_1\right\},$$

where $\bar{B}_1$ is the closed unit ball in $\mathbb{R}^n$. For $y \in K$, we have $y + \tilde{\theta}_h a \in K + \bar{B}_1$ because $|\tilde{\theta}_h| \leq |\theta_h| \leq |h| < 1$. Thus,

$$\left|\frac{(\tau_{-ha}\psi)(y) - \psi(y)}{h} - D_a\psi(y)\right| \leq |h| M_K \quad \forall h \in (-1,1)\backslash\{0\},\ \forall y \in K.$$

This implies that $\frac{\tau_{-ha}\psi - \psi}{h}$ converges to $D_a\psi$ uniformly on every compact subset of $\mathbb{R}^n$ as $h \to 0$. Applying this result for $\psi = D^\beta \phi$, where β is any multi-index, we get

$$D^\beta\left(\frac{\tau_{-ha}\phi - \phi}{h}\right) \to D^\beta(D_a\phi)$$

uniformly on every compact subset of $\mathbb{R}^n$ as $h \to 0$. By Proposition 6.8, $\frac{\tau_{-ha}\phi - \phi}{h}$ converges to $D_a\phi$ in $C^\infty(\mathbb{R}^n)$ as $h \to 0$.

(ii) Put $A = (\mathrm{supp} u) + (\mathrm{supp}\phi)$. Then A is a compact subset of $\mathbb{R}^n$. We need to show that $\mathrm{supp}(u * \phi) \subset A$. Take any $x \in \mathbb{R}^n \backslash A$. We show that $(u * \phi)(x) = 0$. Because $\tau_x\check{\phi}(y) = \phi(x - y)$, we have $\mathrm{supp}(\tau_x\check{\phi}) = x - \mathrm{supp}\phi$. Since $x \notin (\mathrm{supp} u + \mathrm{supp}\phi)$, $(x - \mathrm{supp}\phi) \cap \mathrm{supp} u = \emptyset$. Thus, $\mathrm{supp}(\tau_x\check{\phi}) \cap \mathrm{supp}\phi = \emptyset$. By Part (i) of Proposition 6.17, $u(\tau_x\check{\phi}) = 0$. Thus, $(u * \phi)(x) = 0$. ☐

From now on, the same notation $\langle \cdot, \cdot \rangle$ is used to denote either the pairing between $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{D}(\mathbb{R}^n)$, or the paring between $\mathscr{E}'(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$.

Proposition 6.22 (Dirac measure)**.** *Define a map $\delta_0 : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$, $\delta_0(\phi) = \phi(0)$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$. We have the following statements.*

(i) $\delta_0 \in \mathscr{E}'(\mathbb{R}^n)$ with supp$\delta_0 = \{0\}$.

(ii) $\delta * \phi = \phi$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$.

Proof. (i) Let W be the union of all open sets ω in $\mathbb{R}^n$ in which δ_0 vanishes. Then $\mathrm{supp}\delta_0 = \mathbb{R}^n \backslash W$ by definition. We want to show that $W = \mathbb{R}^n \backslash \{0\}$. Take any $x \in W$. There is a neighborhood $\omega \in W$ of x such that $\delta_0(\phi) = 0$ for all $\phi \in \mathscr{D}(\omega)$. Thus, $\phi(0) = 0$ for all $\phi \in \mathscr{D}(\omega)$. If $0 \in \omega$ then we can choose a bump function ψ supported in a small ball centered at 0 such that $\psi(0) = 1$. This is a contradiction. Thus, $0 \notin \omega$. Thus, $\omega \subset \mathbb{R}^n \backslash \{0\}$. Hence, $W \subset \mathbb{R}^n \backslash \{0\}$. Take any open set $\omega \subset \mathbb{R}^n \backslash \{0\}$ and $\phi \in \mathscr{D}(\omega)$. Then $\delta_0(\phi) = \phi(0) = 0$. This means δ_0 vanishes in ω. Thus, $\omega \subset W$. Thus, $\mathbb{R}^n \backslash \{0\} \subset W$. Therefore, $W = \mathbb{R}^n \backslash \{0\}$.

(ii) For each $x \in \mathbb{R}^n$, $(\delta_0 * \phi)(x) = \delta_0(\tau_x\check{\phi}) = \tau_x\check{\phi}(0) = \phi(x)$. Hence, $\delta_0 * \phi = \phi$. ☐

Define a function $\eta : \mathbb{R}^n \to \mathbb{R}$,

$$\eta(x) = \begin{cases} \exp\left(\dfrac{1}{|x|^2 - 1}\right) & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

For each $\epsilon > 0$, we put $\eta_\epsilon(x) = \epsilon^{-n}\eta(\epsilon^{-1}x)$ for all $x \in \mathbb{R}^n$. Then $\eta \in \mathscr{D}(\mathbb{R}^n)$ and suppη=$\bar{B}_1$, the closed unit ball in $\mathbb{R}^n$. Also, $\eta_\epsilon \in \mathscr{D}(\mathbb{R}^n)$ and suppη_ϵ=$\bar{B}_\epsilon$. We refer to the family $\{\eta_\epsilon\}_{\epsilon>0}$ as an *approximate identity on* $\mathbb{R}^n$.

For each function $\phi \in \mathbb{R}^n$, we know that $\phi * \eta_\epsilon \in \mathscr{D}(\mathbb{R}^n)$ and supp$(\phi * \eta_\epsilon) \subset$ (suppϕ)$+\bar{B}_\epsilon$ for all $\epsilon > 0$. For every multi-index α, we have $D^\alpha(\phi * \eta_\epsilon) = (D^\alpha\phi) * \eta_\epsilon$ which converges to $D^\alpha\phi$ uniformly in $\mathbb{R}^n$ as $\epsilon \to 0$ (see [Adm75, p.29]). Thus $\phi * \eta_\epsilon \to \phi$ in $\mathscr{D}(\mathbb{R}^n)$ as $\epsilon \to 0$.

Proposition 6.23. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\{\eta_\epsilon\}_{\epsilon>0}$ be the approximate identity on $\mathbb{R}^n$ as defined above. Then for every $\phi \in \mathscr{D}(\mathbb{R}^n)$,*

$$\langle u * \eta_\epsilon, \phi\rangle \to \langle u, \phi\rangle \text{ as } \epsilon \to 0.$$

Proof. First, we show that $\langle u * \eta_\epsilon, \phi\rangle = \langle u, \phi * \eta_\epsilon\rangle$. For every $m \in \mathbb{N}$, we partition the space $\mathbb{R}^n$ into cubes of side $\frac{1}{m}$. One way of partitioning yields cubes of the form

$$\left[\frac{i_1}{m}, \frac{i_1+1}{m}\right] \times \left[\frac{i_2}{m}, \frac{i_2+1}{m}\right] \times \ldots \times \left[\frac{i_n}{m}, \frac{i_n+1}{m}\right]$$

for $i_1, i_2, \ldots, i_n \in \mathbb{Z}$. We number those cubes in an arbitrary way as $Q_{1,m}, Q_{2,m}, Q_{3,m}, \ldots$ Then

$$\text{diam}Q_{i,m} = \sqrt{\left(\frac{1}{m}\right)^2 + \ldots + \left(\frac{1}{m}\right)^2} = \frac{\sqrt{n}}{m}, \qquad |Q_{i,m}| = \frac{1}{m^n}.$$

Let $x_{i,m}$ be the center of the cube $Q_{i,m}$. For each multi-index β, we put

$$M_{\epsilon,\beta} = \max\left\{|D_x D_y^\beta(\eta_\epsilon(y-x)\phi(x))| : \ x, y \in \mathbb{R}^n\right\}$$

where D_x denotes the gradient with respect to $x = (x_1, \ldots, x_n)$ and D_y^β denotes the β'th partial derivative with respect to $y = (y_1, \ldots, y_n)$. For each $m \in \mathbb{N}$, we put

$$f_m(y) = \sum_{i=1}^{\infty} \eta_\epsilon(y - x_{i,m})\phi(x_{i,m})|Q_{i,m}| \quad \forall y \in \mathbb{R}^n.$$

Note that only finitely many terms in this series can be nonzero because suppϕ is bounded. More specifically, if supp$\phi \subset [0, N]^n$ then there are at most $(Nm)^n$ nonzero terms. We have $f_m \in \mathscr{D}(\mathbb{R}^n)$ and supp$f_m \subset$ (suppη_ϵ) + (suppϕ) for all $m \in \mathbb{N}$. For each multi-index β,

$$D^\beta f_m(y) = \sum_{i=1}^{\infty} D^\beta\eta_\epsilon(y - x_{i,m})\phi(x_{i,m})|Q_{i,m}| \quad \forall y \in \mathbb{R}^n.$$

Thus,

$$D^\beta f_m(y) - ((D^\beta\eta_\epsilon) * \phi)(y) = \sum_{i=1}^{\infty} \left[D^\beta\eta_\epsilon(y - x_{i,m})\phi(x_{i,m})|Q_{i,m}| - \int\limits_{Q_{i,m}} D^\beta\eta_\epsilon(y - x)\phi(x)dx\right]. \tag{6.11}$$

By Mean Value Theorem, there exists $x^*_{i,m,y} \in Q_{i,m}$ such that

$$\int\limits_{Q_{i,m}} D^\beta \eta_\epsilon(y-x)\phi(x)dx = D^\beta \eta_\epsilon(y - x^*_{i,m,y})\phi(x^*_{i,m,y})|Q_{i,m}|.$$

Then (6.11) becomes

$$D^\beta f_m(y) - ((D^\beta \eta_\epsilon) * \phi)(y) =$$
$$= \sum_{\substack{i=1 \\ Q_{i,m}\subset[0,N]^n}}^{\infty} \left[\underbrace{D^\beta \eta_\epsilon(y - x_{i,m})\phi(x_{i,m}) - D^\beta \eta_\epsilon(y - x^*_{i,m,y})\phi(x^*_{i,m,y})}_{\{1\}} \right] |Q_{i,m}|. \tag{6.12}$$

We have

$$|\{1\}| \leq \left(\max_{x\in\mathbb{R}^n} |D^\beta \eta_\epsilon(y-x)\phi(x)| \right) |x_{i,m} - x^*_{i,m,y}| \leq M_{\epsilon,\beta}\mathrm{diam}(Q_{i,m}) = \frac{\sqrt{n}}{m} M_{\epsilon,\beta}.$$

Then from (6.12) we get

$$\begin{aligned} \left|D^\beta f_m(y) - ((D^\beta \eta_\epsilon) * \phi)(y)\right| &\leq \sum_{\substack{i=1 \\ Q_{i,m}\subset[0,N]^n}}^{\infty} \frac{\sqrt{n}}{m} M_{\epsilon,\beta}|Q_{i,m}| \\ &\leq (Nm)^n \frac{\sqrt{n}}{m} M_{\epsilon,\beta} \frac{1}{m^n} \\ &= \frac{N^n\sqrt{n}}{m} M_{\epsilon,\beta} \quad \forall y \in \mathbb{R}^n. \end{aligned}$$

Thus, $D^\beta f_m \to (D^\beta \eta_\epsilon) * \phi = D^\beta(\eta_\epsilon * \phi)$ uniformly in $\mathbb{R}^n$ as $m \to \infty$. This is true for every multi-index β. Hence, $f_m \to \eta_\epsilon * \phi$ in $C^\infty(\mathbb{R}^n)$ according to Proposition 6.8. Moreover, $f_m \in \mathscr{D}_K$ for all $m \in \mathbb{N}$ where $K = (\mathrm{supp}\eta_\epsilon) + (\mathrm{supp}\phi)$. By Proposition 6.13, $f_m \to \eta_\epsilon * \phi$ as $m \to \infty$.

Since $u \in \mathscr{D}'(\mathbb{R}^n)$, we have

$$\langle u, \eta_\epsilon * \phi \rangle = \lim_{m\to\infty} \langle u, f_m \rangle. \tag{6.13}$$

We have

$$\begin{aligned} f_m(y) &= \sum_{i=1}^{\infty} \eta_\epsilon(y - x_{i,m})\phi(x_{i,m})|Q_{i,m}| \\ &= \sum_{i=1}^{\infty} \tau_{x_{i,m}}(\check{\eta}_\epsilon(y))\phi(x_{i,m})|Q_{i,m}|. \end{aligned}$$

Thus,

$$\langle u, f_m \rangle = \sum_{i=1}^{\infty} \left\langle u, \tau_{x_{i,m}}(\hat{\eta}_\epsilon) \right\rangle \phi(x_{i,m})|Q_{i,m}|.$$

Hence,

$$\lim_{m\to\infty} \langle u, f_m\rangle = \int\limits_{\mathbb{R}^n} \langle u, \tau_x(\hat{\eta}_\epsilon)\rangle \, \phi(x)dx = \int\limits_{\mathbb{R}^n} (u * \eta_\epsilon)(x)\phi(x)dx = \langle u * \eta_\epsilon, \phi\rangle . \quad (6.14)$$

By (6.13) and (6.14), we get $\langle u * \eta_\epsilon, \phi\rangle = \langle u, \phi * \eta_\epsilon\rangle$. By the remark before Proposition 6.23, we have $\phi*\eta_\epsilon \to \phi$ in $\mathscr{D}(\mathbb{R}^n)$. Therefore, $\lim\limits_{\epsilon\to 0} \langle u * \eta_\epsilon, \phi\rangle = \lim\limits_{\epsilon\to 0} \langle u, \phi * \eta_\epsilon\rangle = \langle u, \phi\rangle$. ☐

Proposition 6.24. *Let $u \in \mathscr{E}'(\mathbb{R}^n)$ and $\{\eta_\epsilon\}_{\epsilon>0}$ be the approximate identity on $\mathbb{R}^n$ as defined on Page 73. Put $v_\epsilon = v * \eta_\epsilon$ for each $\epsilon > 0$. We have the following statements.*

(i) $\mathrm{supp} v_\epsilon \subset (\mathrm{supp} v) + \bar{B}_\epsilon$ *for all* $\epsilon > 0$.

(ii) For each $\delta > 0$, there exists $\lambda > 0$ such that $\mathrm{supp} v \subset (\mathrm{supp} v_\epsilon) + \bar{B}_\delta$ *for all* $0 < \epsilon < \lambda$.

Proof. (i) By Part (i) of Proposition 6.21, $\mathrm{supp} v_\epsilon \subset (\mathrm{supp} v) + (\mathrm{supp}\eta_\epsilon)$. Because $\mathrm{supp}\eta_\epsilon = \bar{B}_\epsilon$, we get $\mathrm{supp} v_\epsilon \subset (\mathrm{supp} v) + \bar{B}_\epsilon$.

(ii) Suppose otherwise. Then there exist $\delta > 0$ and a decreasing sequence (ϵ_m) which converges to 0 such that $\mathrm{supp} v \not\subset \mathrm{supp} v_{\epsilon_m} + \bar{B}_\delta$ for all $m \in \mathbb{N}$. Thus, there exists $x_m \in \mathrm{supp} v \backslash (\mathrm{supp} v_{\epsilon_m} + \bar{B}_\delta)$. Because $\mathrm{supp} v$ is compact, there exists a convergent subsequence (x_{m_k}). By replacing the sequence (x_n) by the subsequence (x_{m_k}), we can assume $x_m \to x_0 \in \mathrm{supp} v$. We have

$$\begin{aligned} \mathrm{dist}(x_0, \mathrm{supp} v_{\epsilon_m}) &\geq \mathrm{dist}(x_m, \mathrm{supp} v_{\epsilon_m}) - |x_m - x_0| \\ &> \delta - |x_m - x_0|. \end{aligned} \quad (6.15)$$

There exists $m_0 \in \mathbb{N}$ such that $|x_m - x_0| < \delta/2$ for all $m > m_0$. Then (6.15) implies

$$\mathrm{dist}(x_0, \mathrm{supp} v_{\epsilon_m}) > \delta - \frac{\delta}{2} = \frac{\delta}{2} \quad \forall m > m_0.$$

For every $y \in B_{\delta/4}(x_0)$ and $m > m_0$, we have

$$\mathrm{dist}(y, \mathrm{supp} v_{\epsilon_m}) > \mathrm{dist}(x_0, \mathrm{supp} v_{\epsilon_m}) - |x_0 - y| > \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4}.$$

Thus, $y \in \mathbb{R}^n \backslash \mathrm{supp} v_{\epsilon_m}$. Thus, $v_{\epsilon_m}(y) = 0$. Hence, $v_{\epsilon_m} = 0$ in $B_{\delta/4}(x_0)$. For each $\phi \in \mathscr{D}(B_{\delta/4}(x_0))$, we have

$$\langle v_{\epsilon_m}, \phi\rangle = \int\limits_{B_{\delta/4}} v_{\epsilon_m}\phi dx = 0 \quad \forall m > m_0. \quad (6.16)$$

By Proposition 6.23,

$$\langle v, \phi\rangle = \lim_{m\to\infty} \langle v_{\epsilon_m}, \phi\rangle . \quad (6.17)$$

From (6.16) and (6.17), we conclude that $\langle v, \phi\rangle = 0$ for all $\phi \in \mathscr{D}(B_{\delta/4}(x_0))$. Thus, v vanishes in $B_{\delta/4}(x_0)$. This is a contradiction because $x_0 \in \mathrm{supp} v$. ☐

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