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Math 8583: Theory of PDE

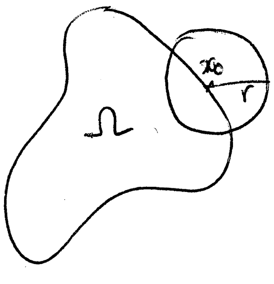
Homework #2



(1) Let $\Omega$ be an open connected subset of $\mathbb{R}^n$ such that $\partial\Omega$ is of class $C^2$. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a harmonic function such that

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } (\partial\Omega) \cap B_r(x_0),$$

for some $x_0 \in \partial\Omega$ and $r > 0$. We'll show that $u \equiv 0$ in $\Omega$.




Define $v : \Omega \cup B_r(x_0) \to \mathbb{R}$, $v(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in B_r(x_0) \setminus \Omega. \end{cases}$

Because $u = 0$ on $(\partial\Omega) \cap B_r(x_0)$, the function $v$ defined above is continuous. Moreover, because $u$ is bounded on $\bar{\Omega} \cap \bar{B}_r$, it is bounded on $\bar{\Omega} \cap \bar{B}_r$. Thus, $v|_{B_r(x_0)}$ is bounded. In particular, $v \in L^1_{loc}(B_r(x_0))$.

Take $\phi \in C_0^\infty(B_r(x_0))$ arbitrarily. We'll show that $\int_{B_r(x_0)} v \Delta\phi \, dx = 0$.

Because $v = 0$ in $B_r(x_0) \setminus \Omega$, we have

$$\int_{B_r(x_0)} v \Delta\phi \, dx = \int_{B_r(x_0) \cap \Omega} v \Delta\phi \, dx = \int_{B_r(x_0) \cap \Omega} u \Delta\phi \, dx$$

$$= \int_{\Omega} u \Delta\phi \, dx \quad (\text{because } \phi \equiv 0 \text{ in } \Omega \setminus B_r(x_0))$$

Thus, $\int_{B_r(x_0)} v \Delta\phi \, dx = \int_{\Omega} u \Delta\phi \, dx \quad (1)$

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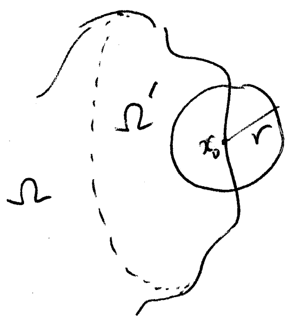
Let $\Omega'$ be an open bounded subset of $\Omega$ such that $\partial\Omega'$ is of class $C^1$ and $\Omega \cap B_r(x_0) \subset \Omega'$. Because $\phi \equiv 0$ in $\Omega \setminus \Omega'$, we have

$$\int_{\Omega} u \Delta \phi \, dx = \int_{\Omega'} u \Delta \phi \, dx \qquad (2),$$




We have $u|_{\Omega'}, \phi|_{\Omega'} \in C^2(\Omega') \cap C^1(\overline{\Omega'})$. Moreover, $u\Delta\phi - \phi\Delta u$ is continuous in $\Omega'$ with compact support.

Thus, $u\Delta\phi - \phi\Delta u \in L^1(\Omega')$. In addition, $\Omega'$ is bounded and is of class $C^1$. Hence, we can apply Green's formula (Lemma 1.16, page 9, Saponov's lecture notes):

$$\int_{\Omega'} u\Delta\phi \, dx - \underbrace{\int_{\Omega'} \phi \Delta u \, dx}_{\{1\}} = \underbrace{\int_{\partial\Omega'} u \frac{\partial \phi}{\partial \nu} dS}_{\{2\}} - \underbrace{\int_{\partial\Omega'} \phi \frac{\partial u}{\partial \nu} dS}_{\{3\}} \qquad (3)$$

We have $\{1\} = 0$ because $u$ is harmonic,

$$\{2\} = \int_{(\partial\Omega') \cap B_r(x_0)} u \frac{\partial\phi}{\partial\nu} dS \quad (\text{because } \phi \equiv 0 \text{ in } B_r\cancel{(x_0) \cap \partial\Omega'}\ (\partial\Omega')\setminus B_r(x_0))$$

$$= 0 \quad (\text{because } u = 0 \text{ on } (\partial\Omega') \cap B_r(x_0)).$$

$$\{3\} = \int_{(\partial\Omega') \cap B_r(x_0)} \phi \frac{\partial u}{\partial \nu} dS \quad (\text{because } \phi \equiv 0 \text{ in } B_r\cancel{(x_0) \cap \partial\Omega'}\ (\partial\Omega')\setminus B_r(x_0))$$

$$= 0 \quad (\text{because } \frac{\partial u}{\partial \nu} = 0 \text{ on } (\partial\Omega') \cap B_r(x_0))$$

Thus, (3) is simply $\int_{\Omega'} u\Delta\phi \, dx = 0$. Then by (2), $\int_{\Omega} u \Delta\phi \, dx = 0$.

Then by (1), $\int_{B_r(x_0)} v \Delta \phi \, dx = 0$. Then by Weyl's lemma (Lemma 1.19, page 11, Safonov's lecture notes), $v \in C^\infty(B_r(x_0))$ and $v$ is harmonic.

By the definition of $v$, we know that $v = 0$ in $B_r(x_0) \setminus \bar{\Omega}$, which is a nonempty open subset since $x_0 \in \partial\Omega$. In addition, $B_r(x_0)$ is connected. Then by the uniqueness of continuation of harmonic functions (Corollary 1.14, page 9, Safonov's lecture notes), we get $v \equiv 0$ in $B_r(x_0)$. 10/10

Consequently, $u = 0$ in $\Omega \cap B_r(x_0)$, which is a nonempty open subset of $\cancel{B_r(x_0)}$ $\Omega$. In addition, $\Omega$ is connected. By the uniqueness of continuation of harmonic functions, we get $u \equiv 0$ in $\Omega$.

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(2) Let $u$ be a harmonic function in the unit ball $B_1(x_0)$ in $\mathbb{R}^n$.

We'll show that $|\nabla u(x_0)| \leq n \left( \sup_{B_1(x_0)} u - u(x_0) \right)$.

First, by replacing $u$ by the function $v(x) = u(x + x_0) - u(x_0)$ for $x \in B_1(0)$, we can assume $x_0 = 0$ and $u(x_0) = 0$. Now we have that $u$ is harmonic in $B_1(0) \subset \mathbb{R}^n$ and $u(0) = 0$. We'll show that

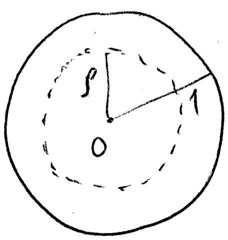
$$|\nabla u(0)| \leq n \sup_{B_1(0)} u \qquad (*)$$

If $\sup_{B_1(0)} u = \infty$ then $(*)$ is automatically true. Consider the case

$$\sup_{B_1(0)} u = A < \infty$$

Note that $A \geq u(0) = 0$.

Because $u$ is harmonic in $B_1(0)$, by the smoothness theorem (Theorem 1.10, page 6, Safonov's lecture notes), $u \in C^\infty(B_1(0))$. Thus, every partial derivative of $u$ also belongs to $C^\infty(B_1(0))$. For each index $1 \le i \le n$, we have $\Delta(D_i u) = D_i(\Delta u) = D_i(0) = 0$ in $B_1(0)$. Thus, $D_i u$ is harmonic. For each $0 < \rho < 1$, $\overline{B(0,\rho)} \subset B_1(0)$.




By the Mean Value Theorem (Theorem 1.1, page 1, Safonov's lecture notes), we have

$$D_i u(0) = \frac{1}{|B_\rho(0)|} \int_{B_\rho(0)} D_i u(x)\, dx.$$

Put $\vec{v} = (0, \dots, \underset{\substack{\uparrow \\ i}}{u}, \dots 0)$ we have $\nabla \cdot \vec{v} = D_i u$. Then

$$D_i u(0) = \frac{1}{|B_\rho(0)|} \int_{B_\rho(0)} \nabla \cdot \vec{v}\, dx$$

$$= \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} \vec{v} \cdot \vec{\nu}\, dS \quad \text{(by Divergence theorem)}$$

$$= \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} u \nu_i\, dS$$

where $\nu_i$ is the i'th component of the unit normal vector $\vec{\nu}$ on $\partial B_\rho(0)$.

Thus, $$D_i u(0) = \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} (u - A)\nu_i\, dS + \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} A \nu_i\, dS \qquad (1)$$

(Recall that $A = \sup\limits_{B_1(0)} u < \infty$)

Put $\vec{w} = (0, \dots, \underset{\substack{\uparrow \\ i}}{A}, \dots 0)$. Then

$$\int_{\partial B_\rho(0)} A \nu_i \, dS = \int_{\partial B_\rho(0)} \vec{w} \cdot \vec{\nu} \, dS = \int_{B_\rho(0)} \mathrm{div}(\vec{w}) \, dx = 0 \quad (\text{since } \vec{w} \text{ is a constant field})$$

Then (1) becomes $$D_i u(0) = \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} (u - A) \nu_i \, dS \qquad (2)$$

With $\nabla u(0) = (D_1 u(0), \dots, D_n u(0))$ and $\vec{\nu} = (\nu_1, \dots, \nu_n)$, we can rewrite (2) in form of vectors.

$$\nabla u(0) = \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} (u - A) \vec{\nu} \, dS$$

Thus, $$|\nabla u(0)| = \frac{1}{|B_\rho(0)|} \left| \int_{\partial B_\rho(0)} (u - A) \vec{\nu} \, dS \right|$$

Now we'll use the inequality $\left| \int_{\partial B_\rho(0)} \vec{f} \, dS \right| \leq \int_{\partial B_\rho(0)} |\vec{f}| \, dS$ for $\vec{f} = (u - A)\vec{\nu}$.

We then get

$$|\nabla u(0)| \leq \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} (A - u) \underbrace{|\vec{\nu}|}_{1} \, dS$$

$$= \frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} (A - u) \, dS$$

$$= \underbrace{\frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} A \, dS}_{\{1\}} - \underbrace{\frac{1}{|B_\rho(0)|} \int_{\partial B_\rho(0)} u \, dS}_{\{2\}} \qquad (3)$$

Because $u$ is harmonic, $\int_{\partial B_\rho(0)} u \, dS = |\partial B_\rho(0)| \, u(0) = 0$. Thus $\{2\} = 0$.

$$\{1\} = A\frac{|\partial B_\rho(0)|}{|B_\rho(0)|} = \frac{A\,|\partial B_1(0)|\,\rho^{n-1}}{|B_1(0)|\,\rho^n} = \frac{A}{\rho}\,\frac{|\partial B_1(0)|}{|B_1(0)|}.$$

We have $|B_1(0)| = \displaystyle\int_{B_1(0)} dx = \int_0^1\left(\int_{\partial B_\rho(0)} dS\right)d\rho = \int_0^1 |\partial B_\rho(0)|\,d\rho$

$$= \int_0^1 |\partial B_1(0)|\,\rho^{n-1}\,d\rho = |\partial B_1(0)|\,\frac{\rho^n}{n}\Big|_0^1 = \frac{|\partial B_1(0)|}{n}.$$

Thus, $\dfrac{|\partial B_1(0)|}{|B_1(0)|} = n$ and $\{1\} = \dfrac{An}{\rho}$.

Therefore, (5) gives us $|\nabla u(0)|_1 \leq \{1\}_1 + \{2\}_1 = \dfrac{An}{\rho} = \dfrac{An}{\gamma}$.

This inequality is supposed to be true for all $0<\rho<1$. Letting $\rho \to 1^-$,

we get $|\nabla u(0)| \leq An = n \sup\limits_{B_1(0)} u$.

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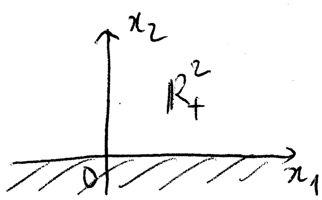
(3) Denote $\mathbb{R}^2_+ = \{(x_1,x_2)\in\mathbb{R}^2 \mid x_2>0\}$.




We are supposed to show that there is no function $u \in C^2(\mathbb{R}^2_+)\cap C(\overline{\mathbb{R}^2_+})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+ \\ u(x_1,0) = x_1^2, & \forall x_1 \in \mathbb{R} \\ u \geq 0 & \text{in } \mathbb{R}^2_+ \end{cases}$$

So far I have not been able to prove it. The hint given by Professor Safonov suggests to use the Comparison Principle. However, this principle is not applicable in our case because the upper-half plane is unbounded.

Nevertheless, I will show that there is no function $u \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \mathbb{R}^2_+ \\ u(x_1, 0) = x_1^2, \quad \forall x_1 \in \mathbb{R} \\ u \geq 0 \text{ in } \mathbb{R}^2_+, \\ u \text{ grows with polynomial order at infinity} \quad (*) \end{cases}$$

The condition $(*)$ specifically means, there are an integer $n \geq 1$ and constants $M, R_0 > 0$ such that $|u(z)| \leq M|z|^n \; \forall z \in H, |z| > R_0$.

Here $H = \{z = x_1 + ix_2 \mid x_2 > 0\}$ is the upper-half plane, and $u = u(z)$ was viewed as a function of complex variable $z = x_1 + ix_2$.

Suppose by contradiction that there is such a function $u$. We can of course assume that $n \geq 3$. Put $v(x_1, x_2) = u(x_1, x_2) - (x_1^2 - x_2^2)$.

The function $(x_1, x_2) \mapsto x_1^2 - x_2^2$ is harmonic. Thus, $v$ is also harmonic.

We have $v(x_1, 0) = u(x_1, 0) - (x_1^2 - 0^2) = x_1^2 - x_1^2 = 0$ for all $x_1 \in \mathbb{R}$.

Moreover,
$$\begin{aligned} |v(x_1, x_2)| &\leq |u(x_1, x_2)| + |x_1^2 - x_2^2| \\ &\leq |u(x_1, x_2)| + x_1^2 + x_2^2 \\ &\leq M|z|^n + |z|^2 \quad \forall z \in H, |z| > R_0. \end{aligned}$$

Because $n \geq 3$, there is a constant $\tilde{M} > M > 0$ such that $|v(z)| \leq \tilde{M}|z|^n$ for all $z \in H$, $|z| > R_0$. We summarize the propedies of $v$ as follows.

$$\begin{cases} \Delta v = 0 \text{ in } H \\ v = 0 \text{ on } \partial H \\ |v(z)| \leq \tilde{M}|z|^n \quad \forall z \in H, |z| > R_0 \quad (**) \end{cases}$$

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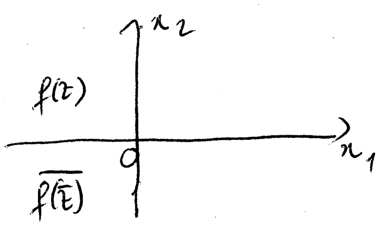
Because is harmonic in $H$, there is a complex analytic function $f: H \to \mathbb{C}$ such that $v = \operatorname{Im}(f)$. Since $v(0) = 0$, we can add a real constant to $f$ so that we can assume $f(0) = 0$. Because the imaginary part of $f$ extends to a continuous function on $\overline{H}$ which vanishes on $\partial H$, we can apply Schwarz's Reflection principle. Accordingly, $f$ has an analytic extension $g: \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} f(z) & \text{if } z \in \overline{H} \\ \overline{f(\overline{z})} & \text{if } z \in \mathbb{C} \setminus \overline{H} \end{cases}$$




We have
$$\operatorname{Im} g(z) = \begin{cases} v(z) & \text{if } z \in \overline{H}, \\ -v(\overline{z}) & \text{if } z \in \mathbb{C} \setminus \overline{H}. \end{cases}$$

Put $h(z) = i g(z)$ for all $z \in \mathbb{C}$. Then $h$ is also an entire function with
$$\operatorname{Re}(h(z)) = -\operatorname{Im}(g(z)) = \begin{cases} -v(z) & \text{if } z \in \overline{H}, \\ v(\overline{z}) & \text{if } z \in \mathbb{C} \setminus \overline{H}. \end{cases}$$

Moreover, $h(0) = i g(0) = i f(0) = 0$. For any $R > 0$ and $z \in B_R(0)$, we have the Poisson's integral formula
$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(h(Re^{i\theta})\right) \frac{Re^{i\theta} + z}{Re^{i\theta} - z} \, d\theta \qquad (1)$$

By the condition $(**)$, we have
$$|\operatorname{Re}(h(Re^{i\theta}))| \leq \max\{|v(Re^{i\theta})|, |v(Re^{-i\theta})|\} \leq \widetilde{M} R^n \quad \forall R > R_0,\ \theta \in [0, 2\pi].$$

Thus, for $R > R_0$ and $z \in B_R(0)$, from (1) we have

$$|h(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |Re(h(Re^{i\theta}))| \left| \frac{Re^{i\theta}+z}{Re^{i\theta}-z} \right| d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \tilde{M} R^n \frac{R+|z|}{R-|z|} d\theta = \tilde{M} R^n \frac{R+|z|}{R-|z|}.$$

For $z \in \mathbb{C}$, $|z| > R_0$, we pick $R = 2|z|$. Then

$$|h(z)| \leq \tilde{M} R^n \frac{2|z|+|z|}{2|z|-|z|} = 3\tilde{M}(2|z|)^n = \underbrace{3 \cdot 2^n \cdot \tilde{M}}_{M_1} |z|^n.$$

Thus, $|h(z)| \leq M_1 |z|^n$ for all $z \in \mathbb{C}$, $|z| > R_0$.

Because $h$ is an entire function, it admits a power series expansion

$$h(z) = a_0 + \sum_{k=1}^{\infty} a_k z^k \qquad \forall z \in \mathbb{C},$$

where $$a_k = \frac{1}{2\pi i} \int_{|\xi|=r} \frac{h(\xi)}{\xi^{k+1}} d\xi \cdot \quad (\text{for any } r>0).$$

Consider $r > R_0$. Then on the circle $\partial B_r(0)$ we have

$$\left| \frac{h(\xi)}{\xi^{k+1}} \right| \leq \frac{M_1 |\xi|^n}{|\xi|^{k+1}} = M_1 r^{n-k-1}.$$

Then by M-L inequality, we have $|a_k| \leq \frac{1}{2\pi}(2\pi r) M_1 r^{n-k-1} = M_1 r^{n-k}$.

If $k > n$ then $\lim_{r\to\infty} r^{n-k} = 0$. Thus $a_k = 0$ for all $k > n$. Thus, $h$ is a polynomial of degree $\leq n$. We have

$$u(z) = v(z) + (x_1^2 - x_2^2) = Re(h(z)) + Re(z^2) = Re(h(z) + z^2) \quad \forall z \in H.$$

Put $h_1(z) = h(z) + z^2$, which is also a polynomial of degree $\leq n$.

Write $h_1(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0$.

Write $b_k = A_k e^{i\beta_k}$ with $A_k \geq 0$, $0 \leq \beta_k < 2\pi$. For each $z \in H$, we write $z = re^{i\theta}$ where $r > 0$ and $0 < \theta < \pi$. Then

$$h_1(z) = A_n e^{i\beta_n} r^n e^{in\theta} + \cdots + A_1 e^{i\beta_1} re^{i\theta} + A_0 e^{i\beta_0}$$

$$= \sum_{k=0}^{n} A_k e^{i\beta_k} r^k e^{ik\theta} = \sum_{k=0}^{n} A_k r^k e^{i(k\theta + \beta_k)}.$$

Thus, $u(z) = Re(h_1(z)) = \sum_{k=0}^{n} A_k r^k \cos(k\theta + \beta_k)$.

~~Let~~ Since $u(x_1, 0) = x_1^2$, $h_1$ is ~~con~~ not a constant polynomial. Let $m \geq 1$ be the degree of $h_1$. Then $A_m > 0$ and

$$u(z) = A_m r^m \cos(m\theta + \beta_m) + \sum_{k=0}^{m-1} A_k r^k \cos(k\theta + \beta_k).$$

Because $u \geq 0$ in $H$, in polar coordinates we have $u(r, \theta) \geq 0$ $\forall r > 0$, $\forall 0 < \theta < \pi$. ~~If $m \geq$~~ The range of $(m\theta + \beta_m)$ as $\theta$ varies from $0$ to $\pi$ is $(\beta_m, m\pi + \beta_m)$, which is an interval of length $m\pi$. If $m \geq 2$ then the range of $(m\theta + \beta_m)$ is $\geq 2\pi$. Thus we can pick $\theta_0 \in (0, \pi)$ such that $\cos(m\theta_0 + \beta_m) < 0$. Then

$$u(r, \theta_0) = A_m r^m \cos(m\theta_0 + \beta_m) + \sum_{k=0}^{m-1} A_k r^k \cos(k\theta_0 + \beta_k)$$

is a polynomial in $r$ of degree $m$ and leading coefficient $A_m \cos(m\theta_0 + \beta_m) < 0$.

Thus, $u(r, \theta_0) < 0$ if $r$ is sufficiently large. This is a contradiction.

Therefore, $m < 2$. Thus $m = 1$. Thus, $h_1(z) = \alpha z + \beta$. Since $u(z) = Re(h_1(z))$, we have $u(x_1, x_2) = a x_1 + b x_2 + c$ for some real constants $a, b, c$. Then $x_1^2 = u(x_1, 0) = a x_1 + c$ $\forall x_1 \in \mathbb{R}$. This is a contradiction. Therefore, there is no function $u \in C^2(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ satisfying

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+ \\ u(x_1, 0) = x_1^2 & \forall x_1 \in \mathbb{R}, \\ u \geq 0 & \text{in } \mathbb{R}^2_+ \\ u \text{ grows with polynomial order at infinity.} \end{cases}$$

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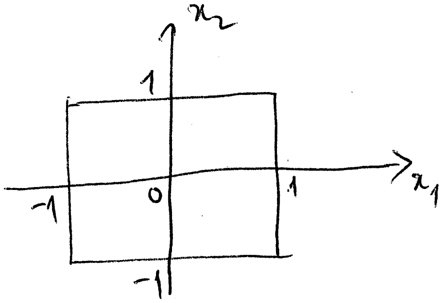
(4) Denote $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$. Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution to the problem




$$\begin{cases} \Delta u = -1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (*)$$

(i) Suppose by contradiction that $u \in C^2(\overline{\Omega})$. Then $\Delta u$ is continuous in $\overline{\Omega}$. Thus $\Delta u(1,1) = -1$. (1)

Because $u(t, 1) = 0$ for all $-1 < t < 1$, we have

$$D_1 u(t, 1) = \frac{d}{dt}\left(u(t,1)\right) = 0 \quad \forall -1 < t < 1.$$

Then

$$D_1^2 u(t,1) = \frac{d}{dt}\left(D_1 u(t,1)\right) = 0 \quad \forall -1 < t < 1.$$

Since $D_1^2 u$ is continuous in $\overline{\Omega}$, we get $D_1^2 u(1,1) = 0$.

Similarly, because $u(1, t) = 0$ for all $-1 < t < 1$, we have

$$D_2 u(1,t) = \frac{d}{dt}\left(u(1,t)\right) = 0 \qquad \forall -1<t<1.$$

Thus, $$D_2^2 u(1,t) = \frac{d}{dt}\left(D_2 u(1,t)\right) = 0 \qquad \forall -1<t<1.$$

Since $D_2^2 u$ is continuous in $\bar{\Omega}$, we get $D_2^2 u(1,1) = 0$. Therefore,

$$\Delta u(1,1) = D_1^2 u(1,1) + D_2^2 u(1,1) = 0.$$

This contradicts with (1).

(ii) By the uniqueness theorem of Poisson equation (Theorem 1.6, page 4, Safonov's lecture notes), Problem (*) has a unique solution in $C^2(\Omega)\cap C(\bar{\Omega})$.

Put $u_1(x_1,x_2) = u(-x_1,x_2)$, $u_2(x_1,x_2) = u(x_1,-x_2)$, $u_3(x_1,x_2) = u(-x_1,-x_2)$ for all $(x_1,x_2)\in\bar{\Omega}$. Then

$$\Delta u_1(x_1,x_2) = \Delta u(-x_1,x_2) = -1,$$

$$\Delta u_2(x_1,x_2) = \Delta u(x_1,-x_2) = -1,$$

$$\Delta u_3(x_1,x_2) = \Delta u(-x_1,-x_2) = -1.$$

Moreover, $u_1|_{\partial\Omega} = u_2|_{\partial\Omega} = u_3|_{\partial\Omega} = 0$. Hence, $u_1$, $u_2$ and $u_3$ are also solutions in $C^2(\Omega)\cap C(\bar{\Omega})$ of Problem (*). By the uniqueness of solutions, we have $u = u_1 = u_2 = u_3$ in $\Omega$. Thus, $u$ depends only on $|x_1|$ and $|x_2|$.

Put $v(x_1,x_2) = u(x_1,x_2) + \frac{x_1^2}{2}$. Then $\Delta v = \Delta u + 1 = 0$ in $\Omega$.

Then $v\in C^\infty(\Omega)$ by the smoothness theorem (Theorem 1.10, page 6, Safonov's lecture notes). Thus, $u\in C^\infty(\Omega)$. We have

$$\Delta(D_1 u) = D_1(\Delta u) = D_1(-1) = 0,$$
$$\Delta(D_2 u) = D_2(\Delta u) = D_2(-1) = 0.$$

Thus $D_1 u$ and $D_2 u$ are harmonic in $\Omega$. Moreover, they are continuous in $\bar{\Omega}$ because $u \in C^1(\bar{\Omega})$.

Because $u$ is continuous in $\bar{\Omega}$, it must attain maximum and minimum in $\bar{\Omega}$. Since $\Delta u < 0$, $u$ attains minimum on $\partial\Omega$ by the Weak Maximum Principle. Thus, $u \geq 0$ in $\Omega$.

We see that $u$ is not constant for $\Delta u = -1$. Thus $u$ attains maximum inside $\Omega$. Now suppose by contradiction that $u(a) = \max_{\Omega} u$ where $a = (a_1, a_2) \neq (0,0)$. We have $\nabla u(a) = 0$. Thus,

$$D_1 u(a) = D_2 u(a) = 0 \qquad (2)$$

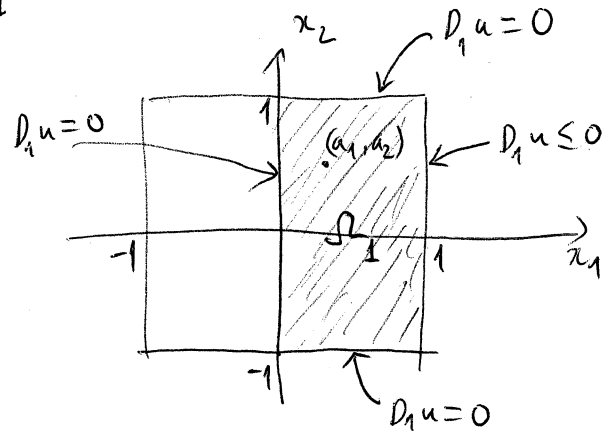
We prove earlier that $u(a)$ depends only on $|a_1|, |a_2|$. Thus we can assume $a_1, a_2 \geq 0$. Since $a \neq 0$, $a_1 > 0$ or $a_2 > 0$. We consider two following cases.

- $\underline{a_1 > 0}$ Denote $\Omega_1 = (0,1)\times(-1,1)$. Then $a \in \Omega_1$.

Because $u(t,1) = u(t,-1) = 0$ for all $t \in (0,1)$, we have

$$D_1 u(t,1) = D_1 u(t,-1) = 0 \qquad \forall t \in (0,1).$$

In other words, $D_1 u = 0$ on the upper and lower edges of $\Omega_1$ in the figure (next page). On the other hand, $u(x_1, x_2) = u(-x_1, x_2)$.



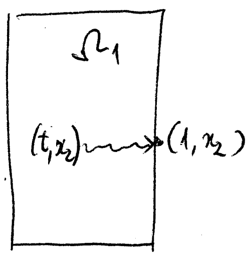

Taking derivative both sides with respect to $x_1$, we get

$$D_1 u(x_1, x_2) = - D_1 u(-x_1, x_2).$$

Thus $D_1 u(0, x_2) = - D_1 u(0, x_2)$, which is simply $D_1 u(0, x_2) = 0$ for all $x_2 \in (-1, 1)$.

In other words, $D_1 u = 0$ on the left edge of $\Omega_1$.

For each $x_2 \in (-1, 1)$ we have




$$D_1 u(1, x_2) = \lim_{t \to 1^-} \frac{u(t, x_2) - u(1, x_2)}{t - 1}$$

$$= \lim_{t \to 1^-} \frac{u(t, x_2)}{t - 1} \leq 0 \quad (\text{since } u \geq 0)$$

Thus, $D_1 u \leq 0$ on the right edge of $\Omega_1$. Therefore, $D_1 u \leq 0$ on the boundary of $\Omega_1$. By the weak Maximum principle,

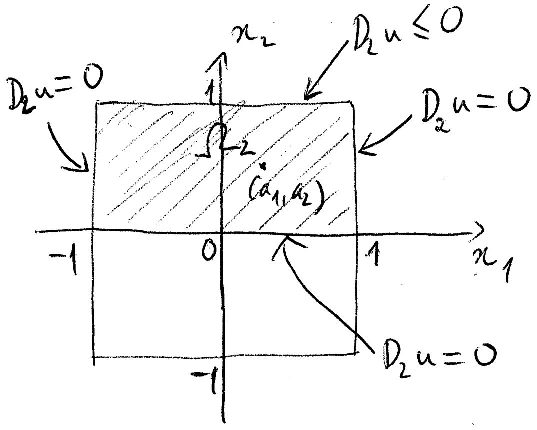
$$\sup_{\Omega_1} D_1 u \leq \sup_{\partial \Omega_1} D_1 u \leq 0.$$

On the other hand, $a \in \Omega_1$ and $D_1 u(a) = 0$ by Eq. (2). Thus, $D_1 u$ attains maximum at an interior point of $\Omega_1$. By the Strong Maximum principle, $D_1 u \equiv \text{const}$ in $\Omega_1$. Thus, $D_1 u \equiv 0$ in $\Omega_1$. For each $(x_1, x_2) \in \Omega_1$,

$$u(x_1, x_2) = \underbrace{u(1, x_2)}_{0} + \int_1^{x_1} \underbrace{D_1 u(t, x_2)}_{0} \, dt = 0$$

Thus, $u = 0$ in $\Omega_1$. This contradicts the fact that $\Delta u = -1$.

- $\underline{a_2 > 0}$ We will follow the same approach as in the previous case.




Denote $\Omega_2=(-1,1)\times(0,1)$. Then $a\in\Omega_2$.

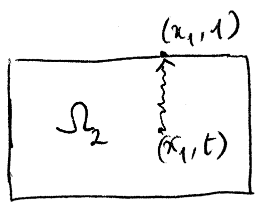
Because $u(-1,t)=u(1,t)=0$ for all $t\in(0,1)$, we have $D_2u(-1,t)=D_2u(1,t)=0, \forall t\in(0,1)$. In other words, $D_2u=0$ on the left and right edges of $\Omega_2$ in the figure. On the other hand,

$$u(x_1,x_2)=u(x_1,-x_2).$$

Taking derivative both sides with respect to $x_2$, we get

$$D_2u(x_1,x_2)=-D_2u(x_1,-x_2)$$

Thus, $D_2u(x_1,0)=0$. In other words, $D_2u=0$ on the lower edge of $\Omega_2$. For each $x_1\in(-1,1)$ we have




$$D_2u(x_1,1)=\lim_{t\to1^-}\frac{u(x_1,t)-u(x_1,1)}{t-1}$$

$$=\lim_{t\to1^-}\frac{u(x_1,t)}{t-1}\le 0 \quad (\text{since } u\ge 0).$$

Thus, $D_2u\le 0$ on the upper edge of $\Omega_2$. Therefore, $D_2u\le0$ on the boundary of $\Omega_2$. By the Weak Maximum Principle,

$$\sup_{\Omega_2} D_2u \le \sup_{\partial\Omega_2} D_2u \le 0.$$

On the other hand, $a\in\Omega_2$ and $D_2u(a)=0$ by Eq. (2). Thus, $D_2u$ attains maximum at an interior point of $\Omega_2$. By the Strong Maximum principle, $D_2u\equiv$ const in $\Omega_2$. Thus, $D_2u\equiv0$ in $\Omega$.

For each $(x_1,x_2)\in\Omega_2$, we have 10/10

$$u(x_1,x_2) = \underbrace{u(x_1,1)}_{0} + \int_1^{x_2} \underbrace{D_2 u(x_1,t)}_{0}\, dt = 0$$

Thus, $u=0$ in $\Omega_2$. This contradicts the fact that $\Delta u = -1$.

(5) Consider $n\in\mathbb{N}$, $n\geq 2$ and $p\in[1,n)$. Put

$$\alpha = \frac{p-\beta}{p}, \text{ where } \beta = \frac{1}{2}\min\{n-p, p\}$$

$$C = \frac{1-\alpha}{n+\alpha-2}, \qquad \nu = \min\left\{1, C, \frac{1}{C+1}\right\}$$

For each $1\leq i,j\leq n$, we define a function $a_{ij}:\mathbb{R}^n\to\mathbb{R}$ as follows.

$$a_{ij}(x) = \begin{cases} C\delta_{ij} + \dfrac{x_i x_j}{|x|^2} & \text{if } x\neq 0 \\ \delta_{ij} & \text{if } x = 0 \end{cases}$$

where $\delta_{ij}$ is the ~~Kroneker~~ ~~symb~~ Kroneker symbol.

First, we'll show that $\alpha\in(0,1)$, $C>0$ and $\nu\in(0,1]$. By definition, $\beta>0$ and $\beta\leq\frac{1}{2}p<p$. Thus, $\frac{p-p}{p}<\frac{p-\beta}{p}<\frac{p-0}{p}$. Hence, $0<\alpha<1$.

Since $n\geq 2$ and $\alpha>0$, $n+\alpha-2>0$. Thus $C>0$. By the definition of $\nu$, we have $0<\nu\leq 1$.

Next, we'll show that $\nu|\xi|^2\leq\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\leq\nu^{-1}|\xi|^2$ for all $\xi, x\in\mathbb{R}^n$. For $x=0$, we have $\sum_{i,j=1}^n a_{ij}(0)\xi_i\xi_j = \sum_{i,j=1}^n \delta_{ij}\xi_i\xi_j = |\xi|^2$.

Since $0<\nu\leq 1$, we have $\nu|\xi|^2 \leq |\xi|^2 = \sum a_{ij}(0)\xi_i\xi_j \leq \nu^{-1}|\xi|^2$.

Consider the case $x\neq 0$. We have

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j = \sum_{i,j=1}^{n}\left(C\delta_{ij} + \frac{x_i x_j}{|x|^2}\right)\xi_i\xi_j$$

$$= C\sum_{i,j=1}^{n}\delta_{ij}\,\xi_i\xi_j + \sum_{i,j=1}^{n}\frac{(x_i\xi_i)(x_j\xi_j)}{|x|^2}$$

$$= C|\xi|^2 + |x|^{-2}\left(\sum_{i=1}^{n} x_i\xi_i\right)^2 \qquad (1)$$

Then, $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C|\xi|^2 \geq \nu|\xi|^2$. Moreover,

$$\left(\sum_{i=1}^{n} x_i\xi_i\right)^2 \overset{\text{Schwarz}}{\leq} \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n}\xi_i^2\right) = |x|^2|\xi|^2$$

Then (1) gives us the estimate $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq C|\xi|^2 + |x|^{-2}\left(|x|^2|\xi|^2\right)$

$$= (C+1)|\xi|^2$$

$$\leq \nu^{-1}|\xi|^2.$$

Next, consider the function $u(x) = 1-|x|^\alpha$ for $x\in\mathbb{R}^n$. We see that $u$ is smooth on $\mathbb{R}^n\setminus\{0\}$. For $x\neq 0$, we'll show that

$$D_{ij}u(x) = -\alpha\delta_{ij}|x|^{\alpha-2} + \alpha(2-\alpha)x_i x_j|x|^{\alpha-4} \qquad (2)$$

With $|x| = (x_1^2+x_2^2+\dots+x_n^2)^{1/2}$, we have $D_i(|x|) = \frac{x_i}{|x|}$ for $1\leq i\leq n$.

Then $D_i u = -D_i(|x|^\alpha) = -\alpha|x|^{\alpha-1}D_i(|x|)$

$$= -\alpha|x|^{\alpha-1}\frac{x_i}{|x|} = -\alpha x_i|x|^{\alpha-2}$$

Thus, $D_i u(x) = -\alpha x_i|x|^{\alpha-2}\quad \forall x\neq 0.$

If $j \neq i$ then
$$
\begin{aligned}
D_{ij}u = D_j(D_i u) &= -\alpha x_i D_j(|x|^{\alpha-2}) \\
&= -\alpha(\alpha-2) x_i |x|^{\alpha-3} D_j(|x|) \\
&= \alpha(2-\alpha) x_i |x|^{\alpha-3} \frac{x_j}{|x|} \\
&= \alpha(2-\alpha) x_i x_j |x|^{\alpha-4} \qquad (3)
\end{aligned}
$$

If $j = i$ then
$$
\begin{aligned}
D_{ii}u = D_i(D_i u) &= D_i(-\alpha x_i |x|^{\alpha-2}) \\
&= -\alpha\left(|x|^{\alpha-2} + x_i D_i(|x|^{\alpha-2})\right) \\
&= -\alpha\left(|x|^{\alpha-2} + (\alpha-2) x_i |x|^{\alpha-3} D_i(|x|)\right) \\
&= -\alpha\left(|x|^{\alpha-2} + (\alpha-2)|x|^{\alpha-3} x_i \frac{x_i}{|x|}\right) \\
&= -\alpha|x|^{\alpha-2} + \alpha(2-\alpha) x_i^2 |x|^{\alpha-4} \qquad (4),
\end{aligned}
$$

Combining the formulae (3) and (4), we get Eq. (2).

Next, we'll show that $D_{ij}u \in L^p(B_1)$ where $B_1$ is the open unit ball in $\mathbb{R}^n$. By (2), for $x \neq 0$ we have
$$
\begin{aligned}
|D_{ij}u(x)| &= \left|-\alpha\delta_{ij}|x|^{\alpha-2} + \alpha(2-\alpha) x_i x_j |x|^{\alpha-4}\right| \\
&\leq \alpha|x|^{\alpha-2} + \alpha(2-\alpha)|x_i||x_j||x|^{\alpha-4} \\
&\leq \alpha|x|^{\alpha-2} + \alpha(2-\alpha)|x||x||x|^{\alpha-4} \\
&= [\alpha + \alpha(2-\alpha)]|x|^{\alpha-2}
\end{aligned}
$$

Thus, it suffices to show that the function $x \mapsto |x|^{\alpha-2}$ is in $L^p(B_1)$.

That is to show $\displaystyle\int_{B_1} (|x|^{\alpha-2})^p \, dx < \infty$.

We have $\displaystyle\int_{B_1} |x|^{(\alpha-2)p}\,dx = \int_0^1 \left( \int_{\partial B_\rho} |x|^{(\alpha-2)p}\,dS \right) d\rho$

$$= \int_0^1 \left( \int_{\partial B_\rho} \rho^{(\alpha-2)p}\,dS \right) d\rho = \int_0^1 \rho^{(\alpha-2)p}\,|\partial B_\rho|\,d\rho$$

$$= \sigma_n \int_0^1 \rho^{(\alpha-2)p+n-1}\,d\rho,$$

where $\sigma_n = |\partial B_1|$. To show that this integral is finite, it suffices to show that $(\alpha-2)p+n-1 > -1$. By the definition of $\alpha$, we have

$$(\alpha-2)p+n-1 = \left(\frac{p-\beta}{p}-2\right)p+n-1 = (p-\beta-2p)+n-1$$
$$= n-p-\beta-1$$
$$\geqslant n-p-\frac{n-p}{2}-1$$
$$= \frac{n-p}{2}-1 > -1.$$

Thus, we have proved that $D_{ij}u \in L^p(B_1)$. Next, we'll show that

$$\sum_{i,j=1}^n a_{ij}(x)\,D_{ij}u(x) = 0 \qquad \forall x \neq 0.$$

For $x \neq 0$, we have

$$\sum_{i,j=1}^n a_{ij}(x)\,D_{ij}u(x) = \sum_{i,j=1}^n \left(C\delta_{ij} + x_i x_j |x|^{-2}\right)\left(-\alpha\delta_{ij}|x|^{\alpha-2} + \alpha(2-\alpha)x_i x_j |x|^{\alpha-4}\right)$$

$$= |x|^{\alpha-4}\sum_{i,j=1}^n \left(C\delta_{ij} + x_i x_j |x|^{-2}\right)\left(-\alpha\,\delta_{ij}|x|^2 + \alpha(2-\alpha)x_i x_j\right)$$

$$= |x|^{\alpha-4}\sum_{i,j=1}^n \left(-C\delta_{ij}\alpha|x|^2 - \alpha\,\delta_{ij}x_i x_j + C\alpha(2-\alpha)\delta_{ij}x_i x_j + \alpha(2-\alpha)x_i^2 x_j^2|x|^{-2}\right)$$

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$$= |x|^{\alpha-4}\Big[-C\alpha\sum_{i,j=1}^{n}\delta_{ij}|x|^2 - \alpha\sum_{i,j=1}^{n}\delta_{ij}x_ix_j + C\alpha(2-\alpha)\sum_{i,j=1}^{n}\delta_{ij}x_ix_j + $$

$$+\alpha(2-\alpha)|x|^{-2}\sum_{i,j=1}^{n}x_i^2x_j^2\Big]$$

$$= |x|^{\alpha-4}\left[-C\alpha n|x|^2 - \alpha|x|^2 + C\alpha(2-\alpha)|x|^2 + \alpha(2-\alpha)|x|^2\right]$$

$$= \alpha|x|^{\alpha-2}\left[-Cn-1+C(2-\alpha)+2-\alpha\right]$$

$$= \alpha|x|^{\alpha-2}\left[C(2-\alpha-n)+1-\alpha\right]$$

$$= \alpha(n+\alpha-2)|x|^{\alpha-2}\Big(\underbrace{-C+\frac{1-\alpha}{n+\alpha-2}}_{=0}\Big)$$

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$$= 0.$$

Therefore, $\sum_{i,j=1}^{n} a_{ij}(x)\, D_{ij}u(x) = 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

$\Sigma = 50/50$