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Math 8583: Theory of PDE

Homework #3

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(1) Denote by B_1 the open unit ball in $\mathbb{R}^n$. Suppose that $u \in C^4(\overline{B_1})$ is a function satisfying
$$\begin{cases} \Delta^2 u := \Delta(\Delta u) = 0 & \text{in } B_1, \\ u = |Du| = 0 & \text{on } B_1. \end{cases}$$

We'll show that $u \equiv 0$ in B_1.

Denote $v = \Delta u \in C^2(\overline{B_1})$. Because $u, v \in C^2(\overline{B_1})$ and ∂B_1 is in class C^1, by Green's formula (Lemma 1.16, page 9, Safonov's lecture notes), we have
$$\int_{B_1} (v\Delta u - u\Delta v)\,dx = \int_{\partial B_1} \left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right) d\sigma \qquad (1)$$
where ν denotes the (outward) unit vectors on ∂B_1. On ∂B_1 we have
$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = (Du)\cdot\nu = 0$$
In B_1, $\Delta v = \Delta(\Delta u) = 0$. Thus Eq. (1) becomes $\int_{B_1} v\Delta u\,dx = 0$, which is equivalent to $\int_{B_1} v^2 dx = 0$. Since v is continuous in B_1, $v \equiv 0$ in B_1.

Consequently, $u \in C^2(\overline{B_1})$ satisfies
$$\begin{cases} \Delta u = 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

Since u is harmonic in B_1 and continuous on $\overline{B_1}$, it attains maximum and minimum values in $\overline{B_1}$ at points on ∂B_1. Thus,
$$\min_{\overline{B_1}} u = \max_{\overline{B_1}} u = 0$$
This means $u \equiv 0$ in B_1.

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(2) For each $r > 0$, we denote $B_r = \{x \in \mathbb{R}^n : |x| < r\}$.
Let $u \in C^2(B_2)$ be a function satisfying $\begin{cases} \Delta u = 0 & \text{in } B_2, \\ u > 0 & \text{in } B_2. \end{cases}$
we'll show that there exists a number $N = N(n)$ such that

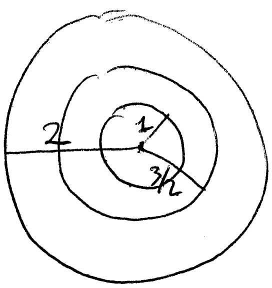
$$|D(\ln u)| \leq N \quad \text{in } B_1.$$

For each $1 \leq i \leq n$, by the chain rule we have

$$D_i(\ln u) = (D_i u)\frac{1}{u} = \frac{D_i u}{u}.$$

Thus, $D(\ln u) = \frac{Du}{u}$. We want to find $N = N(n) > 0$ such that

$$|Du(x)| \leq N u(x) \quad \forall x \in B_1.$$



Put $\Omega = B_{3/2}$ and $\delta = 1/2$. Then

$$\begin{aligned} \Omega^\delta &:= \{x \in \Omega : dist(x, \partial\Omega) > \delta\} \\ &= \{x \in B_{3/2} : dist(x, \partial B_{3/2}) > \tfrac{1}{2}\} \\ &= \{x \in B_{3/2} : \tfrac{3}{2} - |x| > \tfrac{1}{2}\} \\ &= B_1. \end{aligned}$$

Applying Theorem 1.12, page 7, Saponov's lecture notes (Estimation for derivatives) we have $\sup_{\Omega^\delta} |Du| \leq \frac{n}{\delta} \sup_{\Omega} |u|$. Thus,

$$\sup_{B_1} |Du| \leq 2n \sup_{B_{3/2}} |u| = 2n \sup_{B_{3/2}} u \qquad (1)$$

Put $\Gamma = B_2$ and $\varepsilon = \frac{1}{2}$. We have

$$\Gamma^\varepsilon := \{x \in \Gamma : dist(x, \partial\Gamma) > \varepsilon\} = \{x \in B_2 : dist(x, \partial B_2) > \tfrac{1}{2}\}$$
$$= \{x \in B_2 : 2 - |x| > \tfrac{1}{2}\} = B_{3/2}.$$

Since $u > 0$ in Γ, we can apply Harnack Inequality (Theorem 1.7, page 4, Safonov's lecture notes) to function u in domain Γ, namely

$$\sup_{\Gamma^\varepsilon} u \leq K \inf_{\Gamma} u, \qquad (2)$$

where $K > 0$ depends only on n and $\frac{\varepsilon}{\operatorname{diam}\Gamma}$. Because $\frac{\varepsilon}{\operatorname{diam}\Gamma} = \frac{1/2}{4} = \frac{1}{8}$,

K depends only on n. Then (2) implies

$$\sup_{B_{3/2}} u \leq K(n)\, u(x) \quad \forall x \in B_{3/2} \qquad (3)$$

Combining (1) and (3), we have $\sup_{B_1} |Du| \leq 2n K(n)\, u(x)$ $\forall x \in B_{3/2}$.

Hence, for each $x \in B_1$, $|Du(x)| \leq \sup_{B_1} |Du| \leq 2n K(n)\, u(x)$.

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Therefore, we can choose $N = N(n) = 2n K(n)$.

(3) For each $1 \leq i,j \leq n$, we consider a function $a_{ij} : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$. Suppose that the matrix (a_{ij}) is symmetric, i.e. $a_{ij} = a_{ji}$, and satisfies the uniform parabolicity condition

$$\nu |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \nu^{-1} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \forall t > 0. \qquad (*)$$

with a constant $\nu \in (0,1]$. For any $\alpha, \beta > 0$, we denote

$$K_{\alpha,\beta}(t,x) := t^{-\alpha} \exp\left(-\frac{|x|^2}{\beta t}\right) \quad \forall t > 0\ \forall x \in \mathbb{R}^n.$$

We'll show that there exist positive constants $\alpha_1, \beta_1, \alpha_2, \beta_2$ depending only on n and ν such that

$$L K_{\alpha_1,\beta_1}(t,x) \geq 0 \quad \text{and} \quad L K_{\alpha_2,\beta_2}(t,x) \leq 0 \quad \forall t > 0, \forall x \in \mathbb{R}^n,$$

where $L := \partial_t - \sum_{i,j=1}^n a_{ij} D_{ij}$.

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We have
$$\partial_t K_{\alpha,\beta}(t,x) = \partial_t \left[t^{-\alpha} \exp\left(-\frac{|x|^2}{\beta t}\right)\right]$$
$$= (-\alpha)\, t^{-\alpha-1} \exp\left(-\frac{|x|^2}{\beta t}\right) + t^{-\alpha}\left(\frac{|x|^2}{\beta t^2}\right)\exp\left(-\frac{|x|^2}{\beta t}\right)$$
$$= -\alpha t^{-1} K_{\alpha,\beta}(t,x) + \frac{|x|^2}{\beta t^2} K_{\alpha,\beta}(t,x)$$
$$= \frac{1}{\beta^2 t^2} K_{\alpha,\beta}(t,x)\left(-\alpha\beta^2 t + \beta |x|^2\right) \qquad (1)$$

For each $1 \le j \le n$, we have
$$D_j K_{\alpha,\beta}(t,x) = t^{-\alpha} D_j \left[\exp\left(\frac{-|x|^2}{\beta t}\right)\right]$$
$$= t^{-\alpha} D_j\left(-\frac{|x|^2}{\beta t}\right)\exp\left(\frac{-|x|^2}{\beta t}\right)$$
$$= -\frac{2x_j}{\beta t} K_{\alpha,\beta}(t,x) \qquad (2)$$

If $1 \le i \le n$ and $i \ne j$, we have
$$D_{ij} K_{\alpha,\beta}(t,x) = D_i\left(-\frac{2x_j}{\beta t} K_{\alpha,\beta}(t,x)\right) = \frac{-2x_j}{\beta t} D_i K_{\alpha,\beta}(t,x)$$
$$\overset{(2)}{=} \frac{-2x_j}{\beta t}\left(-\frac{2x_i}{\beta t} K_{\alpha,\beta}(t,x)\right) = \frac{4x_i x_j}{\beta^2 t^2} K_{\alpha,\beta}(t,x) \qquad (3)$$

If $i = j$, we have
$$D_{jj} K_{\alpha,\beta}(t,x) = D_j\left(-\frac{2x_j}{\beta t} K_{\alpha,\beta}(t,x)\right) = \frac{-2}{\beta t} K_{\alpha,\beta}(t,x) - \frac{2x_j}{\beta t} D_j K_{\alpha,\beta}(t,x)$$
$$\overset{(2)}{=} -\frac{2}{\beta t} K_{\alpha,\beta}(t,x) - \frac{2x_j}{\beta t}\left(-\frac{2x_j}{\beta t} K_{\alpha,\beta}(t,x)\right)$$
$$= \left(-\frac{2}{\beta t} + \frac{4x_j^2}{\beta^2 t^2}\right) K_{\alpha,\beta}(t,x) \qquad (4)$$

Combining (3) and (4) together, we can write
$$D_{ij} K_{\alpha,\beta}(t,x) = \left(\frac{-2}{\beta t}\delta_{ij} + \frac{4x_i x_j}{\beta^2 t^2}\right) K_{\alpha,\beta}(t,x),$$
where δ_{ij} is the Kronecker's symbol.

Hence, $D_{ij} K_{\alpha,\beta}(t,x) = \frac{1}{\beta^2 t^2} K_{\alpha,\beta}(t,x)\left(-2\beta t \delta_{ij} + 4x_i x_j\right)$ $\quad (5)$

From (1) and (5) we have

$$L K_{\alpha,\beta}(t,x) = \partial_t K_{\alpha,\beta}(t,x) - \sum_{i,j=1}^{n} a_{ij}(t,x) D_{ij} K_{\alpha,\beta}(t,x)$$

$$= \frac{1}{\beta^2 t^2} K_{\alpha,\beta}(t,x)\Big[\underbrace{-\alpha\beta^2 t + \beta|x|^2 - \sum_{i,j=1}^{n} a_{ij}\left(-2\beta t \delta_{ij} + 4x_i x_j\right)}_{A}\Big]$$

We have $A = \beta t\big(\underbrace{-\alpha\beta + 2\sum_{i,j=1}^{n} a_{ij}\delta_{ij}}_{\{1\}}\big) + \big(\underbrace{\beta|x|^2 - 4\sum_{i,j=1}^{n} a_{ij} x_i x_j}_{\{2\}}\big)$

Take any $1 \le k \le n$. From the parabolicity condition $(*)$, if we take $\xi = (0,\dots,\underset{\underset{k}{\uparrow}}{1},\dots 0)$ then $(*)$ gives $\nu \le a_{kk} \le \nu^{-1}$. Since $\{1\} = -\alpha\beta + 2\sum_{k=1}^{n} a_{kk}$, we have

$$-\alpha\beta + 2n\nu \le \{1\} \le -\alpha\beta + 2n\nu^{-1} \qquad (6)$$

We also have $\nu|x|^2 \le \sum_{i,j=1}^{n} a_{ij} x_i x_j \le \nu^{-1}|x|^2$. Therefore,

$$\beta|x|^2 - 4\nu^{-1}|x|^2 \le \{2\} \le \beta|x|^2 - 4\nu|x|^2$$

Equivalently, $|x|^2(\beta - 4\nu^{-1}) \le \{2\} \le |x|^2(\beta - 4\nu)$ $\qquad (7)$

From (6) and (7) we have

$$A = \beta t\{1\} + \{2\} \geq \beta t(-\alpha\beta + 2n\nu) + |x|^2(\beta - 4\nu^{-1}) \quad \forall t > 0, \forall x \in \mathbb{R}^n.$$

If we choose $\alpha = \frac{n\nu^2}{2}$ and $\beta = 4\nu^{-1}$, we get $-\alpha\beta + 2n\nu = \beta - 4\nu^{-1} = 0$; then $A \geq 0$. From (6) and (7), we also have

$$A = \beta t\{1\} + \{2\} \le \beta t(-\alpha\beta + 2n\nu^{-1}) + |x|^2(\beta - 4\nu) \quad \forall t > 0, \forall x \in \mathbb{R}^n$$

If we choose $\alpha = \frac{n\nu^{-2}}{2}$ and $\beta = 4\nu$, we get $-\alpha\beta + 2n\nu^{-1} = \beta - 4\nu = 0$;

then $A \leq 0$, and thus $LK_{\alpha,\beta}(t,x) \leq 0$ for all $t>0$, $x \in \mathbb{R}^n$.

In conclusion, we can choose

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$$\alpha_1 = \frac{n\nu^2}{2}, \; \beta_1 = 4\nu^{-1} \quad \text{and} \quad \alpha_2 = \frac{n\nu^{-2}}{2}, \; \beta_2 = 4\nu.$$

*Comment: We actually do not need the symmetry of (a_{ij}).

(4) Denote $H_T = (0,T) \times \mathbb{R}^n$. Let $(a_{ij})_{1 \leq i,j \leq n}$ be a matrix of functions $a_{ij} : H_T \to \mathbb{R}$ satisfying the uniform parabolicity condition

$$\nu|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \quad \forall (t,x) \in H_T, \; \cancel{\forall \xi \in \mathbb{R}^n}, \; \forall \xi \in \mathbb{R}^n$$

for some constant $\nu \in (0,1)$. Denote the differential operator $L := \partial_t - \sum_{i,j=1}^n a_{ij} D_{ij}$.

We'll show that the problem

$$\begin{cases} Lu(t,x) = 0 & \forall (t,x) \in H_T, \\ u(0,x) = g(x) & \forall x \in \mathbb{R}^n \end{cases} \qquad (*)$$

where $g \in C(\mathbb{R}^n)$ is a given function, has at most one solution $u \in C^{1,2}(H_T) \cap C(\overline{H_T})$ satisfying $|u(t,x)| \leq N \exp(a|x|^2)$ in H_T for some constant numbers $N, a > 0$.

Suppose that u_1 and u_2 are two solutions in $C^{1,2}(H_T) \cap C(\overline{H_T})$ of Problem $(*)$ such that $|u_1(t,x)| \leq N_1 \exp(a_1|x|^2)$, $|u_2(t,x)| \leq N_2 \exp(a_2|x|^2)$ in H_T, for some constant numbers $N_1, N_2, a_1, a_2 > 0$. Put $v = u_1 - u_2 \in C^{1,2}(H_T) \cap C(\overline{H_T})$.

Then

$$\begin{cases} Lv(t,x) = 0 & \forall (t,x) \in H_T, \\ v(0,x) = g(x) - g(x) = 0 & \forall x \in \mathbb{R}^n \end{cases}$$

Moreover,

$$\begin{aligned} |v(t,x)| \leq |u_1(t,x)| + |u_2(t,x)| &\leq N_1 \exp(a_1|x|^2) + N_2 \exp(a_2|x|^2) \\ &\leq 2(N_1 + N_2) \exp((a_1+a_2)|x|^2) \\ &= \cancel{2}\, N \exp(a|x|^2) \end{aligned}$$

where $N = 2(N_1+N_2) > 0$ and $a = a_1 + a_2 > 0$. In short, we have

$$v \in C^{1,2}(H_T) \cap C(\overline{H_T}),$$

$$|v(t,x)| \leq N \exp(a|x|^2) \quad \forall (t,x) \in H_T \qquad (1)$$

$$\begin{cases} Lv(t,x) = 0 & \text{in } H_T \\ v(0,x) = 0 & \forall x \in \mathbb{R}^n \end{cases} \qquad (**)$$

We now need to show that $v \equiv 0$ in H_T. For any $\alpha, \beta > 0$, we put

$$K_{\alpha,\beta}(t,x) = t^{-\alpha} \exp\left(-\frac{|x|^2}{\beta t}\right) \quad \forall (t,x) \in (0,\infty) \times \mathbb{R}^n$$

as in Problem 3. For each $h > 0$, we denote $H_h = (0,h) \times \mathbb{R}^n$ and

$$w_h(t,x) := K_{\alpha\beta}(h-t, \sqrt{-1}\,x) = (h-t)^{-\alpha} \exp\left(\frac{|x|^2}{\beta(h-t)}\right) \quad \forall (t,x) \in H_h.$$

We have similar computations as in Problem 3.

$$\partial_t w_h(t,x) = \alpha(h-t)^{-\alpha-1} \exp\left(\frac{|x|^2}{\beta(h-t)}\right) + (h-t)^{-\alpha} \frac{|x|^2}{\beta(h-t)^2} \exp\left(\frac{|x|^2}{\beta(h-t)}\right)$$

$$= \alpha(h-t) w_h + \frac{|x|^2}{\beta(h-t)^2} w_h$$

$$= \frac{w_h}{\beta^2(h-t)^2} \left(\alpha\beta^2(h-t) + \beta|x|^2\right) \qquad (2)$$

For $1 \leq i,j \leq n$, we have

$$D_j w_h(t,x) = (h-t)^{-\alpha} D_j \left[\exp\left(\frac{|x|^2}{\beta(h-t)}\right)\right]$$

$$= (h-t)^{-\alpha} \frac{2x_j}{\beta(h-t)} \exp\left(\frac{|x|^2}{\beta(h-t)}\right)$$

$$= \frac{2x_j}{\beta(h-t)} w_h(t,x) \qquad (3)$$

If $i \neq j$ then $\quad D_{ij} w_h = D_i(D_j w_h) \overset{(3)}{=\!=} \dfrac{2x_j}{\beta(h-t)} D_i w_h(t,x)$

$$\overset{(3)}{=\!=} \frac{2x_j}{\beta(h-t)} \frac{2x_i}{\beta(h-t)} w_h = \frac{4x_i x_j}{\beta^2(h-t)^2} w_h \qquad (4)$$

If $i=j$ then

$$D_{jj} w_h = D_j (D_j w_h) \overset{(3)}{=\!=} D_j \left(\frac{2x_j}{\beta(h-t)} w_h \right)$$

$$= \frac{2}{\beta(h-t)} w_h + \frac{2x_j}{\beta(h-t)} D_j w_h$$

$$\overset{(3)}{=\!=} \frac{2}{\beta(h-t)} w_h + \frac{2x_j}{\beta(h-t)} \frac{2x_j}{\beta(h-t)} w_h$$

$$= \frac{w_h}{\beta^2(h-t)^2} \left(2\beta(h-t) + 4x_j^2 \right) \qquad (5)$$

Combining (4) and (5), we have

$$D_{ij} w_h = \frac{w_h}{\beta^2(h-t)^2} \left(2\beta(h-t)\delta_{ij} + 4x_i x_j \right), \qquad (6)$$

where δ_{ij} is the Kronecker's symbol. From (2) and (6) we get

$$Lw_h(t,x) = \partial_t w_h - \sum_{i,j=1}^{n} a_{ij}(t,x) D_{ij} w_h$$

$$= \frac{w_h}{\beta^2(h-t)^2} \left[\alpha\beta^2(h-t) + \beta|x|^2 - \sum_{i,j=1}^{n} a_{ij}(t,x) \left(2\beta(h-t)\delta_{ij} + 4x_i x_j \right) \right]$$

$$= \frac{w_h}{\beta^2(h-t)^2} \Big[\beta(h-t) \underbrace{\Big(\alpha\beta - 2\sum_{i,j=1}^{n} a_{ij}(t,x)\delta_{ij} \Big)}_{\{1\}} + \underbrace{\beta|x|^2 - 4\sum_{i,j=1}^{n} a_{ij}(t,x) x_i x_j}_{\{2\}} \Big]$$

We have $\{1\} = \alpha\beta - 2\sum_{k=1}^{n} a_{kk}(t,x)$. We showed in Problem 3 that $\nu \le a_{kk}(t,x) \le \nu^{-1}$.

Thus, $\alpha\beta - 2n\nu^{-1} \le \{1\} \le \alpha\beta - 2n\nu \qquad (7)$

Also, $\{2\} \geq \beta|x|^2 - 4\nu^{-1}|x|^2 = (\beta - 4\nu^{-1})|x|^2 \qquad (8)$

By (7) and (8) we have

$$Lw_h(t,x) = \frac{w_h}{\beta^2(h-t)^2}\left[\beta(h-t)\{1\} + \{2\}\right]$$

$$\geqslant \frac{w_h}{\beta^2(h-t)^2}\left[\beta(h-t)(\alpha\beta - 2n\nu^{-1}) + (\beta - 4\nu^{-1})|x|^2\right] \qquad (9)$$

Choose $\alpha = \frac{n}{2}$ and $\beta = 4\nu^{-1}$. Then $RHS(9) = 0$. Thus, $Lw_h(t,x) \geqslant 0$ for all $(t,x) \in H_h$. Therefore, for each $h > 0$, with

$$w_h(t,x) = (h-t)^{-n/2} \exp\left(\frac{\nu|x|^2}{4(h-t)}\right)$$

we have $Lw_h(t,x) \geqslant 0$ in H_h.

Put $h_0 = \min\left\{\frac{\nu}{8a}, T\right\} > 0$. We have $a < \frac{\nu}{4h_0}$. For each $\varepsilon > 0$,

$$\lim_{R\to\infty} \frac{\varepsilon h_0^{-n/2}\exp\left(\frac{\nu R^2}{4h_0}\right)}{N\exp(aR^2)} = \frac{\varepsilon h_0^{-n/2}}{N}\lim_{R\to\infty}\exp\left(\left(\underbrace{\frac{\nu}{4h_0} - a}_{>0}\right)R^2\right) = \infty.$$

Thus, there exists $R_\varepsilon > 0$ such that

$$\varepsilon h_0^{-n/2}\exp\left(\frac{\nu R^2}{4h_0}\right) > N\exp(aR^2) \qquad \forall R > R_\varepsilon \qquad (10)$$

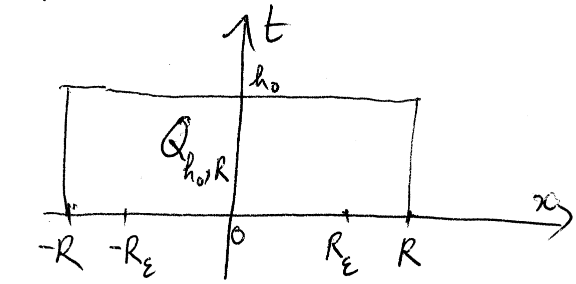
Consider $R > R_\varepsilon$. If $|x| = R$ and $t \in (0, h_0)$ then

$$\varepsilon w_{h_0}(t,x) = \varepsilon(h_0 - t)^{-n/2}\exp\left(\frac{\nu R^2}{4(h_0-t)}\right)$$

$$\geqslant \varepsilon h_0^{-n/2}\exp\left(\frac{\nu R^2}{4(h_0-t)}\right)$$

$$\geqslant \varepsilon h_0^{-n/2}\exp\left(\frac{\nu R^2}{4h_0}\right) \geqslant N\exp(aR^2) \quad \text{(by (10))}$$

$$= N\exp(a|x|^2) \geqslant \pm v(t,x).$$



For any $h, R > 0$, we denote

$$Q_{h,R} = (0,h) \times B_R = \{(t,x) \in (0,\infty) \times \mathbb{R}^n \mid 0 < t < h,\ |x| < R\}$$

The parabolic boundary of $Q_{h,R}$ is $\partial_p Q_{h,R} = \{(t,x) \in (0,\infty) \times \mathbb{R}^n \mid |x| = R,\ 0 < t < h\}$

$$\cup \{(0,x) \mid x \in B_R\}.$$

We showed earlier that $\pm v(t,x) \leq \varepsilon w_{h_0}(t,x)$ $\forall\, 0 < t < h_0$, $\forall x \in \mathbb{R}^n$, $|x| \geq R$.

In addition, $\pm v(0,x) = 0 < \varepsilon w_{h_0}(0,x)$ $\forall x \in B_R$. Therefore,

$$\pm v(t,x) \leq \varepsilon w_{h_0}(t,x) \qquad \forall (t,x) \in \partial_p Q_{h_0,R}.$$

Because $Lv = 0 \leq \varepsilon L(w_{h_0}) = L(\varepsilon w_0)$ ~~for~~ and $L(-v) = 0 \leq L(\varepsilon w_0)$, by the Comparison Principle for parabolic equations, we have

$$\pm v(t,x) \leq \varepsilon w_{h_0}(t,x) \qquad \forall (t,x) \in \overline{Q_{h_0,R}}.$$

Thus, $|v(t,x)| \leq \varepsilon w_{h_0}(t,x)$ $\forall (t,x) \in \overline{Q_{h_0,R}}$.

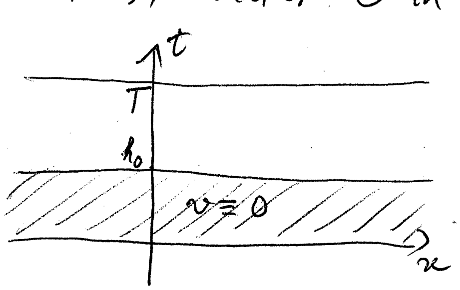
Because this inequality is true for all $R > R_\varepsilon$, we have

$$|v(t,x)| \leq \varepsilon w_{h_0}(t,x) \qquad \forall\, 0 \leq t \leq h_0,\ \forall x \in \mathbb{R}^n$$

Because this inequality is true for all $\varepsilon > 0$, we get

$$|v(t,x)| \leq 0 \qquad \forall\, 0 \leq t \leq h_0,\ \forall x \in \mathbb{R}^n.$$

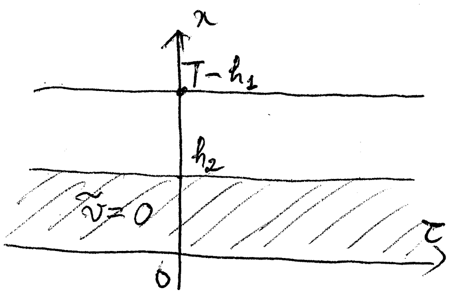
Thus, $v(t,x) = 0$ in $\overline{H_{h_0}}$. Put



$$S = \{0 \leq h \leq T \mid v(t,x) = 0 \ \forall\, 0 \leq t \leq h,\ \forall x \in \mathbb{R}^n\}$$

Then $0, h_0 \in S$. Since v is continuous, there exists $h_1 = \max S > 0$. If $h_1 = T$ then

$v \equiv 0$ in H_T and we are done. Suppose by contradiction that $h_1 < T$.



Put $\tau = t - h_1$ $\forall h_1 \leq t \leq T$ and

$\tilde{v}(\tau, x) = v(t, x)$ $\forall h_1 \leq t \leq T$, $\forall x \in \mathbb{R}^n$.

Then $\tilde{v} \in C^{1,2}(H_{T-h_1}) \cap C(\overline{H_{T-h_1}})$,

$$\tilde{v}(0, x) = v(h_1, x) = 0 \quad \forall x \in \mathbb{R}^n \text{ (because } h_1 \in S\text{)},$$

Put $$\tilde{L}\tilde{v}(\tau, x) := \partial_\tau \tilde{v}(\tau, x) - \sum_{i,j=1}^{n} a_{ij}(t, x) D_{ij}\tilde{v}(\tau, x) \quad \forall (\tau, x) \in H_{T-h_1}.$$

Then $$\tilde{L}\tilde{v}(\tau, x) = \partial_t v(t, x) - \sum_{i,j=1}^{n} a_{ij}(t, x) D_{ij} v(t, x) = L v(t, x) = 0,$$ $$\forall (\tau, x) \in H_{T-h_1}.$$

In addition, $|\tilde{v}(\tau, x)| = |v(t, x)| \leq N \exp(a|x|^2)$ $\forall (\tau, x) \in H_{T-h_1}$.

Therefore, we can apply the result that we have proved for v for $\tilde{v}$. Note that the domain $\overline{H_T}$ of v corresponds to the domain $\overline{H_{T-h_1}}$ of $\tilde{v}$. Namely, if we put $h_2 = \min\left\{\frac{\nu}{8a}, T - h_1\right\} > 0$ then $\tilde{v}(\tau, x) \equiv 0$ in $\overline{H_{h_2}}$.

This implies $v(t - h_1, x) = 0$ for all $h_1 \leq t \leq h_1 + h_2$ and $x \in \mathbb{R}^n$. Thus,

$$v(t, x) = 0 \quad \forall\, 0 \leq t \leq h_1 + h_2, \ \forall x \in \mathbb{R}^n$$

Hence $h_1 + h_2 \in S$. This contradicts the fact that $h_1 = \max S$. Therefore, we have proved that $v \equiv 0$ in $\overline{H_T}$. 15/15.