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Math 8583: Theory of PDE
Homework #4

① Let $g \in C^1(\mathbb{R}^n)$ be a bounded function such that
$$|Dg(x) - Dg(y)| \le K|x-y|^\alpha \quad \forall x, y \in \mathbb{R}^n$$
for some constants $K \geq 0$ and $\alpha \in (0,1)$. Let u be a bounded classical solution to the problem
$$\begin{cases} u_t = \Delta_x u & \forall (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \forall x \in \mathbb{R}^n. \end{cases} \qquad (*)$$
We'll show that $|u(x,t) - u(x,s)| \le NK|t-s|^{\frac{1+\alpha}{2}}$ for all $x \in \mathbb{R}^n$, $t, s \in [0,\infty)$, with a constant $N = N(n,\alpha) > 0$.

By Theorem 2.1(i), Safonov's lecture notes, page 2-2, and the uniqueness theorem (Theorem 2.6, page 2-12), Problem (*) has a unique bounded solution, which is given by
$$u(x,t) = \int_{\mathbb{R}^n} \Gamma(y,t)\, g(x-y)\, dy \qquad \forall (x,t) \in \mathbb{R}^n \times (0,\infty)$$
where $\Gamma(y,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4t}} \quad \forall (y,t) \in \mathbb{R}^n \times (0,\infty)$. ✓

Put $z = t^{-1/2} y$. Then $dz = t^{-n/2} dy$, and
$$u(x,t) = \int_{\mathbb{R}^n} \Gamma(y,t) g(x-y)\,dy = \int_{\mathbb{R}^n} \Gamma(t^{1/2} z, t)\, g(x - t^{1/2} z)\, t^{n/2}\, dz$$
$$= \int_{\mathbb{R}^n} \Gamma(z,1) (t^{1/2})^{-n} g(x - t^{1/2} z)\, t^{n/2}\, dz$$
$$= \int_{\mathbb{R}^n} \Gamma(z,1)\, g(x - t^{1/2} z)\, dz$$
Therefore, $u(x,t) = \int_{\mathbb{R}^n} \Gamma(z,1)\, g(x - t^{1/2} z)\, dz \qquad (1)$

Note that for the case $u(x,t) \equiv 1$ we obtain the identity

$$\int_{\mathbb{R}^n} \Gamma(z,1)\,dz = 1 \qquad (2)$$

Consequently, the function $e^{-\frac{|z|^2}{4}} \in L^1(\mathbb{R}^n)$. (3)

We'll prove the following property, which will be used many times later on.

$$\int_{\mathbb{R}^n} \Gamma(z,1)\,|z|^\beta\,dz < \infty \qquad \forall \beta > 0 \qquad (**)$$

This inequality will be proved if we can show that $\int_{|z|>R} e^{-\frac{|z|^2}{4}} |z|^\beta\,dz < \infty$ for some number $R>0$. We know that there exists a number $R>0$ depending on β such that $t^\beta < e^{t/8}$ for all $t>R$. For $|z| > R^{1/\beta}$, we have $|z|^\beta > R$ and thus $|z|^\beta < e^{\frac{|z|^2}{8}}$. Then

$$\int_{|z|>R} e^{-\frac{|z|^2}{4}} |z|^\beta\,dz \le \int_{|z|>R} e^{-\frac{|z|^2}{4}} e^{\frac{|z|^2}{8}}\,dz = \int_{|z|>R} e^{-\frac{|z|^2}{8}}\,dz \qquad (4)$$

Put $\tilde{z} = z/\sqrt{2}$. Then $\text{RHS}(4) = \int_{|\tilde{z}|>\frac{R}{\sqrt{2}}} e^{-\frac{|\tilde{z}|^2}{4}} (\sqrt{2})^n\,d\tilde{z} < \infty$ (by (3)).

Therefore, (**) is proved.

For any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, we have

$$\begin{aligned} u(x,t) - u(x,0) &= \int_{\mathbb{R}^n} \Gamma(z,1)\, g(x - t^{1/2} z)\,dz - g(x) && \text{(by (1))} \\ &= \int_{\mathbb{R}^n} \Gamma(z,1)\, g(x - t^{1/2} z)\,dz - \int_{\mathbb{R}^n} \Gamma(z,1)\, g(x)\,dz && \text{(by (2))} \\ &= \int_{\mathbb{R}^n} \Gamma(z,1)\big(g(x - t^{1/2} z) - g(x)\big)\,dz && (5) \end{aligned}$$

Consider the map $h(s) = g(x - st^{1/2}z)$ for all $s \in [0,1]$. Since $g \in C^1(\mathbb{R}^n)$, h is continuous on $[0,1]$ and differentiable on $(0,1)$. By Mean-Value Theorem, there exists $\theta \in (0,1)$ depending on x, z, t such that $h'(\theta) = h(1) - h(0)$. Thus,

$$g(x - t^{1/2}z) - g(x) = h(1) - h(0) = h'(\theta) = \frac{d}{ds}[g(x - \theta t^{1/2}z)] = -t^{1/2}z \cdot Dg(x - \theta t^{1/2}z).$$

Replacing this identity into (5), we get

$$\begin{aligned} u(x,t) - u(x,0) &= \int_{\mathbb{R}^n} \Gamma(z,1)(-t^{1/2})z \cdot Dg(x - \theta t^{1/2}z)\,dz \\ &= -t^{1/2}\int_{\mathbb{R}^n} \Gamma(z,1)z \cdot Dg(x - \theta t^{1/2}z)\,dz \\ &= -t^{1/2}\Big[\int_{\mathbb{R}^n} \Gamma(z,1)z \cdot (Dg(x - \theta t^{1/2}z) - Dg(x))\,dz + \\ &\qquad + \underbrace{\int_{\mathbb{R}^n} \Gamma(z,1)z \cdot Dg(x)\,dz}_{\{1\}}\Big] \end{aligned} \tag{6}$$

We have $\{1\} = \left(\int_{\mathbb{R}^n} \Gamma(z,1)z\,dz\right) \cdot Dg(x)$. Note that by $(**)$, $\Gamma(z,1)|z| \in L^1(\mathbb{R}^n)$.

Put $\tilde{z} = -z$. Then $d\tilde{z} = (-1)^n dz$ and

$$\int_{\mathbb{R}^n} \Gamma(\tilde{z},1)\tilde{z}\,d\tilde{z} = \int_{\mathbb{R}^n} \Gamma(-z,1)(-z)\,|(-1)^n|\,dz = -\int_{\mathbb{R}^n} \Gamma(z,1)z\,dz.$$

Thus, $\int_{\mathbb{R}^n} \Gamma(z,1)z\,dz = 0$. This means $\{1\} = 0$. Then (6) reduces to

$$u(x,t) - u(x,0) = -t^{1/2}\int_{\mathbb{R}^n} \Gamma(z,1)z \cdot (Dg(x - \theta t^{1/2}z) - Dg(x))\,dz$$

Hence, $|u(x,t) - u(x,0)| \le t^{1/2}\int_{\mathbb{R}^n} \Gamma(z,1)\,|z|\,|Dg(x - \theta t^{1/2}z) - Dg(x)|\,dz$

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$$\leq t^{1/2} \int_{\mathbb{R}^n} \Gamma(z,1)|z|\, K|0-\theta t^{1/2} z|^{\alpha}\, dz = K\left(\int_{\mathbb{R}^n} \Gamma(z,1)|z|^{\alpha+1}\, dz\right) t^{\frac{1+\alpha}{2}} \qquad (7)$$

Put $N = N(n,\alpha) = \int_{\mathbb{R}^n} \Gamma(z,1)\, |z|^{\alpha+1} dz > 0$.

Note that $N < \infty$ by $(**)$. Then by (7), we have

$$|u(x,t) - u(x,0)| \leq NK t^{\frac{1+\alpha}{2}} \quad \forall x \in \mathbb{R}^n,\ \forall t \geq 0. \qquad (8)$$

Now take any $t_0 > 0$, we'll show that

$$|u(x,t+t_0) - u(x,t_0)| \leq NK t^{\frac{1+\alpha}{2}} \quad \forall x \in \mathbb{R}^n,\ \forall t > 0. \qquad (9)$$

For any $x \in \mathbb{R}^n$, we have $|Dg(x) - Dg(0)| \leq K|x-0|^{\alpha} = K|x|^{\alpha}$.

Put $K_0 = |Dg(0)| \geq 0$. Then

$$|Dg(x)| \leq |Dg(x) - Dg(0)| + |Dg(0)| \leq K|x|^{\alpha} + K_0 \qquad (10)$$

With this estimate, we'll show that for any $1 \leq i \leq n$,

$$D_i u(x,t) = \int_{\mathbb{R}^n} \Gamma(z,1) D_i g(x - t^{1/2} z)\, dz \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty).$$

For any $x \in \mathbb{R}^n$, $t > 0$ and $s \in (-1,1)\setminus\{0\}$ we have

$$\frac{u(x+se_i, t) - u(x,t)}{s} = \int_{\mathbb{R}^n} \Gamma(z,1)\, \frac{g(x+se_i - t^{1/2}z) - g(x - t^{1/2}z)}{s}\, dz \qquad (11)$$

The integrand goes to $\Gamma(z,1) D_i g(x - t^{1/2}z)$ as s goes to 0. Moreover,

$$\left|\Gamma(z,1)\, \frac{g(x+se_i - t^{1/2}z) - g(x-t^{1/2}z)}{s}\right| = \Gamma(z,1)\, |D_i g(x + \theta s e_i - t^{1/2} z)|$$

(for some $\theta \in (0,1)$ by Mean-Value theorem)

$$\overset{(10)}{\leq} \Gamma(z,1)\left(K|x + \theta s e_i - t^{1/2} z|^{\alpha} + K_0\right)$$

$$\leq \Gamma(z,1)\Big[K(|x|+\underbrace{|\theta s|}_{\leq 1}+t^{1/2}|z|)^\alpha + K_0\Big]$$

$$\leq \Gamma(z,1)\Big\{K\big[(|x|+1)^\alpha + (t^{1/2}|z|)^\alpha\big] + K_0\Big\} \qquad (\text{because } 0<\alpha<1)$$

$$= (K(|x|+1)^\alpha + K_0)\underbrace{\Gamma(z,1)}_{\in L^1(\mathbb{R}^n)} + t^{\alpha/2}\underbrace{\Gamma(z,1)\,|z|^\alpha}_{\in L^1(\mathbb{R}^n)}$$

Thus, the integrand on the right-hand-side of (11) is bounded by an L^1-function, which is independent of s. By Lebesgue's Dominated Convergence theorem, we have

$$\lim_{s\to 0}\frac{u(x+se_i,t)-u(x,t)}{s} = \int_{\mathbb{R}^n}\Gamma(z,1)\,D_i g(x-t^{1/2}z)\,dz.$$

Therefore,

$$D_i u(x,t) = \int_{\mathbb{R}^n}\Gamma(z,1)\,D_i g(x-t^{1/2}z)\,dz \qquad (12)$$

Now we return to proving (9). Define a function $\tilde{g}(x) = u(x,t_0)$ $\forall x\in\mathbb{R}^n$. Then

$$D_i\tilde{g}(x) = D_i u(x,t_0) \overset{(12)}{=} \int_{\mathbb{R}^n}\Gamma(z,1)\,D_i g(x-t_0^{1/2}z)\,dz$$

Thus, $D\tilde{g}(x) = \int_{\mathbb{R}^n}\Gamma(z,1)\,Dg(x-t_0^{1/2}z)\,dz$.

Then for any $x,y\in\mathbb{R}^n$, we have the estimate

$$|D\tilde{g}(x)-D\tilde{g}(y)| = \left|\int_{\mathbb{R}^n}\Gamma(z,1)\big(Dg(x-t_0^{1/2}z)-Dg(y-t_0^{1/2}z)\big)\,dz\right|$$

$$\leq \int_{\mathbb{R}^n}\Gamma(z,1)\,|Dg(x-t_0^{1/2}z)-Dg(y-t_0^{1/2}z)|\,dz$$

$$\leq \int_{\mathbb{R}^n}\Gamma(z,1)\,K|x-y|^\alpha\,dz = K|x-y|^\alpha\underbrace{\int_{\mathbb{R}^n}\Gamma(z,1)\,dz}_{=1 \text{ by } (2)}.$$

Thus, $|D\tilde{g}(x) - D\tilde{g}(y)| \leq K|x-y|^{\alpha} \quad \forall x, y \in \mathbb{R}^n$.

In other words, $\tilde{g}$ satisfies the same properties that g have in the hypotheses of the problem. We see that the function $v(x,t) = u(x, t+t_0)$ for $(x,t) \in \mathbb{R}^n \times [0,\infty)$ is a bounded classical solution to the problem

$$\begin{cases} v_t = \Delta_x v & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = \tilde{g}(x) & \text{in } \mathbb{R}^n \end{cases}$$

Hence, we can apply the estimate (8) to the function v. Namely,

$$|v(x,t) - v(x,0)| \leq NKt^{\frac{1+\alpha}{2}} \quad \forall x \in \mathbb{R}^n, \ \forall t \geq 0 \qquad (13)$$

where N is still given by $N = \int_{\mathbb{R}^n} \Gamma(z,1)|z|^{\alpha+1}dz$.

We can write (3) in terms of u as

$$|u(x,t+t_0) - u(x,t_0)| \leq NKt^{\frac{1+\alpha}{2}} \quad \forall (x,t) \in \mathbb{R}^n \times [0,\infty)$$

Because this estimate is true for all $t_0, t \geq 0$, we can write it in a nicer form as follows

$$|u(x,t) - u(x,s)| \leq NK|t-s|^{\frac{1+\alpha}{2}} \quad \forall x \in \mathbb{R}^n, \ \forall t, s \geq 0$$

where

$$N = N(n,\alpha) = \int_{\mathbb{R}^n} \Gamma(z,1)|z|^{\alpha+1}dz.$$

(2) Let $f \in C(\mathbb{R})$ be a periodic function with periode 1. Let $u(x,t)$ be a bounded classical solution to the problem

$$\begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & \text{in } \mathbb{R} \end{cases} \qquad (*)$$

We'll show that $\lim_{t\to\infty} u(x,t) = \int_0^1 f(y)\,dy \quad \forall x\in\mathbb{R}$ (1)

Put $\beta = \int_0^1 f(y)\,dy$. Then the functions $\tilde{u}(x,t) = u(x,t)-\beta$ and $\tilde{f}(x) = f(x)-\beta$ satisfy

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{in } \mathbb{R}\times(0,\infty), \\ \tilde{u}(x,0) = \tilde{f}(x) & \text{in } \mathbb{R}. \end{cases}$$

Moreover, $\tilde{f}\in C(\mathbb{R})$ is periodic with periode 1 and $\int_0^1 \tilde{f}(x)\,dx = \int_0^1 (f(x)-\beta)\,dx = 0$.

Then, showing (1) is equivalent to showing that $\lim_{t\to\infty} \tilde{u}(x,t) = 0 \quad \forall x\in\mathbb{R}$.

This means that in Problem (*), we could have assumed $\beta = \int_0^1 f(y)\,dy = 0$.

With this additional assumption, we'll show that $\lim_{t\to\infty} \tilde{u}(x,t) = 0$ for all $x\in\mathbb{R}$. By Theorem 2.1 (c), Safonov's lecture notes, page 2-2, the function u is given by

$$u(x,t) = \int_{\mathbb{R}} \Gamma(x-y,t) f(y)\,dy \qquad (2)$$

where $\Gamma(z,t) = (4\pi t)^{-1/2} e^{-\frac{z^2}{4t}}$.

We define a function $F(y) = \int_0^y f(z)\,dz$ for all $y\in\mathbb{R}$. Then $F\in C^1(\mathbb{R})$ and $\frac{d}{dy}(F(y+1)-F(y)) = F'(y+1)-F'(y) = f(y+1)-f(y) = 0$. Thus,

$$F(y+1)-F(y) \equiv \text{const} \quad \text{in } \mathbb{R}.$$

At $y=0$, we have $F(1)-F(0) = F(1) = \int_0^1 f(z)\,dz = 0$ by the additional assumption we made earlier. Therefore, $F(y+1)-F(y)\equiv 0$. In other words, F is periodic with period 1. Consequently, $F(k) = F(0) = 0$ for all $k\in\mathbb{Z}$.

By (2) we have $u(x,t) = \lim_{k\to\infty} \int_{-k}^{k} \Gamma(x-y,t) f(y)\,dy$ (3)

For $k \in \mathbb{Z}$, we have

$$\int_{-k}^{k} \Gamma(x-y,t) f(y)\,dy = \int_{-k}^{k} \Gamma(x-y,t) F'(y)\,dy$$

$$= \underbrace{\Gamma(x-y,t)F(y)\Big|_{y=-k}^{y=k}}_{=0 \text{ since } F(k)=F(-k)=0} - \int_{-k}^{k} F(y)(-\Gamma_x(x-y,t))\,dy$$

$$= \int_{-k}^{k} F(y)\,\Gamma_x(x-y,t)\,dy$$

Then (3) becomes $\quad u(x,t) = \lim_{k\to\infty} \int_{-k}^{k} \Gamma_x(x-y,t) F(y)\,dy \qquad (4)$

Since $\Gamma(x,t) = (4\pi t)^{-1/2} e^{-\frac{x^2}{4t}}$, we have

$$\Gamma_x(x,t) = (4\pi t)^{-1/2}\left(\frac{-2x}{4t}\right) e^{-\frac{x^2}{4t}} = \frac{-x}{2t}\Gamma(x,t)$$

Then $\displaystyle\int_{\mathbb{R}} |\Gamma_x(x,t)|\,dx = \int_{\mathbb{R}} \frac{|x|}{2t}\Gamma(x,t)\,dx = \frac{1}{2t(4\pi t)^{1/2}} \int_{\mathbb{R}} |x| e^{-\frac{x^2}{4t}}\,dx$

$$= \frac{1}{t(4\pi t)^{1/2}} \int_0^\infty x e^{-\frac{x^2}{4t}}\,dx$$

Put $y = \frac{x^2}{4t}$. Then $dy = \frac{x}{2t}dx$, and

$$\int_{\mathbb{R}} |\Gamma_x(x,t)|\,dx = \frac{1}{t(4\pi t)^{1/2}} \int_0^\infty 2t e^{-y}\,dy = \frac{2}{(4\pi t)^{1/2}} \int_0^\infty e^{-y}\,dy$$

$$= \frac{2}{(4\pi t)^{1/2}} \left(-e^{-y}\right)\Big|_0^\infty = \frac{2}{(4\pi t)^{1/2}}$$

Hence, $\displaystyle\int_{\mathbb{R}} |\Gamma_x(x,t)|\,dx = \frac{2}{(4\pi t)^{1/2}} < \infty \qquad (5).$

Thus, $\Gamma_x \in L^1(\mathbb{R})$. Since F is continuous and periodic in $\mathbb{R}$, it is bounded. Then the map $y \mapsto \Gamma_x(x-y,t)F(y)$ is in $L^1(\mathbb{R})$. Then (4) implies

$$u(x,t) = \int_{\mathbb{R}} \Gamma_x(x-y,t) F(y)\,dy = \Gamma_x * F(y) = \int_{\mathbb{R}} \Gamma_x(y,t) F(x-y)\,dy$$

Thus, $|u(x,t)| \leq \int_{\mathbb{R}} |\Gamma_x(y,t)|\,|F(x-y)|\,dy \leq \|F\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\Gamma_x(y,t)|\,dy$

$$= \|F\|_{L^\infty(\mathbb{R})} \frac{2}{(4\pi t)^{1/2}} \quad (\text{by } (5))$$

This estimate is true for all $x \in \mathbb{R}$. Thus, $\lim_{t\to\infty} u(x,t) = 0$. Moreover, the convergence is uniform in $x \in \mathbb{R}$. ✓

(3) Let $u(x,t)$ be a function satisfying

$$\begin{cases} \partial_t u - \Delta_x u + u^p = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ 0 \leq u(x,t) \leq M & \text{in } \mathbb{R}^n \times (0,\infty), \end{cases} \qquad (*)$$

with constants $M > 0$ and $0 < p < 1$. We'll show that there exists $T = T(M,p) > 0$ such that $u \equiv 0$ in $\mathbb{R}^n \times (T,\infty)$.

Put $C = \dfrac{M^{1-p}}{1-p} > 0$. Define a function $v : [0,\infty) \to \mathbb{R}$,

$$v(t) = \begin{cases} [(1-p)(C-t)]^{\frac{1}{1-p}} & \text{for } 0 \leq t < C, \\ 0 & \text{for } t \geq C. \end{cases} \qquad (1)$$

Then v is continuous on $[0,\infty)$. Moreover,

$$v(0) = [(1-p)C]^{\frac{1}{1-p}} = \left[(1-p)\frac{M^{1-p}}{1-p}\right]^{\frac{1}{1-p}} = M.$$

We have

$$v'(t) = \begin{cases} -(1-p)\frac{1}{1-p}[(1-p)(C-t)]^{\frac{1}{1-p}-1} & \text{for } 0 \leq t < C, \\ 0 & \text{for } t > C \end{cases}$$

$$= \begin{cases} -[(1-p)(C-t)]^{\frac{p}{1-p}} & \text{for } 0 \leq t < C, \\ 0 & \text{for } t > C \end{cases}$$

$= -v(t)^p$ for $t \geq 0$, $t \neq C$.

Thus, v is continuously differentiable on $[0,\infty) \setminus \{C\}$. Since v is continuous at C and $v'(t) = -v(t)^p$, the function v' can extend continuously to $t = C$. Therefore, $v \in C^1([0,\infty))$.

Now we view v as a function of two variables, namely

$$v(x,t) := v(t) \quad \forall (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Then $v \in C^{1,2}(\mathbb{R}^n \times [0,\infty))$ and satisfies

$$\begin{cases} \partial_t v = -v^p & \text{in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = M & \text{in } \mathbb{R}^n. \end{cases}$$

Denote the operator $H = \partial_t - \Delta_x$. Then by $(*)$ we get

$$Hu = \partial_t u - \Delta_x u = -u^p \quad \text{in } \mathbb{R}^n \times (0,\infty) \qquad (2)$$

Also, $Hv = \partial_t v - \underbrace{\Delta_x v}_{=0} = -v^p$ in $\mathbb{R}^n \times (0,\infty)$. $\quad (3)$.

Denote $u_0(x) := u(x,0)$ for all $x \in \mathbb{R}^n$. Then u solves the problem

$$\begin{cases} Hu = f_1(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \qquad (**)$$

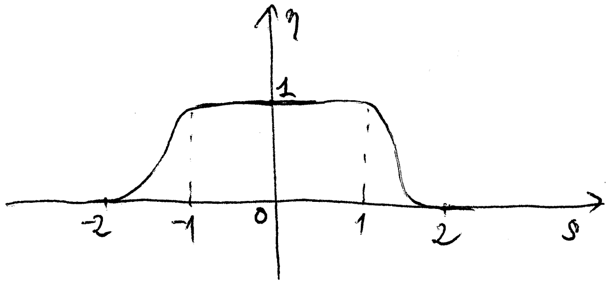
where $f_1(x,t) = -u(x,t)^p$. Because u is a bounded classical solution to the problem $(**)$, by Theorem 2.6, page 2-12, and Theorem 2.3, page 2-4, Safonov's lecture notes, it is given by the formula

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y, t-s) f_1(y,s)\, dy\, ds + \int_{\mathbb{R}^n} \Gamma(x-y,t) u_0(y)\, dy,$$

where $\Gamma(z,t) = (4\pi t)^{-n/2} \exp\left(-\frac{|z|^2}{4t}\right)$ for all $(z,t) \in \mathbb{R}^n \times (0,\infty)$.

Hence,
$$u(x,t) = -\int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-s)\, u(y,s)^p \, dy\, ds + \int_{\mathbb{R}^n} \Gamma(x-y,t)\, u_0(y)\, dy \quad (4)$$

Let $\eta: \mathbb{R} \to \mathbb{R}$ be a function in $C^\infty(\mathbb{R})$ satisfying



$$\begin{cases} 0 \le \eta(s) \le 1 & \forall s \in \mathbb{R}, \\ \eta(s) = 1 & \text{for } |s| \le 1, \\ \eta(s) = 0 & \text{for } |s| \geq 2. \end{cases}$$

For each $r > 0$, we define
$$u_r(x,t) := -\int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-s)\, u(y,s)^p \, dy\, ds + \int_{\mathbb{R}^n} \Gamma(x-y,t)\, u_0(y)\, \eta(r^{-1}|y|)\, dy \quad (5)$$

Then by theorem 2.3, page 2-4, Safonov's lecture notes, u_r solves the equations
$$\begin{cases} Hu_r = -u^p & \text{in } \mathbb{R}^n \times (0,\infty), \\ u_r(x,0) = u_0(x)\, \eta(r^{-1}|x|) & \text{in } \mathbb{R}^n. \end{cases} \quad (***)$$

Because η is bounded and that for each $y \in \mathbb{R}^n$, $\eta(r^{-1}|y|) \to 1$ as $r \to \infty$, by Lebesgue's Dominated Convergence theorem, we have
$$\int_{\mathbb{R}^n} \Gamma(x-y,t)\, u_0(y)\, \eta(r^{-1}|y|)\, dy \xrightarrow{r\to\infty} \int_{\mathbb{R}^n} \Gamma(x-y,t)\, u_0(y)\, dy$$

By comparing (4) and (5), we get
$$\lim_{r\to\infty} u_r(x,t) = u(x,t) \quad (6)$$

Moreover, because $u_0(y) \geq 0$ and $0 \le \eta \le 1$, (4) and (5) imply

$$u_r(x,t) \le u(x,t) \qquad \forall (x,t) \in \mathbb{R}^n \times [0,\infty) \qquad (7).$$

For each $\varepsilon > 0$, we define $v_\varepsilon(x,t) := v(x,t) + \varepsilon = v(t) + \varepsilon$ for $(x,t) \in \mathbb{R}^n \times [0,\infty)$.

By (***), $H(u_r - v_\varepsilon) = Hu_r - Hv = (-u^p) - (-v^p) = v^p - u^p \qquad (8).$

For each pair $R, \tau > 0$, we denote the set $Q_{R,\tau} := B_R(0) \times (0,\tau)$
$= \{(x,t) \in \mathbb{R}^n \times (0,\infty) : |x| < R,\ 0 < t < \tau\}$.

We'll show that with suitable choices of $R > 0$ and $\tau > 0$ (depending on r and ε), $u_r \le v_\varepsilon$ in $Q_{R,\tau}$. Consider two following situations.

▪ <u>Case 1</u> $u_r - v_\varepsilon$ attains maximum in $\overline{Q_{R,\tau}}$ at some $(x_0, t_0) \in Q_\tau$.

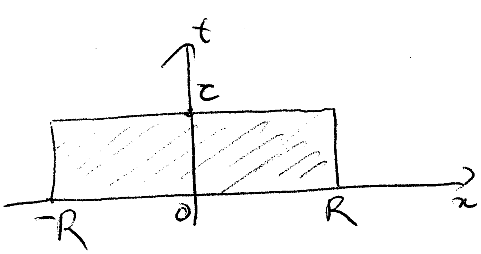
Then $\partial_t (u_r - v_\varepsilon)(x_0,t_0) = 0$, and $\Delta_x (u_r - v_\varepsilon)(x_0,t_0) \le 0$. Then $H(u_r - v_\varepsilon)(x_0,t_0) \ge 0$. Then by (8), $v^p(x_0,t_0) \ge u^p(x_0,t_0)$. Then $u(x_0,t_0) \le v(x_0,t_0)$. Then for any $(x,t) \in Q_\tau$,

$$u_r(x,t) - v_\varepsilon(x,t) \le u_r(x_0,t_0) - v_\varepsilon(x_0,t_0) \overset{(7)}{\le} u(x_0,t_0) - (v(x_0,t_0) + \varepsilon)$$
$$= (\underbrace{u(x_0,t_0) - v(x_0,t_0)}_{\le 0}) - \varepsilon < 0.$$

Therefore, in this case, $u_r \le v_\varepsilon$ in $Q_{R,\tau}$ for any choices of $R > 0$, $\tau > 0$.

▪ <u>Case 2</u> $u_r - v_\varepsilon$ attains maximum in $\overline{Q_{R,\tau}}$ at some point on $\partial Q_{R,\tau}$.

$$\partial Q_{R,\tau} = (B_R(0) \times \{0\}) \cup (\partial B_R(0) \times (0,\tau)) \cup (B_R(0) \times \{\tau\})$$



<u>Consider $(x,t) \in B_R(0) \times \{0\}$</u>

We have

$$u_r(x,0) \overset{(7)}{\le} u(x,0) \le M = v(0) < v(0) + \varepsilon = v_\varepsilon(x,0).$$

Consider $(x,t) \in \partial B_R(0) \times (0,\tau)$

We have $$u_r(x,t) \overset{(5)}{\leq} \int_{\mathbb{R}^n} \Gamma(x-y,t)\, u_0(y)\, \eta(r^{-1}|y|)\, dy$$

$$\leq \int_{B_{2r}(0)} \Gamma(x-y,t)\, u_0(y)\, dy \qquad \left(\text{since } \operatorname{supp}\eta \subset [-2,2] \text{ and } 0 \leq \eta \leq 1\right)$$

$$\leq M \int_{B_{2r}(0)} \Gamma(x-y,t)\, dy$$

$$= M \int_{B_{2r}(0)} (4\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{4t}\right) dy \qquad (9)$$

For $y \in B_{2r}(0)$ and $x \in \partial B_R(0)$ we have

$$|x-y|^2 = |x|^2 + |y|^2 - 2x\cdot y \geq R^2 + |y|^2 - 2R(2r)$$

Suppose $R \geq 8r$. Then $|x-y|^2 \geq R^2 + |y|^2 - 2R(2r) \geq R^2 + |y|^2 - \frac{R^2}{2} = \frac{R^2}{2} + |y|^2$.

Thus, $|x-y|^2 \geq \frac{R^2}{2} + |y|^2$. Then (9) implies

$$u_r(x,t) \leq M \int_{B_{2r}(0)} (4\pi t)^{-n/2} \exp\left(\frac{-\frac{R^2}{2} - |y|^2}{4t}\right) dy$$

$$= M \exp\left(-\frac{R^2}{8t}\right) \int_{B_{2r}(0)} (4\pi t)^{-n/2} \exp\left(-\frac{|y|^2}{4t}\right) dy$$

$$= M \exp\left(-\frac{R^2}{8t}\right) \int_{B_{2r}(0)} \Gamma(y,t)\, dy$$

$$\leq M \exp\left(-\frac{R^2}{8\tau}\right) \underbrace{\int_{\mathbb{R}^n} \Gamma(y,t)\, dy}_{=1}$$

$$= M \exp\left(-\frac{R^2}{8\tau}\right)$$

Suppose that $R > \sqrt{|8\tau \log(\varepsilon^{-1} M)|}$. Then $M \exp\left(-\frac{R^2}{8\tau}\right) < \varepsilon$. Thus $u_r(x,t) < \varepsilon$.

Hence, $u_r(x,t) \leq \varepsilon \leq v(t)+\varepsilon = v_\varepsilon(x,t)$. Therefore if $R > \max\{8r, \sqrt{8\tau \log(\varepsilon^{-1}M)}\}$ then $u_r(x,t) \leq v_\varepsilon(x,t)$.

Consider $(x,t) \in B_R(0) \times \{\tau\}$

In this case, we still have the estimation (9), namely

$$u_r(x,\tau) \leq M \int_{B_{2r}(0)} (4\pi\tau)^{-n/2} \exp\left(-\frac{|x-y|^2}{4\tau}\right) dy$$

$$= M \int_{B_{2r}(x)} (4\pi r)^{-n/2} \exp\left(-\frac{|y|^2}{4\tau}\right) dy$$

$$= M \int_{B_{2r}(x)} \Gamma(y,\tau)\, dy$$

$$\underline{\underline{z=\tau^{-1/2}y}} \quad M \int_{B'_\tau} \Gamma(z,1)\, dz, \qquad (10)$$

where B'_τ is the ball centered at $\tau^{-1/2}x$ with radius $\tau^{-1/2}(2r)$. We have

$$|B'_\tau| = (\tau^{-1/2}(2r))^n\, |B_1(0)|.$$

Since $\Gamma(z,1) \in L^1(\mathbb{R}^n)$, there exists $\alpha_\varepsilon > 0$ such that for any measurable subset $A \subset \mathbb{R}^n$ with $|A| < \alpha_\varepsilon$, $\int_A \Gamma(z,1)\, dz < \varepsilon$. Suppose

$$\tau > (2r)^2 \left(\frac{\alpha_{\varepsilon/M}}{|B_1(0)|}\right)^{-2/n}.$$

Then $|B'_\tau| < \alpha_{\varepsilon/M}$. Then $\int_{B'_\tau} \Gamma(z,1)\, dz < \frac{\varepsilon}{M}$. Then from (10) we get

$$u_r(x,\tau) \leq M \int_{B'_\tau} \Gamma(z,1)\, dz < M\frac{\varepsilon}{M} = \varepsilon$$

Thus, $u_r(x,\tau) < \varepsilon \leq v(\tau)+\varepsilon = v_\varepsilon(x,\tau)$.

In short, given $\varepsilon > 0$ and $r > 0$, we have $u_r \leq v_\varepsilon$ on $\partial Q_{R,\tau}$ provided that

$$\tau > \tau_{n,M,\varepsilon,r} := (2r)^2 \left(\frac{\alpha_{\varepsilon/M}}{|B_1(0)|} \right)^{-2/n} \qquad (11)$$

$$R > R_{M,\varepsilon,r,\tau} := \max \left\{ 8r, \sqrt{|8\tau \log(\varepsilon^{-1}M)|} \right\} \qquad (12)$$

Recall that we are in Case 2, i.e. the function $u_r - v_\varepsilon$ attains maximum in $\overline{Q_{R,\tau}}$ at some point on $\partial Q_{R,\tau}$. We have just showed that with R and τ satisfying (11) and (12), $u_r \leq v_\varepsilon$ on $\partial Q_{R,\tau}$. Therefore, $u_r \leq v_\varepsilon$ in $Q_{R,\tau}$.

Combining Case 1 and Case 2 together, we conclude that with R and τ satisfying (11) and (12), we have

$$u_r(x,t) \leq v_\varepsilon(x,t) \qquad \forall (x,t) \in Q_{R,\tau}.$$

Condition (12) allows ~~us~~ us to take any R sufficiently large. Thus,

$$u_r(x,t) \leq v_\varepsilon(x,t) \qquad \forall x \in \mathbb{R}^n, \ \forall 0 < t < \tau.$$

Now condition (11) allows us to take any τ sufficiently large. Thus,

$$u_r(x,t) \leq \underbrace{v_\varepsilon(x,t)}_{= v(t) + \varepsilon} \qquad \forall x \in \mathbb{R}^n, \ \forall t > 0.$$

Because this is true for all $\varepsilon > 0$, we have $u_r(x,t) \leq v(t) \ \ \forall x \in \mathbb{R}^n, \forall t > 0$.

Then by (6), $u(x,t) = \lim_{r \to \infty} u_r(x,t) \leq v(t)$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$. ✓

Thus, $0 \leq u(x,t) \leq v(t)$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$. By the definition of v at (1), we have $0 \leq u(x,t) \leq v(t) = 0$ for all $(x,t) \in \mathbb{R}^n \times (C,\infty)$. Define $T = T(M,p) := C = \dfrac{M^{1-p}}{1-p} > 0$. Then $u(x,t) = 0$ for all $(x,t) \in \mathbb{R}^n \times (T,\infty)$.

④ For $n \geq 1$, we denote $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Let $f \in C^1(\bar{B})$ and $u \in C^3(\bar{B})$ be functions satisfying

$$\begin{cases} \Delta u = f & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \qquad (*)$$

Put $M_0 = \sup_B |u|$, $\quad K_0 = \sup_B |f|$,

$M_1 = \sup_B |Du|$, $\quad K_1 = \sup_B |Df|$.

We'll find a number $N = N(n) > 0$ such that $M_0 + M_1 \leq N(K_0 + K_1)$.

Consider two cases of n, namely $n = 1$ and $n \geq 2$.

• $\underline{n = 1}$ Then $B = (-1, 1)$. Problem $(*)$ becomes

$$\begin{cases} u'' = f & \text{in } (-1,1) \\ u(-1) = u(1) = 0 \end{cases}$$

We have $u'(t) = \int_{-1}^{t} u''(s)\,ds + \alpha = \int_{-1}^{t} f(s)\,ds + \alpha \qquad (1),$

$$u(x) = \int_{-1}^{x} u'(t)\,dt + \underbrace{u(-1)}_{=0} = \int_{-1}^{x}\left(\int_{-1}^{t} f(s)\,ds + \alpha\right)dt = \int_{-1}^{x}\int_{-1}^{t} f(s)\,ds\,dt + \alpha(x+1) \qquad (2)$$

Since $u(1) = 0$, (2) implies $\int_{-1}^{1}\int_{-1}^{t} f(s)\,ds\,dt + 2\alpha = 0$.

Thus, $|\alpha| = \frac{1}{2}\left|\int_{-1}^{1}\int_{-1}^{t} f(s)\,ds\,dt\right| \leq \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} |f(s)|\,ds\,dt \leq \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} K_0\,ds\,dt = 2K_0$.

Hence, $|\alpha| \leq 2K_0 \qquad (3)$.

By (1), $|u'(t)| \leq \int_{-1}^{t} |f(s)|\,ds + |\alpha| \leq \int_{-1}^{1} K_0\,ds + 2K_0 = 4K_0$.

By (2), $|u(x)| \leq \int_{-1}^{x}\int_{-1}^{t} |f(s)|\,ds\,dt + |\alpha|(x+1) \leq \int_{-1}^{1}\int_{-1}^{1} K_0\,ds\,dt + 2K_0(1+1)$

$$= 8K_0.$$

Thus, $M_0 \leq 4K_0$ and $M_1 \leq 8K_0$. Pick $N = 12$. Then

$$M_0 + M_1 \leq 4K_0 + 8K_0 = 12K_0 \leq 12(K_0 + K_1) = N(K_0 + K_1).$$

$n \geq 2$

First, we'll show that $|u(x)| \leq \frac{K_0}{2n}(1 - |x|^2)$ for $x \in B$. Define the function $w(x) = \frac{K_0}{2n}(1 - |x|^2)$ for all $x \in \overline{B}$. Then $w(x) = \pm u(x) = 0$ on ∂B.

Moreover, $\Delta w = \frac{K_0}{2n}\Delta(1 - |x|^2) = \frac{K_0}{2n}(-2n) = -K_0 \leq -|f| = -|\Delta u|$.

Hence, $\Delta w \leq \Delta(-u)$ and $\Delta w \leq \Delta(u)$ for all $x \in B$. By the Comparison Principle, $-u \leq w$ and $u \leq w$ in B. Therefore,

$$|u(x)| \leq w(x) = \frac{K_0}{2n}(1 - |x|^2) \quad \forall x \in B \qquad (4)$$

As a consequence, $|u(x)| \leq \frac{K_0}{2n}$ for all $x \in B$. Thus $M_0 \leq \frac{K_0}{2n}$ (5).

Next, we'll show that $|Du(x)| \leq K_0$ for all $x \in \partial B$. Take any point $a \in \partial B$. Denote by $(a_1, a_2, \dots, a_n)$ the standard Cartesian coordinates of a. Let A be an orthonormal matrix of size $n \times n$ (to be chosen later). After the rotation by matrix A, we get new Cartesian coordinates $y = Ax$, with $y = (y_1, \dots, y_n)^T$ and $x = (x_1, \dots, x_n)^T$. Define the function

$$\tilde{u}(y_1, \dots, y_n) := u(x_1, \dots, x_n)$$

Let $(\alpha_1, \dots, \alpha_n)$ be the coordinates of a in the $(y_1, \dots, y_n)$-coordinate. We introduce the spherical coordinate $(r, \theta_1, \dots, \theta_{n-1})$ in $\mathbb{R}^n$ excluded a set of measure zero.

$$\begin{cases} y_1 = r\cos\theta_1, \\ y_2 = r\sin\theta_1\cos\theta_2, \\ \dots \\ y_{n-1} = r\sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1}, \\ y_n = r\sin\theta_1 \cdots \sin\theta_{n-2}\sin\theta_{n-1} \end{cases} \qquad (**)$$

where $r>0$, $0<\theta_1,\dots,\theta_{n-2}<\pi$ and $0<\theta_{n-1}<2\pi$.

Now we choose the matrix A such that the point $a(\alpha_1,\dots,\alpha_n)$ corresponds to $(r,\theta_1,\dots,\theta_{n-1})=(1,\frac{\pi}{2},\dots,\frac{\pi}{2})$. With relations $(**)$, we define the map

$$\tilde{\tilde{u}}(r,\theta_1,\dots,\theta_{n-1}):=\tilde{u}(y_1,\dots,y_n)=u(x_1,\dots,x_n)$$

By the chain rule of differentiation, we have

$$\underbrace{\begin{pmatrix}\partial\tilde{\tilde{u}}/\partial r\\ \partial\tilde{\tilde{u}}/\partial\theta_1\\ \vdots\\ \partial\tilde{\tilde{u}}/\partial\theta_{n-1}\end{pmatrix}}_{D\tilde{\tilde{u}}}=\underbrace{\begin{pmatrix}\partial y_1/\partial r & \cdots & \partial y_n/\partial r\\ \partial y_1/\partial\theta_1 & \cdots & \partial y_n/\partial\theta_1\\ \vdots & & \vdots\\ \partial y_1/\partial\theta_{n-1} & \cdots & \partial y_n/\partial\theta_{n-1}\end{pmatrix}}_{J}\underbrace{\begin{pmatrix}\partial\tilde{u}/\partial y_1\\ \partial\tilde{u}/\partial y_2\\ \vdots\\ \partial\tilde{u}/\partial y_n\end{pmatrix}}_{D\tilde{u}}$$

Thus, $D\tilde{u}=J^{-1}D\tilde{\tilde{u}}$ (6).

The first row of $J(a)$ is

$$\left(\frac{\partial y_1}{\partial r},\dots,\frac{\partial y_n}{\partial r}\right)\Bigg|_{(1,\frac{\pi}{2},\dots,\frac{\pi}{2})}=\left(\cos\theta_1,\sin\theta_1\cos\theta_2,\dots,\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1},\sin\theta_1\cdots\sin\theta_{n-1}\right)\Big|_{(1,\frac{\pi}{2},\dots,\frac{\pi}{2})}$$

$$=(0,\dots,0,1).$$

For $1\le i\le n-1$ and $1\le j\le n$, we have $\displaystyle\frac{\partial y_j}{\partial\theta_i}(1,\frac{\pi}{2},\dots,\frac{\pi}{2})=\begin{cases}-1 & \text{if } j=i+1,\\ 0 & \text{if } j\neq i+1.\end{cases}$

Thus, the $(i+1)$'th row of $J(a)$ is

$$\left(\frac{\partial y_1}{\partial\theta_i},\dots,\frac{\partial y_n}{\partial\theta_i}\right)\Bigg|_{(1,\frac{\pi}{2},\dots,\frac{\pi}{2})}=(0,\dots,\underset{\substack{\uparrow\\ i}}{-1},\dots,0)$$

Therefore, $$J(a)=\begin{pmatrix}0 & \cdots & 0 & 1\\ -1 & & & 0\\ & \ddots & & \vdots\\ & & -1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ -I_{n-1} & 0\end{pmatrix}$$

50/50. Excellent! but one can make it shorter.

Hence, $J^{-1}(a) = \begin{pmatrix} 0 & 1 \\ -I_{n-1} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -I_{n-1} \\ 1 & 0 \end{pmatrix}$ (7)

By the chain rule of differentiation,

$$Du(a) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)^T\Big|_a = A\left(\frac{\partial \tilde{u}}{\partial y_1}, \dots, \frac{\partial \tilde{u}}{\partial y_n}\right)^T\Big|_a = A\, D\tilde{u}(a).$$

Thanks to (6), we get

$$Du(a) = A\, J^{-1}(a)\, D\tilde{\tilde{u}}\left(1, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right) \qquad (8)$$

We have $D\tilde{\tilde{u}}\left(1, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right) = \left(\frac{\partial \tilde{\tilde{u}}}{\partial r}, \frac{\partial \tilde{\tilde{u}}}{\partial \theta_1}, \dots, \frac{\partial \tilde{\tilde{u}}}{\partial \theta_{n-1}}\right)\Big|_{(1, \frac{\pi}{2}, \dots, \frac{\pi}{2})}$.

Since $u = 0$ on ∂B, $\tilde{u}(1, \theta_1, \dots, \theta_{n-1}) = 0$ for all $0 < \theta_1, \dots, \theta_{n-2} < \pi$, $0 < \theta_{n-1} < 2\pi$.

Thus, $D\tilde{\tilde{u}}\left(1, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right) = \left(\frac{\partial \tilde{\tilde{u}}}{\partial r}\left(1, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right), 0, \dots, 0\right).$ (9)

By (4), we have $|\tilde{\tilde{u}}(r, \theta_1, \dots, \theta_n)| = |u(x)| \leq \frac{K_0}{2n}(1 - r^2)$.

Thus, $\left|\frac{\tilde{\tilde{u}}(r, \theta_1, \dots, \theta_{n-1}) - \tilde{\tilde{u}}(1, \theta_1, \dots, \theta_{n-1})}{r - 1}\right| = \frac{|\tilde{\tilde{u}}(r, \theta_1, \dots, \theta_n)|}{1 - r} \leq \frac{K_0}{2n}(1 + r) \leq \frac{K_0}{n}$

Letting $r \to 1^-$, we get $\left|\frac{\partial \tilde{\tilde{u}}}{\partial r}(1, \theta_1, \dots, \theta_{n-1})\right| \leq \frac{K_0}{n}$. Thus, (9) implies

$$\left|D\tilde{\tilde{u}}\left(1, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right)\right| \leq \frac{K_0}{n}. \qquad (10)$$

From (8) we have

$$\|Du(a)\| = \left\|A\, J^{-1}(a)\, D\tilde{\tilde{u}}\left(1, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right)\right\| \leq \|A\|\, \|J^{-1}(a)\|\, \left|D\tilde{\tilde{u}}\left(1, \frac{\pi}{2}, \dots, \frac{\pi}{2}\right)\right| \qquad (11)$$

Since A is an orthonormal matrix, $\|A\| = \sqrt{|A_1|^2 + \dots + |A_n|^2} = \sqrt{1 + \dots + 1} = \sqrt{n}$, where $A_1, \dots, A_n$ denote the rows of matrix A. Also,

$$\|J^{-1}(a)\| \overset{(7)}{=\!=\!=} \left\|\begin{pmatrix} 0 & -I_{n-1} \\ 1 & 0 \end{pmatrix}\right\| = \sqrt{1 + (-1)^2 + \dots + (-1)^2} = \sqrt{n}.$$

Then (11) implies $|Du(a)| \leq \sqrt{n}\sqrt{n}\, |D\tilde{u}(1, \frac{\pi}{2}, \dots, \frac{\pi}{2})|$

$$\overset{(10)}{\leq} \sqrt{n}\sqrt{n}\, \frac{K_0}{n} = K_0$$

Therefore, we have proved that $|Du(x)| \leq K_0 \quad \forall x \in \partial B$ (12).

Next, we'll show that $|Du(x)| \leq \sqrt{n}(K_0 + K_1)$ for all $x \in B$. For each $1 \leq i \leq n$, we have $D_i(\Delta u) = D_i f$. Thus, $\Delta(D_i u) = D_i f$. Put

$$v_i := D_i u,$$

$$\phi(x) = \frac{1}{2n} K_1 (1 - |x|^2) + K_0 \quad \forall x \in \overline{B}.$$

For any $x \in \partial B$, we have $|v_i(x)| = |D_i u(x)| \leq |Du(x)| \overset{(12)}{\leq} K_0 = \phi(x)$.

Moreover, $\Delta\phi = \frac{K_1}{2n} \Delta(1 - |x|^2) = \frac{K_1}{2n}(-2n) = -K_1$. Then

$$\Delta\phi(x) \leq -|Df| \leq -(D_i f) = \Delta(-v_i),$$

$$\Delta\phi(x) \leq -|Df| \leq D_i f = \Delta v_i.$$

By the Comparison principle, $-v_i(x) \leq \phi(x)$ and $v_i(x) \leq \phi(x)$ for all $x \in B$.

Thus $|v_i(x)| \leq \phi(x)$, or equivalently $|D_i u(x)| \leq \phi(x)$; for all $x \in B$. Then

$$|Du(x)| = \sqrt{|D_1 u(x)|^2 + \dots + |D_n u(x)|^2} \leq \sqrt{n\,\phi(x)^2} = \sqrt{n}\,\phi(x)$$

$$= \frac{\sqrt{n}}{2n} K_1(1 - |x|^2) + \sqrt{n}\, K_0$$

$$\leq \frac{1}{2\sqrt{n}} K_1 + \sqrt{n}\, K_0$$

$$\leq \sqrt{n}\,(K_1 + K_0). \qquad \forall x \in B.$$

Therefore, $M_1 \leq \sqrt{n}\,(K_1 + K_0)$ (13).

Adding (5) and (13) together, we get $M_0 + M_1 \leq \frac{K_0}{2n} + \sqrt{n}\,(K_1 + K_0) \leq \left(\frac{1}{2n} + \sqrt{n}\right)(K_0 + K_1)$

Therefore, we can choose $N = N(n) = \frac{1}{2n} + \sqrt{n}$ for $n \geq 2$.