Theory of Partial Differential Equations – Math 8583

Take-home final exam

1) Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$, with $n \geq 1$. Define $\Omega^* = \{x \in \mathbb{R}^n : |x|^{-2}x \in \Omega\}$ and $u^* : \Omega^* \to \mathbb{R}$, $u^*(X) = |x|^{2-n}u(|x|^{-2}x)$. Show that u^* is harmonic in Ω^*.

2) Denote $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$. Show that the problem

$$\begin{cases} \Delta u = u^2 \text{ in } B_1, \\ u(x) \to \infty \text{ as } x \to 1^-, \end{cases}$$

has a unique nonnegative solution $u \in C^2(B_1)$.

3) Let $u \in C^2(\mathbb{R})$ be a 1-periodic function.
(a) Show that

$$\int_0^1 (u - u_0)^2 dx \leq \int_0^1 u_x^2 dx,$$

where

$$u_0 = \int_0^1 u(t)dt.$$

(b) Show that

$$\int_0^1 u_x^2 dx \leq \int_0^1 u_{xx}^2 dx.$$

4) Let $u \in C^2(\mathbb{R} \times [0, \infty))$, $a \in C^1(\mathbb{R})$ be such that

$$u_t = a(x)u_{xx} \text{ in } \mathbb{R} \times (0, \infty),$$

$$u(x + 1, t) = u(x, t) \text{ in } \mathbb{R} \times (0, \infty),$$
$$a(x + 1) = a(x) \text{ in } \mathbb{R},$$
$$\nu \leq a(x) \leq \nu^{-1} \text{ in } \mathbb{R},$$

for some constant $\nu \in (0, 1]$. Show that there exists $\mu = \mu(\nu) > 0$ such that

$$\int_0^1 u_x^2(x, t)dx \leq e^{-\mu t} \int_0^1 u_x^2(x, 0)dx \quad \forall t > 0.$$

5) Denote $\mathbb{R}^n_+ = \{(x_1, ..., x_{n-1}, x_n) \in \mathbb{R}^n : x_n > 0\}$. Let $\alpha \in (0, 1)$.

(a) Show that there exists $c = c(n, \alpha) > 0$ such that the function $v(x) = |x|^\alpha + cx_n^\alpha$ satisfies $\Delta v \leq 0$ in $\mathbb{R}^n_+$.
(b) Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex subset. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution to the problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega, \end{cases}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ satisfies $|g(x) - g(y)| \leq |x - y|^\alpha$. Show that there exists $K = K(n, \alpha) > 0$ such that $|u(x) - u(y)| \leq K|x - y|^\alpha$.