Math 8583: Theory of Partial Differential Equations: Fall 2013

Homework #1 (due on Friday, September 27, till 4:30 pm)

50 points are distributed between 5 problems, 10 points each.

1. Show that Laplace's equation is invariant with respect to orthogonal transformations: if $u \in C^2(\Omega)$ satisfies $\Delta u(x) \equiv 0$ in Ω, then $v(y) := u(Ay)$ satisfies $\Delta v(y) \equiv 0$ in the domain $\{y \in \mathbb{R}^n : Ay \in \Omega\}$. Here A is any orthogonal matrix, i.e. its transposed matrix $A^T = A^{-1}$.

2. Let $f = f(x)$ be a continuous function on $[-1, 1]$. Show that the problem

$$u'' + f = 0 \quad \text{in} \quad (-1, 1), \qquad u(\pm 1) = 0, \tag{1}$$

has a unique solution $u \in C^2([-1, 1])$, which has the form

$$u(x) = \int_{-1}^{1} G(x, y)\, f(y)\, dy, \qquad -1 \le x \le 1,$$

where the function $G(x, y)$ (*Green's function* for this problem) does not depend on f. Find the explicit expression of $G(x, y)$ for $-1 \le x, y \le 1$.

3. For a given function $f \in C([-1, 1])$, let $u \in C^2([-1, 1])$ be a solution to the problem (1). Show that for arbitrary constant $\gamma \in (0, 1)$, there exists a constant N depending only on γ, such that

$$\sup_{(-1,1)} d^{-\gamma}|u| \le N \cdot \sup_{(-1,1)} d^{2-\gamma}|f|, \quad \text{where} \quad d = d(x) := 1 - |x|.$$

Show that the above estimate fails for $\gamma = 0$.

Hint. One can either use the previous problem, or the comparison function

$$v = A_1 d^{\gamma} \zeta + A_2(1 - x^2)$$

with some positive constants A_1, A_2 and a smooth function ζ satisfying

$$0 \le \zeta \le 1 \quad \text{on} \quad \mathbb{R}^1, \qquad \zeta(x) \equiv 0 \quad \text{for} \quad |x| \le \frac{1}{3}, \quad \zeta(x) \equiv 1 \quad \text{for} \quad |x| \ge \frac{2}{3}.$$

4. Let $K = \text{const} \ge 0$. Show that the problem

$$U'' + K\,|U'| + 1 = 0 \quad \text{in} \quad (-1, 1), \qquad U(\pm 1) = 0,$$

has a unique solution $U \in C^2([-1, 1])$. Moreover, show that any solution $u \in C^2([-1, 1])$ to the problem

$$u'' + pu' + 1 = 0 \quad \text{in} \quad (-1, 1), \qquad u(\pm 1) = 0, \quad \text{where} \quad |p(x)| \le K,$$

satisfies the estimate $0 \le u \le U$ in $(-1, 1)$.

5. Let $u = u(t, x)$ be a bounded smooth solution to the Cauchy problem

$$u_t = u_{xx} \quad \text{for} \quad t > 0, \qquad u(0, x) \equiv \exp(-x^4).$$

Show that $u(t, 0)$ is not an analytic function at the point $t = 0$.