**Math 8583: Theory of Partial Differential Equations: Fall 2013**

**Homework #2** (due on Wednesday, October 9, till 11:15 am)

50 points are distributed between 5 problems, 10 points each.

**1.** Let $\Omega$ be an open connected set in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class $C^2$. Let $u \in C^2_{loc}(\Omega) \cap C^1(\bar{\Omega})$ be a harmonic function in $\Omega$, such that

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad (\partial\Omega) \cap B_r(x_0)$$

for some $x_0 \in \partial\Omega$ and $r > 0$, where $\nu$ is the exterior unit normal to $\partial\Omega$. Show that $u \equiv 0$ in $\Omega$.

**2.** Let $u$ be a harmonic function in the unit ball

$$B_1(x_0) := \{x \in \mathbb{R}^n : \, |x - x_0| < 1\}.$$

Prove the bound for the gradient

$$|\nabla u(x_0)| \le n \cdot \left[\sup_{B_1} u - u(x_0)\right].$$

**3.** Show that there are no functions $u \in C^2_{loc}(\mathbb{R}^2_+) \cap C_{loc}(\overline{\mathbb{R}^2_+})$ satisfying the properties

$$u \ge 0, \quad \Delta u = 0 \quad \text{in} \quad \mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : \, x_2 > 0\}, \quad u(x_1, 0) \equiv x_1^2.$$

**4.** Let $u \in C^2_{loc}(\Omega) \cap C(\bar{\Omega}^1)$ be a solution to the problem

$$\Delta u = -1 \quad \text{in} \quad \Omega := (-1, 1) \times (-1, 1) \subset \mathbb{R}^2, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

**(i)** Show that $u$ cannot belong to $C^2(\bar{\Omega})$.
**(ii)** Show that

$$\sup_{\Omega} u = u(0).$$

**5.** Show that for any $p \in [1, n)$, there exist constants $\alpha \in (0, 1)$, $\nu \in (0, 1]$, and a real symmetric $n \times n$ matrix function $a(x) = [a_{ij}(x)]$ satisfying the uniform ellipticity condition

$$\nu|\xi|^2 \le (a(x)\xi, \xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \nu^{-1}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n \quad \text{and} \quad x \in \mathbb{R}^n,$$

and such that the function $u(x) := 1 - |x|^\alpha$ satisfies

$$Lu := \sum_{i,j=1}^{n} a_{ij} D_{ij} u = 0 \quad \text{a.e. in} \quad B_1 := \{x \in \mathbb{R}^n : \, |x| < 1\},$$

and its derivatives $D_{ij}u \in L^p(B_1)$.

**Remark.** This example shows that the maximum principle does not hold in general for uniformly elliptic equation with discontinuous coefficients in the Sobolev class $W^{2,p}$ when $p < n$.

## Hints

**1**. Use the fact that a function $v \in C(\Omega)$ is harmonic if and only if

$$\int v\Delta\phi\, dx = 0 \quad \text{for any function} \quad \phi \in C_0^\infty(\Omega).$$

Apply this fact to the function $v \in C(B_r(x_0))$, which is defined as follows

$$v \equiv u \quad \text{in} \quad \Omega \cap B_r(x_0), \qquad v \equiv 0 \quad \text{on} \quad B_r(x_0) \setminus \Omega.$$

**2.** Note that all the derivatives $D_i u$ are harmonic in $B_1 := B_1(x_0)$, so that by the mean value and divergence theorems,

$$D_i u(x_0) = \frac{1}{|B_1|} \int\limits_{B_1} D_i u\, dx = \frac{1}{|B_1|} \int\limits_{\partial B_1} u\nu_i\, dS_x.$$

where $\nu_i$ is the $i$-th component of the unit outward normal $\nu$ to $\partial B_1$. Obviously, for $x \in \partial B_1$, we have $\nu(x) = x - x_0$.

**3.** Suppose there exists such a function $u$. For arbitrary $R > 0$, compare $u$ with the solution $v \in C^2_{loc}(\mathbb{R}^2_+) \cap C(\overline{\mathbb{R}^2_+})$ to the problem

$$\Delta v = 0 \quad \text{in} \quad \mathbb{R}^2_+, \quad v(x_1, 0) \equiv x_1^2 \zeta(x_1),$$

where

$$\zeta \in C_0^\infty(-2R, 2R), \quad 0 \le \zeta \le 1, \quad \text{and} \quad \zeta \equiv 1 \quad \text{on} \quad [-R, R].$$

For any function $g \in C(\mathbb{R}^{n-1})$, $n \ge 2$, the solution $v \in C^2_{loc}(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ to the problem

$$\Delta v = 0 \quad \text{in} \quad \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}, \quad v(x', 0) \equiv g(x')$$

is given by the formula

$$v(x) = v(x', x_n) = \int\limits_{\mathbb{R}^{n-1}} \frac{2g(x' - x_n y')}{\sigma_n (1 + |y'|^2)^{n/2}} dy', \quad x' \in \mathbb{R}^{n-1},\ x_n \ge 0,$$

where $\sigma_n = |\partial B_1| = 2\pi^{n/2}/\Gamma(n/2)$.

**4.** By uniqueness, $u(x_1, x_2)$ must be an even function with respect to each of variables $x_1$ and $x_2$. Then consider the functions $v_k = D_k u$ for $k = 1$ and 2.