Math 8583: Theory of Partial Differential Equations: Fall 2013

Homework #3 (due on Wednesday, November 13, till 11:15 am)

50 points are distributed between 4 problems.

1. *(10 points).* Let $u \in C^4(\overline{B_1})$, where $B_1 := \{x \in \mathbb{R}^n : \ |x| < 1\}$. Suppose that

$$\Delta^2 u := \Delta(\Delta u) = 0 \quad \text{in} \quad B_1, \qquad u = |Du| = 0 \quad \text{on} \quad \partial B_1.$$

Show that $u \equiv 0$ in B_1.

2. *(10 points).* Suppose that $u \in C^\infty(B_2)$,

$$u > 0, \quad \Delta u = 0 \quad \text{in} \quad B_2 := \{x \in \mathbb{R}^n : \ |x| < 2\}.$$

Show that $|D(\ln u)| \le N$ in B_1, with a constant N depending only on n.

3. *(15 points).* Let functions $a_{ij} = a_{ij}(t,x)$ be defined for $i, j = 1, \cdots, n$; $t.0, x \in \mathbb{R}^n$, and satisfy the *uniform parabolicity condition*

$$a_{ij} = a_{ji}, \qquad \nu\,|\xi|^2 \le \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \le \nu^{-1}\,|\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{1}$$

with a constant $\nu \in (0,1]$. Consider the functions

$$K_{\alpha,\beta}(t,x) := t^{-\alpha} \exp\Big\{-\frac{|x|^2}{\beta t}\Big\} \quad \text{for} \quad t > 0, \ x \in \mathbb{R}^n.$$

Show that there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$, depending only on n and ν, such that

$$LK_{\alpha_1,\beta_1}(t,x) := \Big(\partial_t - \sum_{i,j=1}^n a_{ij}D_{ij}\Big)K_{\alpha_1,\beta_1}(t,x) \ge 0, \qquad LK_{\alpha_2,\beta_2}(t,x) \le 0 \quad \text{for all} \quad t > 0, \ x \in \mathbb{R}^n.$$

4. *(15 points).* Let $a_{ij} = a_{ij}(t,x)$ be functions satisfying (1) with a constant $\nu \in (0,1]$, and let $g = g(x)$ be a continuous function on $\mathbb{R}^n$. Use the previous result to show that the problem

$$Lu := \Big(\partial_t - \sum_{i,j=1}^n a_{ij}D_{ij}\Big)u = 0 \quad \text{in} \quad H_T := (0,T) \times \mathbb{R}^n, \qquad u(0,x) \equiv g(x)$$

has at most one classical solution $u \in C^{1,2}(H_T) \cap C(\overline{H_T})$ satisfying the inequality

$$|u(t,x)| \le N \cdot \exp\big(a\,|x|^2\big) \quad \text{in} \quad H_T,$$

where N and a are positive constants.

Hint. Use the **comparison principle:** if

$$Lu \le Lv \quad \text{in} \quad Q_{h,R} := (0,h) \times B_R, \quad \text{where} \quad B_R := \{x \in R^n : \ |x| < R\},$$

and

$$u \le v \quad \text{on} \quad \partial_p Q_{h,R} := (\partial Q_{h,R}) \setminus (h \times \overline{B_R}),$$

then $u \le v$ in $Q_{h,R}$. Take $v(t,x) := K_{\alpha,\beta}(h-t, ix)$.