Math 8583: Theory of Partial Differential Equations: Fall 2013

Homework #4 (due on Wednesday, December 4, till 11:15 am)

50 points are distributed between 4 problems.

1. *(14 points).* Let $g(x)$ be a bounded continuously differentiable function on $\mathbb{R}^n$, satisfying the inequality

$$|Dg(x) - Dg(y)| \le K |x-y|^{\alpha} \qquad \text{for all} \quad x, y \in \mathbb{R}^n$$

with some constants $K \ge 0$ and $\alpha \in (0,1)$. Let $u(x,t)$ be a bounded solution to the Cauchy problem

$$u_t = \Delta_x u \qquad \text{in} \qquad \mathbb{R}^n \times (0,\infty), \qquad\qquad u(x,0) \equiv g(x).$$

Show that

$$|u(x,t) - u(x,s)| \le NK |t-s|^{\frac{1+\alpha}{2}} \qquad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad t, s \in [0,\infty),$$

with a constant $N = N(n, \alpha)$.

2. *(12 points).* Let $u(x,t)$ be a bounded solution of the problem

$$u_t = u_{xx} \quad \text{in} \quad R^1 \times (0,\infty), \qquad u(x,0) = f(x),$$

where $f \in C(R^1)$, and $f(x) \equiv f(x+1)$. Show that there exists

$$\lim_{t\to\infty} u(x,t) = \int_0^1 f(y)\, dy.$$

Hints: From uniqueness it follows $u(x,t) \equiv u(x+1,t)$. By differentiation, one can show that

$$I(t) := \int_0^1 u(x,t)\, dx \equiv I(0 = \int_0^1 f(y)\, dy.$$

3. *(12 points).* Suppose $u(x,t)$ satisfies

$$0 \le u \le M, \quad u_t - \Delta u + u^p = 0 \quad \text{in} \quad R^n \times (0,\infty)$$

with constants $M > 0$, $0 < p < 1$. Show that $u \equiv 0$ on $\mathbb{R}^n \times (T,\infty)$ for some $T = T(M,p) > 0$.

Hint: Use as a barrier function the solution $v = v(t)$ of the problem

$$\frac{dv}{dt} + v^p = 0, \qquad\qquad v(0) = M.$$

Some adjustments are needed when the comparison principle is applied to unbounded domains.

4. *(12 points).* Let u and f be smooth functions satisfying

$$\Delta u = f \qquad \text{in} \quad B = \{x \in \mathbb{R}^n : \, |x| < 1\}, \qquad\qquad u = 0 \quad \text{on} \quad \partial B.$$

Derive the estimate $M_0 + M_1 \le N \cdot (K_0 + K_1)$ with a constant N depending only on n, where

$$M_0 = \sup_B |u|, \qquad M_1 = \sup_B |Du|, \qquad K_0 = \sup_B |f|, \qquad K_1 = \sup_B |Df|.$$

Hints: Step 1. $|u(x)| \le N_0 K_0 \left(1 - |x|^2\right)$.
Step 2. $|Du| \le N_1 K_0$ on ∂B.
Step 3. Differentiate the equality $\Delta u = f$.