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Math 8583: Theory of PDE

Take-home final

(3) Let $u \in C^2(\mathbb{R})$ be a function such that $u(x) = u(x+1)$ for all $x \in \mathbb{R}$.

(a) We'll show that $\int_0^1 (u-u_0)^2 dx \leq \int_0^1 u_x^2 dx$, where $u_0 = \int_0^1 u(t)dt$.

Because u is a function of only one variable, we can either write u_x or u' for the derivative of u. We have

$$u(x) - u_0 = \int_0^1 u(x)\,dt - \int_0^1 u(t)\,dt = \int_0^1 (u(x) - u(t))\,dt = \int_0^1 \int_t^x u'(s)\,ds\,dt.$$

Thus, $|u(x) - u_0| \leq \int_0^1 \left| \int_t^x u'(s)ds \right| dt \qquad (1)$

For $0 \leq t \leq x \leq 1$, $\left| \int_t^x u'(s)\,ds \right| \leq \int_t^x |u'(s)|\,ds \leq \int_0^1 |u'(s)|\,ds$.

For $0 \leq x < t < 1$, $\left| \int_t^x u'(s)\,ds \right| = \left| \int_x^t u'(s)\,ds \right| \leq \int_x^t |u'(s)|\,ds \leq \int_0^1 |u'(s)|\,ds$.

In both cases, we have $\left| \int_t^x u'(s)ds \right| \leq \int_0^1 |u'(s)|ds$. Then from (1) we get

$$|u(x) - u_0| \leq \int_0^1 \int_0^1 |u'(s)|\,ds\,dt = \int_0^1 |u'(s)|\,ds \quad \forall x \in [0,1].$$

Then by Schwarz's inequality,

$$|u(x) - u_0|^2 \leq \left(\int_0^1 |u'(s)|\,ds \right)^2 \leq \left(\int_0^1 ds \right) \left(\int_0^1 |u'(s)|^2 ds \right) = \int_0^1 |u'(s)|^2 ds.$$

Taking the integral over $x \in [0,1)$ of both sides, we get

$$\int_0^1 |u(x) - u_0|^2 dx \leq \int_0^1 \int_0^1 |u'(s)|^2 ds\,dx = \int_0^1 |u'(s)|^2 ds.$$

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(b) We'll show that $\int_0^1 u_x^2\,dx \le \int_0^1 u_{xx}^2\,dx$. We have

$$\int_0^1 u_x^2\,dx = \int_0^1 u_x u_x\,dx = u_x u\Big|_{x=0}^{x=1} - \int_0^1 u_{xx} u\,dx = u_x(1)u(1) - u_x(0)u(0) - \int_0^1 u_{xx} u\,dx.$$

Since u is 1-periodic, $u_x(1) = u_x(0)$ and $u(1) = u(0)$. Thus, the above equation is simply

$$\int_0^1 u_x^2\,dx = -\int_0^1 u_{xx} u\,dx. \qquad (2)$$

We have

$$\begin{aligned} RHS(2) &= -\int_0^1 u_{xx}(u - u_0)\,dx - u_0 \int_0^1 u_{xx}\,dx \\ &= -\int_0^1 u_{xx}(u-u_0)\,dx - u_0 \underbrace{\left(u_x(1) - u_x(0)\right)}_{=0} \\ &= -\int_0^1 u_{xx}(u-u_0)\,dx. \end{aligned}$$

Thus, (2) becomes

$$\int_0^1 u_x^2\,dx = -\int_0^1 u_{xx}(u-u_0)\,dx \qquad (3)$$

By Schwarz's inequality,

$$\begin{aligned} |RHS(3)| &= \left|\int_0^1 u_{xx}(u-u_0)\,dx\right| \\ &\le \left(\int_0^1 u_{xx}^2\,dx\right)^{1/2}\left(\int_0^1 (u-u_0)^2\,dx\right)^{1/2} \\ &\le \left(\int_0^1 u_{xx}^2\,dx\right)^{1/2}\left(\int_0^1 u_x^2\,dx\right)^{1/2} \quad \text{(by part (a))} \end{aligned}$$

Thus, (3) implies

$$\int_0^1 u_x^2\,dx \le \left(\int_0^1 u_{xx}^2\,dx\right)^{1/2}\left(\int_0^1 u_x^2\,dx\right)^{1/2}.$$

Therefore, $\left(\int_0^1 u_x^2\,dx\right)^{1/2} \le \left(\int_0^1 u_{xx}^2\,dx\right)^{1/2}$. Equivalently,

$$\int_0^1 u_x^2\,dx \le \int_0^1 u_{xx}^2\,dx.$$

④ Let $u \in C^2(\mathbb{R}\times[0,\infty))$, $a \in C^1(\mathbb{R})$ such that $u_t = a(x)u_{xx}$ in $\mathbb{R}\times[0,\infty)$

and $\begin{cases} u(x+1, t) \equiv u(x,t), \\ a(x+1) \equiv a(x), \\ \nu \leq a(x) \leq \nu^{-1}, \end{cases}$ for some constant $\nu \in (0,1]$.

We'll show that there exists $\mu = \mu(\nu) > 0$ such that

$$\int_0^1 u_x^2(x,t)\,dx \leq e^{-\mu t}\int_0^1 u_x^2(x,0)\,dx \qquad (1)$$

for all $t>0$.

(1) is equivalent to $e^{\mu t}\int_0^1 u_x^2(x,t)\,dx \leq \int_0^1 u_x^2(x,0)\,dx$, which can be written as $f(t)\leq f(0)$, where $f(t) = e^{\mu t}\int_0^1 u_x^2(x,t)\,dx$. Since $f\in C^1([0,\infty))$, it suffices to find $\mu(\nu)>0$ such that $f'(t)\leq 0$ for all $t>0$. Because $u\in C^2(\mathbb{R}\times[0,\infty))$, we have $f'(t) = \frac{d}{dt}\left[e^{\mu t}\int_0^1 u_x^2(x,t)\,dx\right]$

$$= \mu e^{\mu t}\int_0^1 u_x^2(x,t)\,dx + e^{\mu t}\int_0^1 \frac{d}{dt}\left[u_x^2(x,t)\right]dx$$

$$= \left[\mu\int_0^1 u_x^2(x,t)\,dx + 2\underbrace{\int_0^1 u_x(x,t)\,u_{tx}(x,t)\,dx}_{\{1\}}\right]e^{\mu t} \qquad (2)$$

By the integration-by-part formula,

$$\{1\} = \underbrace{u_x(x,t)\,u_t(x,t)\Big|_{x=0}^{x=1}}_{\{2\}} - \int_0^1 u_{xx}(x,t)\,u_t(x,t)\,dx$$

We have $\{2\} = u_x(1,t)u_t(1,t) - u_x(0,t)u_t(0,t)$. Since u is 1-periodic, so are u_x and u_t. Thus, $\{2\}=0$. Then $\{1\} = -\int_0^1 u_{xx}u_t\,dx = -\int_0^1 a(x)u_{xx}^2\,dx$

Since $a(x) \geq \nu > 0$, we get $\{1\} \leq -\nu \int_0^1 u_{xx}^2(x,t)\,dx$.

By Problem 3 above, $\int_0^1 u_x^2\,dx \leq \int_0^1 u_{xx}^2(x,t)\,dx$. Thus, $\{1\} \leq -\nu \int_0^1 u_x^2\,dx$.

Substituting this estimate into (2), we get

$$f'(t) \leq \left[\mu \int_0^1 u_x^2\,dx + 2\int_0^1 -\nu u_x^2\,dx\right] e^{\mu t}$$

$$= (\mu - 2\nu)\, e^{\mu t} \int_0^1 u_x^2\,dx$$

Choose $\mu = 2\nu$. Then $f'(t) \leq 0$ for all $t \geq 0$. Therefore, f is a decreasing function on $[0, \infty)$ and thus $f(t) \leq f(0)$ for all $t \geq 0$.

(5) Denote $\mathbb{R}^n_+ := \{(x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n : x_n > 0\}$.

Let $\alpha \in (0,1)$ be a fixed constant. We'll show that there is a constant $c = c(n,\alpha) > 0$ such that the function $v(x) = |x|^\alpha + c x_n^\alpha$ satisfies $\Delta v \leq 0$ in $\mathbb{R}^n_+$.

We know that $\frac{\partial |x|}{\partial x_i} = \frac{x_i}{|x|}$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $1 \leq i \leq n$. Thus,

$$\frac{\partial |x|^\alpha}{\partial x_i} = \alpha |x|^{\alpha-1} \frac{\partial |x|}{\partial x_i} = \alpha x_i |x|^{\alpha-2},$$

$$\frac{\partial^2 |x|^\alpha}{\partial x_i} = \alpha \frac{\partial}{\partial x_i}\left(x_i |x|^{\alpha-2}\right) = \alpha\left(|x|^{\alpha-2} + x_i \frac{\partial |x|^{\alpha-2}}{\partial x_i}\right)$$

$$= \alpha\left(|x|^{\alpha-2} + x_i(\alpha-2)|x|^{\alpha-3} \frac{\partial |x|}{\partial x_i}\right)$$

$$= \alpha |x|^{\alpha-4}\left(|x|^2 + (\alpha-2)x_i^2\right) \qquad (1)$$

In $\mathbb{R}^n_+$, we have $\frac{\partial (x_n^\alpha)}{\partial x_n} = \alpha x_n^{\alpha-1}$, $\quad \frac{\partial^2 (x_n^\alpha)}{\partial x_n} = \alpha(\alpha-1) x_n^{\alpha-2}$. $\qquad (2)$

Then for $1 \leq i \leq n-1$, $\frac{\partial^2 v}{\partial x_i^2} = \frac{\partial^2 |x|^\alpha}{\partial x_i^2} \overset{(1)}{=\!=} \alpha |x|^{\alpha-4}\left(|x|^2 + (\alpha-2)x_i^2\right)$. $\qquad (3)$

For $i=n$, $\dfrac{\partial^2 v}{\partial x_n^2} = \dfrac{\partial |x|^\alpha}{\partial x_n^2} + c\,\dfrac{\partial (x_n^\alpha)}{\partial x_n^2} \overset{(1),(2)}{=\!=} \alpha |x|^{\alpha-4}\left(|x|^2 + (\alpha-2)x_n^2\right)$
$$+ c\alpha(\alpha-1)x_n^{\alpha-2}. \qquad (4)$$

By (3) and (4) we get

$$\Delta v = \sum_{i=1}^{n-1} \frac{\partial^2 v}{\partial x_i^2} + \frac{\partial^2 v}{\partial x_n^2} = \sum_{i=1}^{n} \alpha |x|^{\alpha-4}\left(|x|^2 + (\alpha-2)x_i^2\right) + c\alpha(\alpha-1)x_n^{\alpha-2}$$

$$= \alpha |x|^{\alpha-4}\left(n|x|^2 + (\alpha-1)|x|^2\right) + c\alpha(\alpha-1)x_n^{\alpha-2}$$

$$= \underbrace{\alpha(n+\alpha-1)|x|^{\alpha-2}}_{\{1\}} + c\alpha(\alpha-1)x_n^{\alpha-2} \qquad (5)$$

Since $|x| \geq x_n$, and $\alpha - 2 < 0$, we have $|x|^{\alpha-2} \leq x_n^{\alpha-2}$. Since $n+\alpha-1 \geq 1+\alpha-1 = \alpha > 0$,

$$\{1\} \leqslant \alpha(n+\alpha-1)x_n^{\alpha-2}.$$

Thus, (5) implies $\Delta v \leq \{1\} + c\alpha(\alpha-1)x_n^{\alpha-2} \leq \alpha(n+\alpha-1)x_n^{\alpha-2} + c\alpha(\alpha-1)x_n^{\alpha-2}$

$$= \alpha(1-\alpha)\left(\frac{n+\alpha-1}{1-\alpha} - c\right)x_n^{\alpha-2}.$$

Choose $c = c(n,\alpha) = \dfrac{n+\alpha-1}{1-\alpha} > 0$. Then $\Delta v \leq 0$ in $\mathbb{R}^n_+$.

(b) Let Ω be an open bounded convex subset of $\mathbb{R}^n$, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a classical solution to the problem
$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

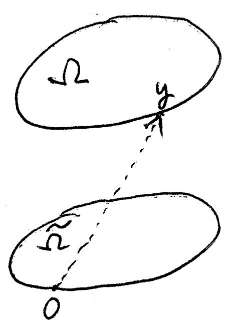
where $g : \mathbb{R}^n \to \mathbb{R}$ satisfies $|g(x) - g(y)| \leq |x-y|^\alpha$ for all $x, y \in \mathbb{R}^n$, with a constant $\alpha \in (0,1)$. We'll show that there exists a constant $K = K(n,\alpha) > 0$ such that $|u(x) - u(y)| \leqslant K|x-y|^\alpha$ for all $x \in \Omega$ and $y \in \partial\Omega$.

Fix $y \in \partial\Omega$. Put $\tilde{\Omega} = \{z - y : z \in \Omega\}$,
$$\tilde{u}(x) = u(x+y) - u(y) \quad \forall x \in \tilde{\Omega},$$
$$\tilde{g}(x) := g(x+y) - g(y), \quad \forall x \in \cancel{\tilde{\Omega}}\ \mathbb{R}^n.$$

Then $\begin{cases} \Delta \tilde{u} = 0 & \text{in } \tilde{\Omega}, \\ \tilde{u} = \tilde{g} & \text{on } \partial\tilde{\Omega}, \end{cases}$

and $|\tilde{g}(x) - \tilde{g}(z)| = |g(x+y) - g(z+y)| \leq |x-z|^\alpha$ for all $x, z \in \mathbb{R}^n$.

Moreover, $\tilde{u}(0) = \tilde{g}(0) = 0$. If we can find a constant $K = K(n,\alpha) > 0$ such that $|\tilde{u}(x)| \leq K|x|^\alpha$ for all $x \in \tilde{\Omega}$ then

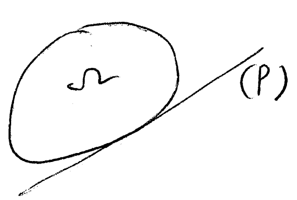


$$|u(x) - u(y)| \leq K|x-y|^\alpha, \quad \forall x \in \Omega.$$

In other words, we can assume from the beginning that $y = 0$, $u(0) = 0$ and $g(0) = 0$.

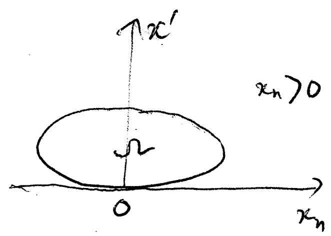
We have $|g(x)| = |g(x) - g(0)| \leq |x - 0|^\alpha = |x|^\alpha \ \forall x \in \mathbb{R}^n$. We want to find a constant $K = K(n,\alpha) > 0$ such that $|u(x)| \leq K|x|^\alpha \ \forall x \in \Omega$.

Because Ω is convex and $0 \in \partial\Omega$, by the Supporting Hyperplane theorem, there exists a hyperplane (P) such that Ω lies entirely in one half of the space $\mathbb{R}^n$ divided by (P).



We choose a Cartesian coordinate system $(\underbrace{x_1, \dots, x_{n-1}}_{x'}, x_n)$ in $\mathbb{R}^n$ such that $(P) = \{(x', x_n) \in \mathbb{R}^n : x_n = 0\}$,

$$\Omega \subset \mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}.$$



By Part (a), the function $v(x) = |x|^\alpha + c x_n^\alpha$ where $c = c(n,\alpha) = \dfrac{n+\alpha-1}{1-\alpha} > 0$ satisfies $\Delta v \lesssim 0$ in $\mathbb{R}^n_+$.

~~Thus,~~ ~~ut~~ Moreover, $|u(x)| \leq |g(x)| \leq |x|^\alpha \leq |x|^\alpha + c x_n^\alpha = v(x) \quad \forall x \in \partial\Omega$.

Thus, $\begin{cases} \Delta v \le 0 = \Delta u & \text{in } \Omega, \\ u \le v & \text{on } \partial\Omega \end{cases}$ and $\begin{cases} \Delta v \le 0 = \Delta(-u) & \text{in } \Omega, \\ -v \le v & \text{on } \partial\Omega. \end{cases}$

By the comparison principle, $u \le v$ in Ω and $-u \le v$ in Ω. Thus, $|u| \le v$ in Ω.

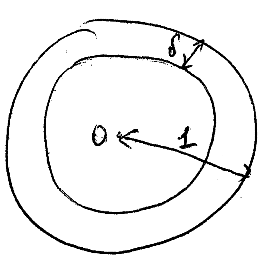
This implies $|u(x)| \le v(x) = |x|^\alpha + c x_n^\alpha \le |x|^\alpha + c|x|^\alpha = (1+c)|x|^\alpha$, $\forall x \in \Omega$.

Therefore, we found that $K = 1 + c = 1 + \dfrac{n+\alpha-1}{1-\alpha} = \dfrac{n}{1-\alpha}$.

(2) Denote $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$. Consider the problem of finding $u \in C^2(B_1)$ such that

$$\begin{cases} \Delta u = u^2 & \text{in } B_1, \\ u(x) \to \infty & \text{as } |x| \to 1^-. \end{cases} \qquad (*)$$

The second condition is interpreted as follows. For every $M > 0$, there exists $\delta \in (0,1)$ such that $u(x) > M$ for all $x \in B_1$, $|x| > 1 - \delta$.



We'll show that Problem $(*)$ has at most one non-negative solution, i.e. a solution u such that $u(x) \ge 0$ for all $x \in B_1$. We divide the solution into 3 steps.

Step 1: Denote the nonlinear differential operator $Tu = \Delta u - u^2$. We'll show that T satisfies the "comparison principle" on any open bounded subset Ω of $\mathbb{R}^n$ in sense that if $u_1, u_2 \in C^2(\bar\Omega) \cap C(\Omega)$ satisfies $Tu_1 \ge Tu_2$ in Ω and $0 \le u_1 \le u_2$ on $\partial\Omega$ then $u_1 \le u_2$ in Ω.

Put $u_0 = u_1 - u_2 \in C^2(\Omega) \cap C(\bar\Omega)$. Since $\bar\Omega$ is compact, u_0 attains maximum in $\bar\Omega$. If $u_0(a) = \max\limits_{\Omega} u_0(x)$ for some $a \in \Omega$ then $\Delta u_0(a) \le 0$. This means

$$0 \geq \Delta(u_1-u_2)(a) = \Delta u_1(a) - \Delta u_2(a) = u_1(a)^2 - u_2(a)^2 = u_0(a)\,(u_1(a)+u_2(a)).$$

Because $u_1(a), u_2(a) \geq 0$, we must have $u_0(a) \leq 0$. Then $u_0(x) \leq u_0(a) \leq 0$ $\forall x \in \Omega$.

Therefore, $u_1(x) \leq u_2(x)$ $\forall x \in \Omega$. If $u_0(b) = \max_{\bar\Omega} u_0(x)$ for some $b \in \partial\Omega$, then

$$u_0(x) \leq u_0(b) = u_1(b) - u_2(b) \leq 0 \quad \forall x \in \Omega.$$

Thus, $u_1(x) \leq u_2(x)$ for all $x \in \Omega$.

<u>Step 2</u>: For any $u \in C^2(B_1)$ satisfying $\Delta u = u^2$, we denote for each $r > 0$,

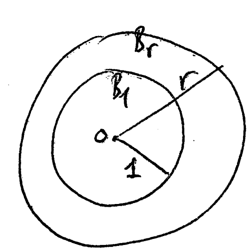
$$B_r = \{x \in \mathbb{R}^n : |x| < r\} \quad \text{and} \quad v_r(x) = \frac{1}{r^2} u\left(\frac{x}{r}\right) \quad \forall x \in B_r.$$

Then $\Delta v_r(x) \equiv v_r^2(x)$. Indeed, by the definition of v_r,

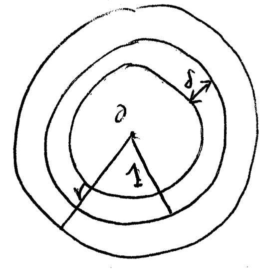
$$\Delta v_r(x) = \frac{1}{r^2}\,\frac{1}{r^2}\,\Delta u\left(\frac{x}{r}\right) = \frac{1}{r^4}\left(u\left(\frac{x}{r}\right)\right)^2 = \left(\frac{1}{r^2}\,u\left(\frac{x}{r}\right)\right)^2 = v_r^2(x), \quad \forall x \in B_r.$$

<u>Step 3</u>: Now we return to the original problem.

Suppose that $u_1, u_2 \in C^2(B_1)$ are two non-negative solutions to Problem (*). We'll show that $u_1 = u_2$ in B_1. Take any $r > 0$, we put $v(x) = \frac{1}{r^2} u_2\left(\frac{x}{r}\right)$ for all $x \in B_r$. By Step 2, $v \in C^2(B_r)$ and $\Delta v = v^2$ in B_r. In particular,



$$\begin{cases} v \in C^2(B_1) \cap C(\overline{B_1}), \\ v(x) \geq 0 \quad \forall x \in \overline{B_1}, \\ Tv = \Delta v - v^2 \equiv 0 \quad \text{in } B_1. \end{cases}$$



Put $M = \max_{\overline{B_1}} v(x)$. Because $u_1(x) \to \infty$ as $|x| \to 1^-$, there exists $\delta \in (0,1)$ such that $u_1(x) > M$ for all $x \in B_1$, $|x| > 1-\delta$.

For every $r_0 \in (1-\delta, 1)$, we have $u_1 \in C^2(B_{r_0}) \cap C(\overline{B_{r_0}})$ and

$$Tu_1 = \Delta u_1 - u_1^2 = 0 = Tv \quad \text{in } B_{r_0}.$$

Moreover, for any $x \in \partial B_{r_0}$, $u_1(x) > M \geq v(x) \geq 0$. Therefore, by Step 1, we get $u_1(x) \geq v(x)$ for all $x \in B_{r_0}$. In other words, $v(x) \leq u_1(x)$ for all $x \in B_1$, $|x| < r_0$. Because this inequality is true for all $r_0 \in (1-\delta, 1)$, we have $v(x) \leq u_1(x)$ $\forall x \in B_1$.

Thus, $\frac{1}{r^2} u_2\left(\frac{x}{r}\right) \leq u_1(x)$ $\forall x \in B_1$, $\forall r > 1$. Now fix $x \in B_1$. We have

$$u_2(x) = \lim_{r \to 1^+} \frac{1}{r^2} u_2\left(\frac{x}{r}\right) \leq u_1(x).$$

Thus, $u_2 \leq u_1$ in B_1. Now we can repeat the same procedure as above, but switch u_1 and u_2. Then we get $u_1 \leq u_2$ in B_1. Therefore, $u_1 = u_2$ in B_1.

1 Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$, with $n \geq 1$. Define $\Omega^* = \{x \in \mathbb{R}^n : |x|^{-2} x \in \Omega\}$ and $u^* : \Omega^* \longrightarrow \mathbb{R}$, $u^*(x) = |x|^{2-n} u(|x|^{-2} x)$. We'll show that u^* is also harmonic in Ω^*.

Denote $w(x) = |x|^{2-n}$ and $v(x) = |x|^{-2} x$. We have $v(x) = (v_1(x), \dots, v_n(x))$ where $v_i(x) = x_i |x|^{-2}$. We'll use the fact that $\Delta w = 0$ in $\mathbb{R}^n \setminus \{0\}$. We have

$$u^*(x) = w(x)\, u(v(x)) \quad \forall x \in \Omega^*.$$

Then for $1 \leq i \leq n$, we compute

$$\frac{\partial u^*}{\partial x_i} = \frac{\partial w}{\partial x_i}(x)\, u(v(x)) + w(x) \frac{\partial}{\partial x_i}\left[u(v(x))\right]$$

$$= \frac{\partial w}{\partial x_i}(x)\, u(v(x)) + w(x) \sum_{j=1}^{n} \frac{\partial u}{\partial y_j}(v(x)) \frac{\partial v_j}{\partial x_i}$$

Then

$$\frac{\partial^2 u^*}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}(x)\, u(v(x))\right) + \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(w(x)\frac{\partial u}{\partial x_j}(v(x))\frac{\partial v_j}{\partial x_i}(x)\right)$$

$$= \frac{\partial^2 w}{\partial x_i^2}(x)\, u(v(x)) + \frac{\partial w}{\partial x_i}(x)\frac{\partial}{\partial x_i}[u(v(x))] + \sum_{j=1}^{n}\frac{\partial w}{\partial x_i}\frac{\partial u}{\partial x_j}(v(x))\frac{\partial v_j}{\partial x_i}(x)$$

$$+\sum_{j=1}^{n} w(x)\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}(v(x))\right)\frac{\partial v_j}{\partial x_i}(x) + \sum_{j=1}^{n} w(x)\frac{\partial u}{\partial x_j}(v(x))\frac{\partial^2 v_j}{\partial x_i^2}(x)$$

$$= \frac{\partial^2 w}{\partial x_i^2}(x)\, u(v(x)) + 2\sum_{j=1}^{n}\frac{\partial w}{\partial x_i}\frac{\partial v_j}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(v(x)) +$$

$$+ w(x)\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial x_k \partial x_j}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_i} + w(x)\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial^2 v_j}{\partial x_i^2}(x)$$

Thus,

$$\Delta u^*(x) = \sum_{i=1}^{n}\frac{\partial^2 u^*}{\partial x_i^2} = \underbrace{\Delta w(x)\, u(v(x))}_{\{1\}} + \underbrace{2\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial w}{\partial x_i}\frac{\partial v_j}{\partial x_i}\right)\frac{\partial u}{\partial x_j}(v(x))}_{\{2\}} +$$

$$+ \underbrace{w(x)\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial x_k \partial x_j}\left(\sum_{i=1}^{n}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_i}\right)}_{\{3\}} + \underbrace{w(x)\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\sum_{i=1}^{n}\frac{\partial^2 v_j}{\partial x_i^2}}_{\{4\}} \qquad (1)$$

- <u>Compute $\{1\}$</u>

  Because $\Delta w \equiv 0$, $\{1\} = 0$. $\qquad (2)$

- <u>Compute $\{2\}$</u>

$$\frac{\partial w}{\partial x_i} = \frac{\partial}{\partial x_i}\left(|x|^{2-n}\right) = (2-n)|x|^{1-n}\frac{\partial |x|}{\partial x_i} = (2-n)\, x_i\, |x|^{-n},$$

$$\frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(x_j |x|^{-2}\right) = \delta_{ij}|x|^{-2} - 2x_i x_j |x|^{-4}, \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$$

Hence,

$$\sum_{i=1}^{n} \frac{\partial w}{\partial x_i}\frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{n} (2-n)x_i|x|^{-n}\left(\delta_{ij}|x|^{-2} - 2x_i x_j |x|^{-4}\right)$$

$$= (2-n)|x|^{-n-2}\sum_{i=1}^{n}\left(x_i\delta_{ij} - 2x_i^2 x_j |x|^{-2}\right)$$

$$= (2-n)|x|^{-n-2}\left(x_j - 2x_j|x|^2|x|^{-2}\right)$$

$$= -(2-n)|x|^{-n-2}x_j$$

Therefore, $\{2\} = 2\sum_{j=1}^{n}\left[-(2-n)|x|^{-n-2}\right]x_j \frac{\partial u}{\partial x_j}(v(x))$

$$= -2(2-n)|x|^{-n-2}\sum_{j=1}^{n} x_j \frac{\partial u}{\partial x_j}(v(x)). \qquad (3)$$

• Compute $\{3\}$

We have $\frac{\partial v_k}{\partial x_i} = |x|^{-2}\left(\delta_{ik} - 2x_i x_k |x|^{-2}\right)$,

$$\frac{\partial v_j}{\partial x_i} = |x|^{-2}\left(\delta_{ij} - 2x_i x_j |x|^{-2}\right).$$

Thus, $\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_i} = |x|^{-4}\left(\delta_{ik} - 2x_i x_k |x|^{-2}\right)\left(\delta_{ij} - 2x_i x_j |x|^{-2}\right)$

$$= |x|^{-4}\left(\delta_{ik}\delta_{ij} - 2x_i\delta_{ij}x_k|x|^{-2} - 2x_i\delta_{ik}x_j|x|^{-2} + 4x_i^2 x_j x_k |x|^{-4}\right).$$

Thus,

$$\sum_{i=1}^{n}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_i} = |x|^{-4}\left(\sum_{i=1}^{n}\delta_{ik}\delta_{ij} - 2\sum_{i=1}^{n}x_i\delta_{ij}x_k|x|^{-2} - 2\sum_{i=1}^{n}x_i\delta_{ik}x_j|x|^{-2} + 4\sum_{i=1}^{n}x_i^2x_jx_k|x|^{-4}\right)$$

$$= |x|^{-4}\left(\delta_{kj} - 2x_jx_k|x|^{-2} - 2x_kx_j|x|^{-2} + 4x_jx_k|x|^{-2}\right)$$

$$= |x|^{-4}\delta_{ij}.$$

Thus, $\{3\} = w(x)\sum_{j,k=1}^{n}\frac{\partial^2 u}{\partial x_k x_j}|x|^{-4}\delta_{ij} = w(x)|x|^{-4}\underbrace{\sum_{k=1}^{n}\frac{\partial^2 u}{\partial x_k^2}}_{=\Delta u = 0}$

Thus, $\{3\} = 0. \qquad (4)$

- Compute {4}

We have
$$\frac{\partial^2 v_j}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\delta_{ij}|x|^{-2} - 2x_i x_j |x|^{-4}\right)$$
$$= \delta_{ij}(-2)|x|^{-3}\frac{\partial |x|}{\partial x_i} - 2\frac{\partial}{\partial x_i}(x_i x_j)|x|^{-4} - 2x_i x_j \frac{\partial}{\partial x_i}(|x|^{-4})$$
$$= -2\delta_{ij} x_i |x|^{-4} - 2(x_j + \delta_{ij} x_j)|x|^{-4} + 8x_i x_j x_i |x|^{-6}$$

Then
$$\sum_{i=1}^{n} \frac{\partial^2 v_j}{\partial x_i^2} = -2x_j|x|^{-4} - 2(nx_j + x_j)|x|^{-4} + 8x_j|x|^2|x|^{-6}$$
$$= \left(-2x_j \cancel{+} -2(n+1)x_j + 8x_j\right)|x|^{-4}$$
$$= 2(2-n)x_j|x|^{-4}.$$

Thus,
$$\{4\} = w(x)\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\,2(2-n)x_j|x|^{-4}$$
$$= 2(2-n)|x|^{-2-n}\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}x_j$$
$$= -\{2\}. \quad (\text{because of } (3)) \qquad (5)$$

Substituting (2), (3), (4), (5) into (1), we get
$$\Delta u^* = 0 + \{2\} + 0 - \{2\} = 0.$$