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Math 8590: Topics in PDE

Homework #1

(A+)

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excellent!

- note the remark concerning cont. of $t \to u(t)$ at $t=0$ ($\in L^\infty_x$)
- nice proof under the smallness! (38)

Consider the model equations

$$\begin{cases} u_{it} + \dfrac{\partial}{\partial x_j}\left(u_i u_j + \dfrac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0 & \forall 1 \le i \le 3, \\ u(x, t_1) = u_0 \end{cases} \quad (I)$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $(x,t) \in \mathbb{R}^3 \times (t_1, t_2)$. We will do the following steps.

(a) Define mild solutions of the Cauchy problem (I).

(b) Outline a proof of a local-in-time existence result.

(c) Discuss the regularity of mild solutions.

(d) Investigate whether the equation has a conserved quantity, i.e. a quantity that doesn't change in time. Use this quantity to show the global-in-time existence of solutions.

We notice that the Cauchy problem (I) has the scaling-invariance property:

$$u \to u_\lambda = \lambda u(\lambda x, \lambda^2 t),$$

$$u_0 \to u_{0\lambda} = \lambda u_0(\lambda x),$$

where λ is any positive parameter. We have $\|u_{0\lambda}\|_{L^p(\mathbb{R}^3)} = \lambda^{1-\frac{3}{p}} \|u_0\|_{L^p(\mathbb{R}^3)}$.

Thus, the critical setting corresponds to the case $p=3$, i.e. $u_0 \in L^3(\mathbb{R}^3)$. We will proceed the steps (a)-(d) in, first, a subcritical setting $u_0 \in L^\infty(\mathbb{R}^3)$, and then the critical setting $u_0 \in L^3(\mathbb{R}^3)$.

Consider the subcritical setting $u_0 \in L^\infty(\mathbb{R}^3)$.

(a) Put $\mathcal{X}_{t_1,t_2} = L^\infty(\mathbb{R}^3 \times (t_1,t_2))$. Then $\mathcal{X}_{t_1,t_2}$ is a Banach space with respect to the norm $\|f\|_{\mathcal{X}_{t_1,t_2}} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{R}^3\times(t_1,t_2)} |f(x,t)|$. The given differential equation can be written as

$$u_{it} - \Delta u_i = -\frac{\partial}{\partial x_j}\left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right) \qquad (1)$$

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Put $G(u)_{ij} = -\left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right)$. Then (1) becomes $u_t - \Delta u = \operatorname{div} G(u)$.

Recall that the heat equation $\begin{cases} u_t - \Delta u = f, & 0<t<T \\ u(x,0) = u_0 \end{cases}$

under some assumptions on the decay of f as $x\to\infty$ has a unique solution

$$u_i(t) = \Gamma(t) * u_{0i} + \int_0^t \Gamma(t-s) * f_i(s)\, ds,$$

where $\Gamma(t) = (4\pi t)^{-3/2}\exp\left(-\frac{|x|^2}{4t}\right)$. Therefore, the heat equation

$$\begin{cases} u_t - \Delta u = f, & t_1 < t < t_2, \\ u(x,t_1) = u_0 . \end{cases}$$

has a solution $u_i(t) = \Gamma(t-t_1) * u_{0i} + \int_{t_1}^t \Gamma(t-s) * f_i(s)\, ds$.

Now we replace f by $\operatorname{div} G(u)$:

$$u_i(t) = \Gamma(t-t_1) * u_{0i} + \int_{t_1}^t \Gamma(t-s) * G(u)_{ij,j}(s)\, ds$$

$$= \Gamma(t-t_1) * u_{0i} + \int_{t_1}^t \Gamma_{,j}(t-s) * G(u)_{ij}(s)\, ds \qquad (2)$$

Put $K_j(x,t) = -\Gamma_{,j}(x,t) = -\frac{\partial \Gamma}{\partial x_j}(x,t) = \frac{2x_j}{(4\pi t)^{5/2}}\exp\left(-\frac{|x|^2}{4t}\right)$, (3) and ✓

$$K(x,t) = (K_1(x,t), K_2(x,t), K_3(x,t)) = \frac{2x}{(4\pi t)^{5/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Then (2) can be written as

$$u_i(t) = \Gamma(t-t_1) * u_{0i} + \int_{t_1}^{t} -K_j(t-s) * G(u)_{ij}\, ds$$

$$= \Gamma(t-t_1) * u_{0i} + \int_{t_1}^{t} K_j(t-s) * \left(u_i(s) u_j(s) + \frac{1}{2}\delta_{ij} |u(s)|^2\right) ds. \qquad (4)$$

Define a bilinear map $B : \mathcal{X}_{t_1,t_2} \times \mathcal{X}_{t_1,t_2} \to \mathcal{X}_{t_1,t_2}$,

$$B(u,v)_i(x,t) = \int_{t_1}^{t} K_j(t-s) * \left(u_i(s) v_j(s) + \frac{1}{2}\delta_{ij}\, u_k(s) v_k(s)\right) ds. \qquad (5)$$

Note that we can write $B(u,v)$ simply as

$$B(u,v) = \int_{t_1}^{t} K(t-s) * \left(u(s) \otimes v(s) + \frac{1}{2} u(s) \cdot v(s) I\right) ds. \qquad (6)$$

Equation (4) can be written as

$$u(t) = \Gamma(t-t_1) * u_0 + B(u,u)(x,t) \qquad (7).$$

We will call a function $u \in \mathcal{X}_{t_1,t_2}$ satisfying the equation (7) a <u>mild solution</u> to the Cauchy problem (I). However, we need to show that B is well-defined, i.e. to show that $B(u,v) \in \mathcal{X}_{t_1,t_2}$. From (6), we have

$$|B(u,v)(x,t)| \leq \int_{t_1}^{t} \left|K(t-s) * \left(u(s) \otimes v(s) + \frac{1}{2}(u(s) \cdot v(s))I\right)\right| ds$$

$$\leq \int_{t_1}^{t} \|K(t-s)\|_{L^1(\mathbb{R}^3)} \frac{3}{2} \|u\|_{\mathcal{X}_{t_1,t_2}} \|v\|_{\mathcal{X}_{t_1,t_2}}\, ds$$

$$= \frac{3}{2} \|u\|_{\mathcal{X}_{t_1,t_2}} \|v\|_{\mathcal{X}_{t_1,t_2}} \int_{t_1}^{t} \|K(t-s)\|_{L^1(\mathbb{R}^3)}\, ds \qquad (8)$$

We have $\int_{t_1}^{t} \|K(t-s)\|_{L^1(\mathbb{R}^3)}\, ds = \int_{0}^{t-t_1} \|K(s)\|_{L^1(\mathbb{R}^3)}\, ds. \qquad (9)$

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$$\|K(s)\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \frac{2|x|}{(4\pi s)^{5/2}} \exp\left(-\frac{|x|^2}{4s}\right) dx$$

$$\underset{z=\frac{x}{\sqrt{s}}}{=} \frac{1}{\sqrt{s}} \underbrace{\int_{\mathbb{R}^3} \frac{2|z|}{(4\pi)^{5/2}} \exp\left(-\frac{|z|^2}{4}\right) dz}_{A_1}$$

$$= \frac{A_1}{\sqrt{s}}. \qquad (10)$$

Then $LHS(9) = \int_0^{t-t_1} \|K(s)\|_{L^1(\mathbb{R}^3)} ds = \int_0^{t-t_1} \frac{A_1}{\sqrt{s}} ds = 2A_1\sqrt{t-t_1}.$

Then (8) implies $|B(u,v)(x,t)| \leq \frac{3}{2} \|u\|_{X_{t_1,t_2}} \|v\|_{X_{t_1,t_2}} 2A_1\sqrt{t-t_1}$

$$= 3 A_1\sqrt{t-t_1}\, \|u\|_{X_{t_1,t_2}} \|v\|_{X_{t_1,t_2}} \qquad (11)$$

$$\leq \underbrace{3A_1\sqrt{t_2-t_1}}_{C} \|u\|_{X_{t_1,t_2}} \|v\|_{X_{t_1,t_2}}$$

Thus, $\|B(u,v)\|_{X_{t_1,t_2}} \leq C\|u\|_{X_{t_1,t_2}} \|v\|_{X_{t_1,t_2}} < \infty.$ (12)

(b) We will outline the proof of a local-in-time existence of a mild solution in X_{t_1,t_2}. Put $U(x,t) = T(t-t_1) * u_0$. Then

$$|U(x,t)| \leq |T(t-t_1) * u_0| \leq \underbrace{\|T(t-t_1)\|_{L^1(\mathbb{R}^3)}}_{=1} \|u_0\|_{L^\infty(\mathbb{R}^3)} = \|u_0\|_{L^\infty}.$$

Thus, $\|U\|_{X_{t_1,t_2}} \leq \|u_0\|_{L^\infty}$. We can write (7) as

$$u = U + B(u,u) \qquad (13).$$

We recall the following lemma from the lecture in class on 02/19/2014:

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Let E be a Banach space and $B: E\times E \to E$ be a bilinear map. Suppose that B is continuous, i.e. there exists a number $C>0$ such that

$$\|B(x,y)\|_E \leq C\|x\|_E\|y\|_E \quad \forall x,y \in E.$$

Consider $a\in E$. If $4C\|a\|_E<1$ then the equation $x = a+B(x,x)$ has a ~~unique~~ solution in the ball $\overline{B}_R = \{x : \|x\| \leq R\}$ with $R = \dfrac{1+\sqrt{1-4C\|a\|_E}}{2C}$.

Moreover, it is the unique solution in that ball and can be obtained by taking the limit of any sequence $\begin{cases} x_0 \in \overline{B}_R, \\ x_{n+1} = a + B(x_n,x_n) \quad \forall n\geq 0. \end{cases}$

We now apply this lemma for $E = \mathcal{X}_{t_1,t_2}$ and $C = 3A_1\sqrt{t_2-t_1}$. Accordingly, if we have $4\cdot 3A_1\sqrt{t_2-t_1}\,\|U\|_{\mathcal{X}_{t_1,t_2}} < 1$ (14)

then (13) has a solution $u\in\mathcal{X}_{t_1,t_2}$, which is unique with respect to the condition

$$\|u\|_{\mathcal{X}_{t_1,t_2}} \leq \frac{1+\sqrt{1-4C\|U\|_{\mathcal{X}_{t_1,t_2}}}}{2C} \qquad (15)$$

Because $\|U\|_{\mathcal{X}_{t_1,t_2}} \leq \|u_0\|_{L^\infty}$, the condition (14) will be satisfied if we have

$$12A_1\sqrt{t_2-t_1}\,\|u_0\|_{L^\infty} < 1 \qquad (16)$$

By (15), we have $\|u\|_{\mathcal{X}_{t_1,t_2}} \leq \dfrac{1+1}{2C} = \dfrac{1}{3A_1\sqrt{t_2-t_1}}$. (17)

Our next goal is to show that u exists in a maximal time interval $[0,T^*)$. But first, we need to show that $u \in C_t L^\infty_x(\mathbb{R}^3\times[t_1,t_2])$, i.e. the

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map $t \in [t_1, t_2] \mapsto u(t) \in L^\infty_x$ is continuous. For $t_1 \le t < t+\tau \le t_2$, by (6) we have

$$B(u,u)(x,t+\tau) - B(u,u)(x,t) = \int_{t_1}^{t+\tau} K(t+\tau-s) * \left(u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right) ds - \int_{t_1}^{t} K(t-s) * \left(u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right) ds.$$

$$= \int_{t_1}^{t} \left(K(t+\tau-s) - K(t-s)\right) * \left(u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right) ds + \int_{t}^{t+\tau} K(t+\tau-s) * \left(u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right) ds$$

Hence,

$$|B(u,u)(x,t+\tau) - B(u,u)(x,t)| \le \int_{t_1}^{t} \left|K(t+\tau-s) - K(t-s)) * (u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right| ds + \int_{t}^{t+\tau} \left|K(t+\tau-s) * \left(u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right)\right| ds$$

$$\le \int_{t_1}^{t} \|K(t+\tau-s) - K(t-s)\|_{L^1} \frac{3}{2}\|u(s)\|^2_{L^\infty} ds + \int_{t}^{t+\tau} \|K(t+\tau-s)\|_{L^1} \frac{3}{2}\|u(s)\|^2_{L^\infty} ds$$

$$\le \left(\int_{t_1}^{t} \|K(t+\tau-s) - K(t-s)\|_{L^1} ds + \int_{t}^{t+\tau} \|K(t+\tau-s)\|_{L^1} ds\right) \frac{3}{2}\|u\|^2_{X_{t_1,t_2}}$$

$$= \Big(\underbrace{\int_0^{t-t_1} \|K(s+\tau) - K(s)\|_{L^1} ds}_{\{1\}} + \underbrace{\int_0^{\tau} \|K(s)\|_{L^1} ds}_{\{2\}}\Big) \frac{3}{2}\|u\|^2_{X_{t_1,t_2}} \qquad (18).$$

By (10), $\{2\} \le \int_0^\tau \frac{A_1}{\sqrt{s}} ds = 2\sqrt{\tau} A_1$. Thus, $\{2\} \to 0$ as $\tau \to 0$. We have

$$\{1\} = \int_0^{t-t_1}\int_{\mathbb{R}^3} |K(x,s+\tau) - K(x,s)| dx ds = \|K(\cdot + (0,\tau)) - K\|_{L^1(\mathbb{R}^3 \times (0,t-t_1))}$$

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This quantity will converge to 0 as $\tau \to 0$ if we can show that $K \in L^1(\mathbb{R}^3 \times (0,T))$ for all $T > 0$. By (10) we have

$$\int_0^T \int_{\mathbb{R}^3} |K(x,s)|\,dx\,ds = \int_0^T \frac{A_1}{\sqrt{s}}\,ds = 2A_1\sqrt{T} < \infty.$$

Therefore, we have proved that RHS(18) $\to 0$ as $\tau \to 0$. Moreover, this convergence is uniform in $(x,t) \in \mathbb{R}^3 \times [t_1,t_2]$. Thus, $|B(u,u)(x,t+\tau) - B(u,u)(x,t)| \to 0$ uniformly in $(x,t) \in \mathbb{R}^3 \times [t_1,t_2]$ as $\tau \to 0$. On the other hand,

$$\|\Gamma(t+\tau-t_1) * u_0 - \Gamma(t-t_1) * u_0\|_{L^\infty} = \|(\Gamma(t+\tau-t_1) - \Gamma(t-t_1)) * u_0\|_{L^\infty}$$
$$\leq \underbrace{\|\Gamma(t+\tau-t_1) - \Gamma(t-t_1)\|_{L^1}}_{\{3\}} \|u_0\|_{L^\infty}. \qquad (19)$$

We have $\Gamma(s) \in L^1_x$ for all $s > 0$. Thus $\{3\} \to 0$ as $\tau \to 0$. However, it is not clear whether this convergence is uniform in $t \in [t_1,t_2]$. Now that

$$u(t) = \Gamma(t) * u_0 + B(u,u),$$

we conclude that $\|u(t+\tau) - u(t)\|_{L^\infty_x} \to 0$ as $\tau \to 0$. Thus, $u \in C_t L^\infty_x$.

Return to the problem of showing the existence of a mild solution on a maximal time interval $[0,T^*)$. Consider the problem

$$\begin{cases} u_{it} + \dfrac{\partial}{\partial x_j}\left(u_i u_j + \dfrac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0, & t > 0 \\ u(x,0) = u_0 \end{cases} \qquad (\mathrm{II})$$

We proved earlier that if $12A_1\sqrt{T_1 - 0}\,\|u(\cdot,0)\|_{L^\infty} < 1$ then (II) has a unique mild solution $u \in L^\infty(\mathbb{R}^3 \times [0,T_1])$ such that

$$\|u\|_{\mathcal{X}_{0,T_1}} \leq \frac{1 + \sqrt{1 - 4C\|\Gamma(t) * u(\cdot,0)\|_{\mathcal{X}_{0,T_1}}}}{2C},$$

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where $C = 3A_1\sqrt{T_1 - 0}$. Moreover, $u(\cdot, T_1) \in L^\infty_x$ because $u \in C_t L^\infty_x(\mathbb{R}^3 \times [0, T_1])$.

Then (II) has a unique mild solution on $[T_1, T_2]$, where

$$12 A_1 \sqrt{T_2 - T_1}\, \| u(\cdot, T_1)\|_{L^\infty} < 1,$$

such that
$$\|u\|_{X_{T_1,T_2}} \leq \frac{1 + \sqrt{1 - 4C' \| \Gamma(t - T_1) * u(\cdot, T_1)\|_{X_{T_1,T_2}}}}{2C'}$$

where $C' = 3A_1\sqrt{T_2 - T_1}$. Continuing this process, we get a unique mild solution on a maximal time interval $[0, T^*)$ where $T^* = \lim_{n\to\infty} T_n \leq \infty$.

(c) We will discuss 2 regularity properties.

1) If $u_0, \nabla u_0 \in L^\infty(\mathbb{R}^3)$ then the mild solution to the Cauchy problem (I) is also a classical solution.

2) If $u_0, \nabla u_0 \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ then $u \in L^\infty(\mathbb{R}^3 \times (t_1, t_2)) \cap L^\infty_t H^1_x(\mathbb{R}^3 \times (t_1, t_2))$. ✓

<u>"Proof" of the first regularity property</u>

We recall that the mild solution was defined to be the Duhamel solution to the heat equation $u_t - \Delta u = \operatorname{div} G(u)$. To show that

$$u_{it} + \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0,$$

we only need to show that

(i) For each $x \in \mathbb{R}^3$, the function $t \in [t_1, t_2] \mapsto u(x, t)$ is continuous on $[t_1, t_2]$ and differentiable on (t_1, t_2).

(ii) For each $t \in (t_1, t_2)$, the function $x \in \mathbb{R}^3 \mapsto u(x, t)$ is twice differentiable.

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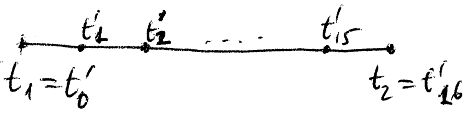
We have $u(x,t) = \Gamma(t-t_1) * u_0 + B(u,u)$. For $i=1,2,3$ and $h \in (-1,1)\setminus\{0\}$, denote

$$\Delta_i^h u(x,t) = \frac{u(x+he_i) - u(x)}{h}.$$

Then $\Delta_i^h u(x,t) = \Gamma(t-t_1) * \Delta_i^h u_0 + B(\Delta_i^h u, u) + B(u, \Delta_i^h u)$. (20)

Put

$$t'_k = t_1 + k\,\frac{t_2 - t_1}{16} \qquad \forall\, 0 \le k \le 16.$$



We have

$$\|B(\Delta_i^h u, u)\|_{X_{t_1,t'_1}} \le 3A_1\sqrt{t'_1 - t_1}\,\|\Delta_i^h u\|_{X_{t_1,t'_1}}\|u\|_{X_{t_1,t'_1}} \quad (\text{by } (12))$$

$$\le 3A_1\frac{\sqrt{t_2-t_1}}{4}\|\Delta_i^h u\|_{X_{t_1,t'_1}}\|u\|_{X_{t_1,t_2}}$$

$$\le \frac{1}{4}\|\Delta_i^h u\|_{X_{t_1,t'_1}} \quad (\text{by } (17))$$

Then by (20), we have

$$|\Delta_i^h u| \le |\Gamma(t-t_1)*\Delta_i^h u_0| + |B(\Delta_i^h u, u)| + |B(u, \Delta_i^h u)|$$

$$\le \underbrace{\|\Gamma(t-t_1)\|_{L^1(\mathbb{R}^3)}}_{=1}\|\Delta_i^h u_0\|_{L^\infty} + \frac{1}{4}\|\Delta_i^h u\|_{X_{t_1,t'_1}} + \frac{1}{4}\|\Delta_i^h u\|_{X_{t_1,t'_1}}$$

$$\le \|\Delta_i^h u_0\|_{L^\infty} + \frac{1}{2}\|\Delta_i^h u\|_{X_{t_1,t'_1}} \qquad \forall t \in (t_1, t'_1).$$

Thus, $\|\Delta_i^h u\|_{X_{t_1,t'_1}} \le 2\|\nabla u_0\|_{L^\infty}$ $\forall h \in (-1,1)\setminus\{0\}$. Therefore, $u(t) \in W^{1,\infty}(\mathbb{R}^3)$ for all $t \in (t_1, t'_1)$ and $\|D_i u\|_{X_{t_1,t'_1}} \le 2\|\nabla u_0\|_{L^\infty}$ (21).

Thus, $\|\nabla u\|_{X_{t_1,t'_1}} = \|\sqrt{D_i u\, D_i u}\|_{X_{t_1,t'_1}} \le 2\sqrt{3}\|\nabla u_0\|_{L^\infty}$.

Similarly, $\|\nabla u\|_{X_{t'_1,t'_2}} \le 2\sqrt{3}\,\|\nabla u(\cdot, t'_1)\| \le (2\sqrt{3})^2\|\nabla u_0\|_{L^\infty}$

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$\|\nabla u\|_{X_{t'_{15},t_2}} \le (2\sqrt{3})^{16}\|\nabla u_0\|_{L^\infty}$.

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Thus, $u(t) \in W^{1,\infty}(\mathbb{R}^3)$ for all $t \in (t_1, t_2)$. Perhaps, by taking higher derivatives with respect to x_i's of the equation $u(t) = \Gamma(t-t_1) * u_0 + B(u,u)$, we can show

✓ that $u(t) \in W^{m,\infty}(\mathbb{R}^3)$ for all $m \in \mathbb{N}$ and $t \in (t_1, t_2)$. By Sobolev's imbedding

✓ theorems, $u(t)$ is a smooth function in $x \in \mathbb{R}^3$. Thus (ii) is proved.

We know by (4) that $u_i(t) = \Gamma(t-t_1) * u_{0i} + \int_{t_1}^{t} K_j(t-s) * \left(u_i(s) u_j(s) + \frac{1}{2}\delta_{ij} |u(s)|^2\right) ds$.

(away from $t = t_1$) From the theory of heat equations, the map $\Gamma(t-t_1) * u_{0i}$ is smooth in t. In Part (b), we showed that $u \in C_t L_x^\infty(\mathbb{R}^3 \times [t_1, t_2])$. Thus, for each $x \in \mathbb{R}^3$, the map $t \in [t_1, t_2]$ (open int.) $\mapsto u(x,t)$ is continuous. It seems (must take open int.) to be true that the map

$$t \in (t_1, t_2) \mapsto \int_{t_1}^{t} K_j(t-s) * \left(u_i(s) u_j(s) + \frac{1}{2} |u(s)|^2\right) ds$$

is differentiable. Therefore, (i) is proved.

Proof of the second regularity property

We suppose that $u_0, \nabla u_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. In Part (b), we noticed that u is the limit of the sequence (u^n) where

$$\begin{cases} u^0 \equiv 0, \\ u^{n+1}(t) = \Gamma(t-t_1) * u_0 + B(u^n, u^n). \end{cases}$$

This sequence is contained in the ball $\overline{B}_R$ with $R = \dfrac{1 + \sqrt{1 - 4C\|\Gamma(t-t_0) * u_0\|_{\mathcal{X}_{t_1,t_2}}}}{2C}$

and $C = 3A_1\sqrt{t_2 - t_1}$. Consequently, $$\|u^n\|_{\mathcal{X}_{t_1,t_2}} \leq R \leq \frac{1}{C} = \frac{1}{3A\sqrt{t_2 - t_1}} \qquad (22)$$

We have $u^1(t) = \Gamma(t-t_1) * u_0 + B(u^0, u^0) = \Gamma(t-t_1) * u_0$.

Then $\|u^1(t)\|_{L^2} \leq \underbrace{\|\Gamma(t-t_1)\|_{L^1(\mathbb{R}^3)}}_{=1} \|u_0\|_{L^2} = \|u_0\|_{L^2} \quad \forall t \in (t_1, t_2)$.

Also, $\nabla[u^1(t)] = \Gamma(t-t_1) * \nabla u_0$. Thus,

$$\|\nabla[u^1(t)]\|_{L^2} \leq \|\Gamma(t-t_1)\|_{L^1(\mathbb{R}^3)} \|\nabla u_0\|_{L^2} = \|\nabla u_0\|_{L^2} \quad \forall t \in (t_1, t_2).$$

Thus, $u^1 \in L^\infty_t H^1_x$. For each $n \geq 0$, we put $v_n(t) = u^{n+1}(t) - u^n(t)$. Then $v_0 \in L^\infty_t H^1_x$.

We have
$$\begin{aligned} v_n(t) = u^{n+1}(t) - u^n(t) &= B(u^n, u^n) - B(u^{n-1}, u^{n-1}) \\ &= B(u^n, u^n - u^{n-1}) + B(u^n - u^{n-1}, u^{n-1}) \\ &= B(u^n, v_{n-1}) + B(v_{n-1}, u^{n-1}). \qquad (23) \end{aligned}$$

Put $\gamma_n = \sup_{t \in (t_1, t_2)} \|v_n(t)\|_{L^2_x}$. Then $\gamma_0 < \infty$. We have

$$\|B(u^n, v_{n-1})\|_{L^2_x} = \left\| \int_{t_1}^t K(t-s) * \left(u^n(s) \otimes v_{n-1}(s) + \frac{1}{2} u^n(s) \cdot v_{n-1}(s) I\right) ds \right\|_{L^2_x} \qquad (24)$$

We apply the following inequality, which is an integral form of Cauchy-Schwarz inequality: $\left\| \int_a^b g(x,t)\, dt \right\|_{L^2_x} \leq \int_a^b \|g(x,t)\|_{L^2_x}\, dt$. ✓

Then (24) implies

$$\begin{aligned} \|B(u^n, v_{n-1})\|_{L^2_x} &\leq \int_{t_1}^t \|K(t-s) * (u^n(s) \otimes v_{n-1}(s) + \tfrac{1}{2} u^n(s) \cdot v_{n-1}(s) I)\|_{L^2_x}\, ds \qquad (25) \\ &\leq \int_{t_1}^t \|K(t-s)\|_{L^1_x} \|u^n(s) \otimes v_{n-1}(s) + \tfrac{1}{2} u^n(s) \cdot v_{n-1}(s) I\|_{L^2_x}\, ds \\ &\overset{(10)}{\leq} \int_{t_1}^t \frac{A_1}{\sqrt{t-s}} \frac{3}{2} \|u^n(s)\|_{L^\infty_x} \|v_{n-1}(s)\|_{L^2_x}\, ds \\ &\leq \frac{3}{2} \|u^n\|_{\mathcal{X}_{t_1,t_2}} \sup_{s \in (t_1,t_2)} \|v_{n-1}(s)\| \int_{t_1}^t \frac{A_1}{\sqrt{t-s}}\, ds \end{aligned}$$

$$= 3\sqrt{t-t_1}\,\|u^n\|_{\mathcal{X}_{t_1,t_2}} \sup_{s\in(t_1,t)} \|v_{n-1}(s)\|.$$

For $t\in(t_1,t_1')$, we have

$$\|B(u^n, v_{n-1})\|_{L^2_x} \le 3\sqrt{t-t_1}\,\|u^n\|_{\mathcal{X}_{t_1,t_2}} \sup_{s\in(t_1,t)} \|v_{n-1}(s)\|$$
$$\le 3\sqrt{t_1'-t_1}\,\|u^n\|_{\mathcal{X}_{t_1,t_2}} \sup_{s\in(t_1,t_1')} \|v_{n-1}(s)\|$$
$$\overset{(22)}{\le} \frac{\sqrt{t_1'-t_1}}{\sqrt{t_2-t_1}} \sup_{s\in(t_1,t_1')} \|v_{n-1}(s)\|$$
$$= \frac{1}{4} \sup_{s\in(t_1,t_1')} \|v_{n-1}(s)\|$$

Similarly, $\|B(v_{n-1}, u^{n-1})\|_{L^2_x} \le \frac{1}{4} \sup_{s\in(t_1,t_1')} \|v_{n-1}(s)\| \quad \forall t\in(t_1,t_1')$.

Then (23) implies $\|v_n(t)\|_{L^2_x} \le \|B(u^n, v_{n-1})\|_{L^2_x} + \|B(v_{n-1}, u^{n-1})\|_{L^2_x}$

$$\le \frac{1}{2} \sup_{s\in(t_1,t_1')} \|v_{n-1}(s)\|_{L^2_x} \qquad \forall t\in(t_1,t_1').$$

Similarly, $\|v_n(t)\|_{L^2_x} \le \frac{1}{2} \sup_{s\in(t_j',t_{j+1}')} \|v_{n-1}(s)\|_{L^2_x} \quad \forall 1\le j\le 15,\ \forall t\in(t_j', t_{j+1}')$.

Thus, $\gamma_n \le \frac{1}{2}\gamma_{n-1}$. Then $\gamma_n<\infty$ for all $n\in\mathbb{N}$ and the series $\sum_{n=1}^{\infty}\gamma_n$ converges.

Thus, the sequence (u^n) is a Cauchy sequence in $L^\infty_t L^2_x(\mathbb{R}^3\times(t_1,t_2))$. Thus, its limit $u\in L^\infty_t L^2_x$.

Next, we'll show that $\nabla[u(t)]\in L^\infty_t L^2_x$. For $i=1,2,3$, we have

$$D_i[u(t)] = D_i\left[\Gamma(t-t_1)*u_0 + B(u,u)\right] = \Gamma(t-t_1)*D_i u_0 + B(D_i u, u) + B(u, D_i u).$$

By (21), $\|D_i u\|_{\mathcal{X}_{t_1,t_1'}}<\infty$. Thus,

$$\|D_i u(t)\|_{L^2_x} \le \underbrace{\|\Gamma(t-t_1)\|_{L^1_x}}_{=1} \|D_i u_0\|_{L^2_x} + \|B(D_i u, u)\|_{L^2_x} + \|B(u, D_i u)\|_{L^2_x} \qquad (26)$$

By the virtue of (25), we have

$$\|B(D_i u, u)\|_{L^2_x} \leq \int_{t_1}^{t} \|K(t-s)\|_{L^1_x} \left\| u(s)\otimes D_i u(s) + \frac{1}{2} u(s)\cdot D_i u(s) I \right\|_{L^2_x} ds$$

$$\overset{(10)}{\leq} \int_{t_1}^{t} \frac{A_1}{\sqrt{t-s}} \frac{3}{2} \|D_j u(s)\|_{L^\infty_x} \|u(s)\|_{L^2_x} ds$$

$$\leq \left(\int_{t_1}^{t} \frac{A_1}{\sqrt{t-s}} ds \right) \frac{3}{2} \|\nabla u\|_{L^\infty(\mathbb{R}^3\times(t_1,t_2))} \|u\|_{L^\infty_t L^2_x}$$

$$\leq 3A_1\sqrt{t_2-t_1} \underbrace{\|\nabla u\|_{L^\infty(\mathbb{R}^3\times(t_1,t_2))}}_{<\infty \text{ because of } (21)} \|u\|_{L^\infty_t L^2_x}. \quad \forall t\in(t_1,t_2)$$

Thus, $B(D_i u, u)\in L^\infty_t L^2_x$. Similarly, $B(u, D_i u)\in L^\infty_t L^2_x$. Then (26) implies $D_i u \in L^\infty_t L^2_x$. Therefore, we conclude that $u\in L^\infty_t H^1_x(\mathbb{R}^3\times(t_1,t_2))$. ✓

(d) We will identify a conserved quantity associate with a mild solution to the problem

$$\begin{cases} u_{it} + \dfrac{\partial}{\partial x_j}\left(u_i u_j + \dfrac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0 & \forall t\in(t_1,t_2) \\ u(\cdot,t_1) = u_0 \end{cases} \qquad (I)$$

and show that the classical solution exists for $t\in(0,\infty)$ when a smallness condition of u_0 is satisfied. By Part (b), if $12A_1\sqrt{t_2-t_1}\,\|u_0\|_{L^\infty} < 1$ then (I) has a mild solution. By the first regularity property in Part (c), this is also a classical solution of (I). Assume that $u_0, \nabla u_0 \in L^2_x \cap L^\infty_x$. By the second regularity property in Part (c), $u\in L^\infty(\mathbb{R}^3\times(t_1,t_2)) \cap L^\infty_t H^1_x(\mathbb{R}^3\times(t_1,t_2))$. Multiplying both sides of the differential equation of (I) by u_i (the sum over $i=1,2,3$ is understood) and taking integration over $\mathbb{R}^3$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 dx + \underbrace{\int_{\mathbb{R}^3} \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right) u_i \, dx}_{\{4\}} + \int_{\mathbb{R}^3} |\nabla u|^2 dx = 0 \qquad (27)$$

We have

$$\{4\} = -\int_{\mathbb{R}^3} \left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right) u_{i,j}\, dx = -\int_{\mathbb{R}^3} u_i u_j u_{i,j}\, dx - \frac{1}{2}\int_{\mathbb{R}^3} \delta_{ij}|u|^2 u_{i,j}\, dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^3} (u_i u_i)_{,j} u_j \, dx - \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 u_{i,i}\, dx$$

$$= +\frac{1}{2}\int_{\mathbb{R}^3} u_i u_i u_{j,j}\, dx - \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 u_{i,i}\, dx$$

$$= 0.$$

Then (27) becomes $\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx = 0$.

Taking integration both sides over $[t_1, t]$, we get

$$\int_{\mathbb{R}^3} |u(x,t)|^2 dx + \int_{t_1}^{t}\int_{\mathbb{R}^3} |\nabla u(x,s)|^2 dx\, ds = \int_{\mathbb{R}^3} |u_0|^2 dx. \qquad (28)$$

Therefore, LHS (28) is a <u>conserved quantity</u>. Moreover, $\|u(t)\|_{L^2_x} \le \|u_0\|_{L^2_x}$ for all $t \in [t_1, t_2]$. Now we consider the problem

$$\begin{cases} u_{it} + \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0, & t > 0 \\ u(\cdot, 0) = u_0, \end{cases} \qquad (II)$$

where $u_0, \nabla u_0 \in L^\infty_x \cap L^2_x$. We rule out the case $u_0 \equiv 0$ because in that case $u \equiv 0$ is obviously a solution of (II). For $t \ge 0$, we put

$$V(t) = \|u(t)\|_{L^\infty_x},$$

$$W(t) = \|u(t)\|_{L^2_x}.$$

Let $[0, T^*)$ be the maximal time-interval of existence to the problem (II).

By the continuation method as described in Part (6), a necessary condition for $T^* < \infty$ is that $\lim\limits_{t \to (T^*)^-} V(t) = \infty$. We'll show that under some smallness condition of u_0, this possibility doesn't happen. By the definition of a mild solution,

$$u(t) = \Gamma(t) * u_0 + \int_0^t K(t-s) * \left(u(s) \otimes u(s) + \frac{1}{2} |u(s)|^2 I\right) ds.$$

Thus,

$$|u(t)| \le \underbrace{\|\Gamma(t)\|_{L^1(\mathbb{R}^3)}}_{=1} \|u_0\|_{L^\infty} + \int_0^t \underbrace{\left| K(t-s) * \left(u(s) \otimes u(s) + \frac{1}{2} |u(s)|^2 I\right)\right|}_{\{5\}} ds \qquad (29)$$

There are two ways to estimate $\{5\}$. On one hand,

$$\{5\} \le \|K(t-s)\|_{L^\infty_x} \left\|u(s) \otimes u(s) + \frac{1}{2} |u(s)|^2 I\right\|_{L^1_x}$$

$$\le \frac{3}{2} \|K(t-s)\|_{L^\infty_x} \|u(s)\|^2_{L^2_x}$$

$$\le \frac{3}{2} \|u_0\|_{L^2_x} \|K(t-s)\|_{L^\infty_x}. \qquad (30)$$

Recall that $|K(x,t)| = \dfrac{2|x|}{(4\pi t)^{5/2}} \exp\left(-\dfrac{|x|^2}{4t}\right) = f(r)$, where $r = |x|$.

$$f'(r) = (4\pi t)^{-5/2} \frac{2t - r^2}{t} \exp\left(-\frac{r^2}{4t}\right).$$

r	0		$\sqrt{2t}$		∞
$f'(r)$		$+$	0	$-$	
$f(r)$	0	$\nearrow$		$\searrow$	0

$$\|K(t)\|_{L^\infty} = \max_{r > 0} f(r) = f(\sqrt{2t}) = \frac{2\sqrt{2}\, e^{-1/2}}{(4\pi)^{5/2}} \frac{1}{t^2}$$

Then (30) becomes:

$$\{5\} \leq \underbrace{\frac{3\sqrt{2}\, e^{-1/2}}{(4\pi)^{5/2}}}_{A_2} \frac{\|u_0\|_{L^2}^2}{(t-s)^2} = \frac{A_2 W(0)^2}{(t-s)^2}. \quad (31)$$

On the other hand,
$$\{5\} \leq \|K(t-s)\|_{L^1_x} \|u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\|_{L^\infty_x}$$
$$\overset{(10)}{\leq} \frac{A_1}{\sqrt{t-s}} \frac{3}{2} \|u(s)\|_{L^\infty_x}^2$$
$$\leq \frac{3A_1}{2\sqrt{t-s}} V(s)^2. \quad (32)$$

By (31) and (32),
$$\{5\} \leq \min\left\{ \frac{A_2 W(0)^2}{(t-s)^2}, \frac{3A_1 V(s)^2}{2\sqrt{t-s}} \right\}.$$

Then (29) ~~becomes~~ implies
$$|u(t)| \leq \|u_0\|_{L^\infty} + \int_0^t \min\left\{ \frac{A_2 W(0)^2}{(t-s)^2}, \frac{3A_1 V(s)^2}{2\sqrt{t-s}} \right\} ds \qquad \forall x \in \mathbb{R}^3.$$

Hence,
$$V(t) \leq V(0) + \int_0^t \min\left\{ \frac{A_2 W(0)^2}{(t-s)^2}, \frac{3A_1 V(s)^2}{2\sqrt{t-s}} \right\} ds \qquad \forall t \in [0, T^*).$$

In part (b), we showed that $u \in C_t L^\infty$ (away from 0). Thus, V is continuous on $[0, T^*)$.

Suppose that there exists a continuous function $\varphi: [0, T^*) \to \mathbb{R}$ such that $\varphi(0) > V(0)$ and
$$\varphi(t) \geq V(0) + \int_0^t \min\left\{ \frac{A_2 W(0)^2}{(t-s)^2}, \frac{3A_1 \varphi(s)^2}{2\sqrt{t-s}} \right\} ds. \quad (33)$$

Then $V(t) < \varphi(t)$ for all $t \in [0, T^*)$. Indeed, suppose otherwise. Then there exists $t_0 \in (0, T^*)$ such that $\varphi(t_0) \geq V(t_0)$. ~~and~~ By the continuity of φ and V, t_0 can be chosen to be minimum. Then $\varphi(t_0) = V(t_0)$ and $\varphi(s) > V(s)$ for all

$0 \le s < t_0$. We have

$$\varphi(t_0) \geq V(0) + \int_0^{t_0} \min\left\{ \frac{A_2 W(0)^2}{(t_0-s)^2}, \frac{3A_1 \varphi(s)^2}{2\sqrt{t_0-s}} \right\} ds$$

$$\geq V(0) + \int_0^{t_0} \min\left\{ \frac{A_2 W(0)^2}{(t_0-s)^2}, \frac{3A_1 V(s)^2}{2\sqrt{t_0-s}} \right\} ds$$

$$\geq V(t_0).$$

This means the equalities must hold. This happens only if

$$\min\left\{ \frac{A_2 W(0)^2}{(t_0-s)^2}, \frac{3A_1 \varphi(s)^2}{2\sqrt{t_0-s}} \right\} = \frac{A_2 W(0)^2}{(t_0-s)^2} \quad \text{for ~~all~~ almost every } s \in (0,t_0).$$

This is impossible because $\int_0^{t_0} \frac{A_2 W(0)^2}{(t_0-s)^2} ds = \infty$. (Note that $W(0) > 0$ because $u_0 \not\equiv 0$).

We choose $\varphi(t) \equiv (1+A)V(0)$ where $A > 0$ is a constant to be determined. Then (33) is equivalent to

$$A\, V(0) \geq \int_0^t \min\left\{ \frac{A_2 W(0)^2}{(t-s)^2}, \frac{3A_1(1+A)^2 V(0)^2}{\sqrt{t-s}} \right\} ds$$

$$= \int_0^t \min\left\{ \frac{A_2 W(0)^2}{s^2}, \frac{3A_1(1+A)^2 V(0)^2}{\sqrt{s}} \right\} ds. \qquad (34)$$

We have $\frac{A_2 W(0)^2}{s^2} \geq \frac{3A_1(1+A)^2 V(0)^2}{\sqrt{s}} \iff s \le s_0 = \left(\frac{2A_2 W(0)^2}{3A_1(1+A)^2 V(0)^2} \right)^{2/3}$.

Then

$$\int_0^{\infty} \min\left\{ \frac{A_2 W(0)^2}{s^2}, \frac{3A_1(1+A)^2 V(0)^2}{\sqrt{s}} \right\} ds = \int_0^{s_0} \frac{3A_1(1+A)^2 V(0)^2}{\sqrt{s}} ds + \int_{s_0}^{\infty} \frac{A_2 W(0)^2}{s^2} ds$$

$$= 3A_1(1+A)^2 V(0)^2 \sqrt{s_0} + \frac{A_2 W(0)^2}{s_0}$$

$$= \frac{3}{2}(2A_2)^{1/3}[3A_1(1+A)^2]^{2/3}(V(0)W(0)^2)^{1/3} V(0)$$

If we have

$$A\,V(0) \geq \frac{3}{2}(2A_2)^{1/3}[3A_1(1+A)^2]^{2/3}(V(0)W(0)^2)^{1/3} V(0) \qquad (35)$$

then (34) is satisfied for all $t>0$. ~~Then~~ The condition (35) is equivalent to

$$V(0)\,W(0)^2 \le \frac{4}{243 A_1^2 A_2}\,\frac{A^3}{(1+A)^4} \qquad (36)$$

The condition (36) is satisfied for some $A>0$ if and only if

$$V(0)\,W(0)^2 \le \frac{4}{243 A_1^2 A_2}\,\max_{A>0}\frac{A^3}{(1+A)^4} \qquad (37)$$

Put $g(A)=\dfrac{A^3}{(1+A)^4}$. Then $g'(A)=\dfrac{A^2(1+A)^3(3-A)}{(1+A)^8}$.

Thus, $\max_{A>0} g(A) = g(3) = \dfrac{3^3}{4^4}$. Then (37) is equivalent to

$$\|u_0\|_{L^\infty}\,\|u_0\|_{L^2}^2 \le \frac{1}{576 A_1^2 A_2}, \qquad (38)$$

, nice condition! make its connection to $\|u_0\|_{L^3} < \varepsilon$, under which we obtain existence from the L^3_+-theory

where $A_1 = \displaystyle\int_{\mathbb{R}^3} \frac{2|z|}{(4\pi)^{5/2}} \exp\left(-\frac{|z|^2}{4}\right) dz$ and $A_2 = \dfrac{3\sqrt{2}\,e^{-1/2}}{(4\pi)^{5/2}}$.

If the condition (38) is satisfied then there exists a number $A>0$ such that the constant function $\varphi(t) \equiv (1+A)V(0)$ satisfies $\varphi(t) \ge V(t)$ for all $t\in[0,T^*)$. As explained earlier in part (d), if $T^* < \infty$ then $\lim_{t\to(T^*)^-} V(t) = \infty$. This possibility cannot happen in our case. Thus, $T^*=\infty$. Therefore, the problem (II) has a regular global solution.

Nice proof of global ex. for small data based on sub-critical theory and energy est.!

Consider the critical setting $u_0 \in L^3(\mathbb{R}^3)$.

(a) Put $Y_{t_1,t_2} = L^5(\mathbb{R}^5 \times (t_1,t_2))$. Then Y_{t_1,t_2} is a Banach space with respect to the norm

$$\|f\|_{Y_{t_1,t_2}} = \left(\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |f(x,t)|^5 \, dx\, dt\right)^{1/5}.$$ ✓

Consider the Cauchy problem

$$\begin{cases} u_{it} + \dfrac{\partial}{\partial x_j}\left(u_i u_j + \dfrac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0, & t_1 < t < t_2 \\ u(x,t_1) = u_0. \end{cases} \tag{I}$$

We will define mild solutions in this case in a similar manner as in the subcritical setting. Define a bilinear map $B: Y_{t_1,t_2} \times Y_{t_1,t_2} \to Y_{t_1,t_2}$,

$$B(u,v)_i(x,t) = \int_{t_1}^{t} K_j(t-s) * \left(u_i(s)v_j(s) + \frac{1}{2}\delta_{ij}\, u_k(s)v_k(s)\right) ds. \tag{39}$$

A function $u \in Y_{t_1,t_2}$ satisfying the equation

$$u(t) = \Gamma(t-t_1) * u_0 + B(u,u)(x,t) \tag{40}$$

will be called a <u>mild solution</u> to Problem (I). Now we need to show that B is well-defined and $\Gamma(t-t_1)*u_0 \in Y_{t_1,t_2}$. We have

$$\|\Gamma(t-t_1)*u_0\|_{L^5_x} \le \|\Gamma(t-t_1)\|_{L^{15/13}_x}\|u_0\|_{L^3_x} \tag{41}$$ ✓

due to Young's Inequality for convolution. (Note that $\frac{1}{5}+1 = \frac{1}{15/13} + \frac{1}{3}$).

We have $\Gamma(t) = \dfrac{1}{(4\pi t)^{3/2}} \exp\left(-\dfrac{|x|^2}{4t}\right)$. Thus,

$$\Gamma(t)^{15/13} = \frac{1}{(4\pi t)^{45/26}} \exp\left(-\frac{15|x|^2}{52t}\right),$$

$$\int_{\mathbb{R}^3} \Gamma(t)^{15/13} dx \overset{z = \frac{x}{\sqrt{t}}}{=\!=\!=} \int_{\mathbb{R}^3} \frac{1}{(4\pi)^{45/26}} \frac{1}{t^{3/13}} \exp\left(-\frac{15|z|^2}{52}\right) dz = \frac{\alpha}{t^{3/13}}, \tag{42}$$

where

$$\alpha = \frac{1}{(4\pi)^{45/26}} \int_{\mathbb{R}^3} \exp\left(-\frac{15}{52}|z|^2\right) dz.$$

Thanks to (42), (41) implies

$$\| \Gamma(t-t_1) * u_0 \|_{L^5_x} \leq \frac{\alpha^{13/5} \|u_0\|_{L^3_x}}{(t-t_1)^{1/5}}.$$

Thus, $$\|\Gamma(t-t_1)*u_0\|_{Y_{t_1,t_2}} = \left(\int_{t_1}^{t_2} \|\Gamma(t-t_1)*u_0\|^5_{L^5_x} dt\right)^{1/5} \leq \left(\alpha^{13/5}\|u_0\|^5_{L^3_x} \int_{t_1}^{t_2} \frac{dt}{t-t_1}\right)^{1/5} = \infty.$$

We have <u>failed</u> to show that $\Gamma(t-t_1)*u_0 \in Y_{t_1,t_2}$!!

This means the use of Young's Inequality at (41) doesn't work. A more subtle approach is needed. Anyway, we will continue to show that B is well-defined.

$$B(u,v) = \int_{t_1}^{t} K(t-s) * \left(u(s)\otimes v(s) + \frac{1}{2} u(s)\cdot v(s) I\right) ds \qquad (43)$$

where $$K(x,t) = \frac{2x}{(4\pi t)^{5/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Note that $K(t) \in L^a(\mathbb{R}^3)$ for all $a \geq 1$. In particular, $K(t) \in L^{5/4}(\mathbb{R}^3)$. Because $u(s), v(s) \in L^5(\mathbb{R}^3)$, $u(s)\otimes v(s) + \frac{1}{2}u(s)\cdot v(s) I \in L^{5/2}(\mathbb{R}^3)$. Because $\frac{1}{5}+1 = \frac{1}{5/4} + \frac{1}{5/2}$,

by Young's Inequality for convolution, we have

$$\left\| K(t-s) * \left(u(s)\otimes v(s) + \frac{1}{2} u(s)\cdot v(s) I\right)\right\|_{L^5_x} \leq \|K(t-s)\|_{L^{5/4}_x} \left\| u(s)\otimes u(s) + \frac{1}{2} u(s)\cdot v(s) I\right\|_{L^{5/2}_x} \qquad (44)$$

We have $$\|K(t)\|^{5/4}_{L^{5/4}_x} = \int_{\mathbb{R}^3} \frac{(2|x|)^{5/4}}{(4\pi t)^{25/8}} \exp\left(-\frac{5|x|^2}{16t}\right) dx$$

$$\overset{z = \frac{x}{\sqrt{t}}}{=\!=\!=} \int_{\mathbb{R}^3} \frac{2^{5/4}}{(4\pi)^{25/8} t} |z|^{5/4} \exp\left(-\frac{5|z|^2}{16}\right) dz.$$

Thus, $$\|K(t)\|_{L^{5/4}_x} = \frac{1}{t^{4/5}} \underbrace{\left(\int_{\mathbb{R}^3} \frac{2^{5/4}}{(4\pi)^{25/8}} |z|^{5/4} \exp\left(-\frac{5|z|^2}{16}\right) dz\right)^{4/5}}_{A_3} = \frac{A_3}{t^{4/5}} \qquad (45).$$

✓

Also, $\| u(s)\otimes v(s) + \frac{1}{2} u(s)\cdot v(s) I \|_{L_x^{5/2}} \leq \| u(s)\otimes v(s)\|_{L_x^{5/2}} + \frac{1}{2} \| u(s)\cdot v(s) I\|_{L_x^{5/2}}$

$$\overset{\text{Holder}}{\leq} \|u(s)\|_{L_x^5} \|v(s)\|_{L_x^5} + \frac{1}{2} \|u(s)\|_{L_x^5} \|v(s)\|_{L_x^5}$$

$$= \frac{3}{2} \|u(s)\|_{L_x^5} \|v(s)\|_{L_x^5}. \qquad (46)$$

✓

By (45) and (46), (44) implies

$$\| K(t-s) * (u(s)\otimes v(s) + \frac{1}{2} u(s)\cdot v(s) I \|_{L_x^5} \leq \frac{A_3}{(t-s)^{4/5}} \frac{3}{2} \|u(s)\|_{L_x^5} \|v(s)\|_{L_x^5}. \qquad (47)$$

We will apply the inequality $\left\| \int_{t_1}^{t} g(x,s)\, ds \right\|_{L_x^5} \leq \int_{t_1}^{t} \|g(x,s)\|_{L_x^5} ds$.

We have $\|B(u,v)\|_{L_x^5} \overset{(43)}{\leq} \left\| \int_{t_1}^{t} K(t-s) * \left(u(s)\otimes v(s) + \frac{1}{2} u(s)\cdot v(s) I\right) ds \right\|_{L_x^5}$

$$\leq \int_{t_1}^{t} \| K(t-s) * (u(s)\otimes v(s) + \frac{1}{2} u(s)\cdot v(s) I) \|_{L_x^5} ds$$

$$\overset{(47)}{\leq} \frac{3A_3}{2} \int_{t_1}^{t} \frac{\|u(s)\|_{L_x^5} \|v(s)\|_{L_x^5}}{(t-s)^{4/5}} ds. \qquad (48)$$

Define $u(s) = v(s) = 0$ for all $s \in \mathbb{R} \setminus [t_1, t_2]$. Then (48) implies

$$\| B(u,v)\|_{L_x^5} \leq \frac{3A_3}{2} \int_{\mathbb{R}} \frac{\|u(s)\|_{L_x^5} \|v(s)\|_{L_x^5}}{|t-s|^{4/5}} ds. \qquad (49)$$

Recall the fractional interpolation

For $f \in L^p(\mathbb{R}^n)$ and $I_\kappa f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\kappa}} dy$, then $\|I_\kappa f\|_q \leq C_p \|f\|_p$

where $p > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\kappa}{n} > 0$.

✓

A proof of this inequality can be found in Theorem 4.18, p. 229, the book Bennett-Sharpley "Interpolation of Operators". Now we apply this inequality for $n=1$, $f(s)=\|u(s)\|_{L^5_x}\|v(s)\|_{L^5_x}$, $p=\frac{5}{2}$, $\kappa=\frac{1}{5}$, $q=5$. Then (47) can be written as $\|B(u,v)\|_{L^5_x}\leq\frac{3A_3}{2}I_\kappa f(t)$. Thus,

$$\big\|\,\|B(u,v)\|_{L^5_x}\big\|_{L^5_t}\leq\frac{3A_3}{2}\|I_\kappa f\|_{L^5_t}\leq\frac{3A_3C_{5/2}}{2}\|f\|_{L^{5/2}_t}$$

$$=\frac{3A_3C_{5/2}}{2}\big\|\,\|u(t)\|_{L^5_x}\|v(t)\|_{L^5_x}\big\|_{L^{5/2}_t}$$

$$\overset{\text{Holder}}{\leq}\frac{3A_3C_{5/2}}{2}\big\|\,\|u(t)\|_{L^5_x}\big\|_{L^5_t}\big\|\,\|v(t)\|_{L^5_x}\big\|_{L^5_t}.$$

Therefore, $$\|B(u,v)\|_{\mathcal{Y}_{t_1,t_2}}\leq\underbrace{\frac{3A_3C_{5/2}}{2}}_{\tilde{C}}\|u\|_{\mathcal{Y}_{t_1,t_2}}\|v\|_{\mathcal{Y}_{t_1,t_2}}<\infty. \qquad (50)$$

Note that $\tilde{C}>0$ doesn't depend on t_2-t_1. ✓

Recall that we failed to show that $\Gamma(t-t_1)*u_0 \in Y_{t_1 t_2}$ by using the estimate (41). Now we will show it via a different method.

$$\|\Gamma(t-t_1)*u_0\|_{L^5_t} = \left\| \int_{\mathbb{R}^3} \Gamma(x-y,t-t_1)u_0(y)dy \right\|_{L^5_t}$$

$$\leq \int_{\mathbb{R}^3} \|\Gamma(x-y,t-t_1)\overset{u_0(y)}{\smile}\|_{L^5_t} dy$$

$$= \int_{\mathbb{R}^3} \|\Gamma(x-y,t-t_1)\|_{L^5_t} |u_0(y)|dy. \qquad (51)$$

We have
$$\int_0^\infty |\Gamma(z,s)|^5 ds = \int_0^\infty \frac{1}{(4\pi s)^{15/2}} \exp\left(-\frac{5|z|^2}{4s}\right) ds$$

$$\overset{\tau=\frac{\sqrt{5}|z|}{2\sqrt{s}}}{=\!=\!=} \int_0^\infty \left(\frac{1}{\sqrt{4\pi}}\right)^{15} \left(\frac{2\tau}{\sqrt{5}|z|}\right)^{15} \exp(-\tau^2)\frac{5|z|^2}{2\tau^3} d\tau$$

$$= \frac{1}{|z|^{13}} \frac{5}{2(10\pi)^{15/2}} \int_0^\infty \tau^{12}\exp(-\tau^2)d\tau.$$

Hence,
$$\left(\int_0^\infty |\Gamma(z,s)|^5 ds\right)^{1/5} \leq \frac{1}{|z|^{13/5}} \underbrace{\left(\frac{5}{2(10\pi)^{15/2}} \int_0^\infty \tau^{12}\exp(-\tau^2)d\tau\right)^{1/5}}_{A_4}. \qquad (52)$$

Thus,
$$\|\Gamma(z,t-t_1)\|_{L^5_t} = \left(\int_{t_1}^{t_2} |\Gamma(z,t-t_1)|^5 dt\right)^{1/5} \leq \left(\int_{t_1}^{\infty} |\Gamma(z,t-t_1)|^5 dt\right)^{1/5}$$

$$\overset{s=t-t_1}{=\!=\!=} \left(\int_0^\infty |\Gamma(z,s)|^5 ds\right)^{1/5} \overset{(52)}{\leq} \frac{A_4}{|z|^{13/5}}.$$

Therefore, $\|\Gamma(x-y,t-t_1)\|_{L^5_t} \leq \dfrac{A_4}{|x-y|^{13/5}}$. Then (51) implies

$$\|\Gamma(t-t_1)*u_0\|_{L^5_t} \leq \int_{\mathbb{R}^3} \frac{A_4}{|x-y|^{13/5}} |u_0(y)|dy = A_4 \int_{\mathbb{R}^3} \frac{|u_0(y)|}{|x-y|^{3-\frac{2}{5}}} dy. \qquad (53)$$

Now apply the fractional interpolation inequality (at the bottom of page 21) for $n=3, p=3, \kappa=\frac{2}{5}, q=5$: there exists a numeric constant $C>0$ such that

$$\left\| \int_{\mathbb{R}^3} \frac{|u_0(y)|}{|x-y|^{3-\frac{2}{5}}}\, dy \right\|_{L^5_x} \leq C \|u_0\|_{L^3_x}.$$

Then (53) implies $\| \| \Gamma(t-t_1) * u_0 \|_{L^5_t} \|_{L^5_x} \leqslant A_4 C \|u_0\|_{L^3_x}$. Therefore,

$$\|\Gamma(t-t_1)*u_0\|_{\mathcal{Y}_{t_1,t_2}} \leq A_4 C \|u_0\|_{L^3_x} < \infty. \qquad (54)$$

(b) We will give a proof of local-in-time existence of a mild solution in $\mathcal{Y}_{t_1,t_2}$. Put $\mathcal{U}(x,t) = \Gamma(t-t_1) * u_0$. By (54),

$$\|\mathcal{U}\|_{\mathcal{Y}_{t_1,t_2}} \leq A_5 \|u_0\|_{L^3_x}, \qquad (55)$$

where $A_5 > 0$ is a numeric constant. By (50),

$$\|B(u,v)\|_{\mathcal{Y}_{t_1,t_2}} \leq A_6 \|u\|_{\mathcal{Y}_{t_1,t_2}} \|v\|_{\mathcal{Y}_{t_1,t_2}}, \qquad (56)$$

where $A_6 > 0$ is a numeric constant. Thus, B is a continuous bilinear map. Now we apply the lemma stated on page 5 for $E = \mathcal{Y}_{t_1,t_2}$ and $C = A_6$.

Accordingly, if $4A_6 \|\mathcal{U}\|_{\mathcal{Y}_{t_1,t_2}} < 1$ then the equation $u = \mathcal{U} + B(u,u)$ has a unique solution in the ball $\overline{B_R} = \{ v \in \mathcal{Y}_{t_1,t_2} : \|v\|_{\mathcal{Y}_{t_1,t_2}} \leq R \}$, where

$$R = \frac{1 + \sqrt{1 - 4A_6 \|\mathcal{U}\|_{\mathcal{Y}_{t_1,t_2}}}}{2A_6} \qquad (57)$$

Thanks to (55), the condition $4A_6 \|\mathcal{U}\|_{\mathcal{Y}_{t_1,t_2}} < 1$ will be satisfied if we have

$$4 A_5 A_6 \|u_0\|_{L^3_x} < 1. \qquad (58)$$

If (58) is satisfied then the problem $u = U + B(u,u)$ has a unique solution, called ${}^{t_2}u$, in the ball $\overline{B}_R$, where t_2 is any value greater than t_1. For $t_1 < t_2 < t_3$, we'll show that ${}^{t_3}u|_{(t_1,t_2)} = {}^{t_2}u$. Note that

$$\|{}^{t_3}u\|_{\mathcal{Y}_{t_1,t_2}} \leq \|{}^{t_3}u\|_{\mathcal{Y}_{t_1,t_3}} \leq \frac{1+\sqrt{1-4A_6\|U\|_{\mathcal{Y}_{t_1,t_3}}}}{2A_6} \leq \frac{1+\sqrt{1-4A_6\|U\|_{\mathcal{Y}_{t_1,t_2}}}}{2A_6}.$$

By the uniqueness of mild solutions in the ball $\{u \in \mathcal{Y}_{t_1,t_2} : \|u\|_{\mathcal{Y}_{t_1,t_2}} < R\}$, where R is given in (57), we conclude that ${}^{t_3}u|_{(t_1,t_2)} = {}^{t_2}u$. Therefore, the equation $u = U + B(u,u)$ actually has a global-in-time solution when (58) is satisfied. In other words, if the initial data is sufficiently small in $L^3(\mathbb{R}^3)$ then the Cauchy problem (I) has a global-in-time mild solution.

Now we consider the case when (58) is not satisfied. Note that the condition for the equation $u = U + B(u,u)$ to have a unique solution in $\overline{B}_R$ is

$$4A_6 \|U\|_{\mathcal{Y}_{t_1,t_2}} < 1. \qquad (59)$$

On the way to prove (54), we actually showed that

$$\left(\int_{t_1}^{\infty} \|U\|_{L^5_x}^5 dt\right)^{1/5} \leq A_5 \|u_0\|_{L^3_x} < \infty.$$

Thus, there exists a number $\varepsilon_0 > 0$ such that if $0 < t_2 - t_1 < \varepsilon_0$ then

$$\left(\int_{t_1}^{t_2} \|U\|_{L^5_x}^5 dt\right)^{1/5} < \frac{1}{4A_6}.$$

Then $4A_\delta \|U\|_{Y_{t_1,t_2}} < 1$. Therefore, if $0 < t_2 - t_1 < \varepsilon_\delta$ then the Cauchy problem (I) has a mild solution in Y_{t_1,t_2}.

As in the subcritical setting, we would like to show that u exists in a maximal time-interval $[0,T^*)$. To do so, by the continuation method, we need to show that $u \in C_t L^3_x(\mathbb{R}^3 \times (t_1,t_2], \mathbb{R}^3)$, i.e. the map $t \in (t_1,t_2] \mapsto u(t) \in L^3_x$ is well-defined and continuous. First, we show that $u \in L^\infty_t L^3_x$. We have $u(t) = \Gamma(t-t_1) * u_0 + B(u,u)$. Because $\|\Gamma(t-t_1)*u_0\|_{L^3_x} \leq \|\Gamma(t-t_1)\|_{L^1_x} \|u_0\|_{L^3_x} = \|u_0\|_{L^3_x}$ for all $t \in (t_1,t_2)$, we get $\Gamma(t-t_1)*u_0 \in L^\infty_t L^3_x$. Hence, we can assume $u_0 \equiv 0$. Then

$$u(t) = B(u,u) = \int_{t_1}^{t} K(t-s) * \left(u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right) ds.$$

$$= \int_{t_1}^{t} \Gamma(t-s) * \partial_x \left(u(s)\otimes u(s) + \tfrac{1}{2}|u(s)|^2 I\right) ds.$$

Thus,

$$|u(t)| \leq C \int_{t_1}^{t} \Gamma(t-s) * |u(s) \otimes \partial_x u(s)|\, ds. \qquad (60)$$

By considering the difference quotient $\Delta^i_h u(x,t) = \dfrac{u(x+he_i,t) - u(x,t)}{h}$, we can show that $\sqrt{t-t_1}\,\partial_x u \in L^5_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$. A reference for such a proof is Dong–Du "On the local smoothness of solutions of the NSE", 2007. Note that

$$\|\Gamma(t)\|_{L^a_x} = \frac{C}{t^{3/2a'}} \quad \forall\, 1 \leq a \leq \infty, \qquad (61)$$

where $\frac{1}{a} + \frac{1}{a'} = 1$. From (60), we have

$$\|u(t)\|_{L^3_x} \le C \int_{t_1}^{t} \| \Gamma(t-s) * |u(s) \otimes \partial_x u(s)| \|_{L^3_x} ds$$

$$\overset{\text{Young}}{\le} C \int_{t_1}^{t} \|\Gamma(t-s)\|_{L^{15/14}_x} \|u(s)\otimes \partial_x u(s)\|_{L^{5/2}_x} ds$$

$$\overset{(61)}{\le} C \int_{t_1}^{t} \frac{1}{(t-s)^{1/10}} \|u(s)\|_{L^5_x} \|\partial_x u(s)\|_{L^5_x} ds$$

$$\le C \int_{t_1}^{t} \frac{1}{(s-t_1)^{1/2}(t-s)^{1/10}} \underbrace{\|u(s)\|_{L^5_x}}_{\in L^5_s} \underbrace{\sqrt{s-t_1}\, \|\partial_x u(s)\|_{L^5_x}}_{\in L^5_s} ds$$

$$\overset{\text{Holder}}{\le} C \Bigg(\underbrace{\int_{t_1}^{t} \frac{1}{(s-t_1)^{5/6}(t-s)^{1/6}} ds}_{\{1\}} \Bigg)^{3/5} \|u\|_{L^5_{t,x}} \|\sqrt{s-t_1}\partial_x u\|_{L^5_{t,x}} .$$

$$\{1\} \le \int_{t_1}^{(t+t_1)/2} \frac{1}{(s-t_1)^{5/6}\left(\frac{t-t_1}{2}\right)^{1/6}} ds + \int_{(t+t_1)/2}^{t} \frac{1}{\left(\frac{t-t_1}{2}\right)^{5/6}(t-s)^{1/6}} ds$$

$$\le C(t-t_1)^{-1/6}(t-t_1)^{1/6} + C(t-t_1)^{-5/6}(t-t_1)^{5/6}$$

$$= C.$$

Thus, $\|u(t)\|_{L^3_x} \le C \|u\|_{L^5_{t,x}} \|\sqrt{s-t_1}\partial_x u\|_{L^5_{t,x}} \quad \forall t \in (t_1, t_2)$.

Hence, $u \in L^\infty_t L^3_x$ and $\|u\|_{L^\infty_t L^3_x} \le C \|u\|_{L^5_{t,x}} \|\sqrt{s-t_1}\partial_x u\|_{L^5_{t,x}}$. (62)

Next, we show that $u \in C_t L^3_x(\mathbb{R}^3 \times (t_1, t_2), \mathbb{R}^3)$. For $t_1 < t_0 < t_0 + \tau \le t_2$, we ~~have~~ want to show that $\lim_{\tau\to 0^+} \|u(t_0+\tau) - u(t_0)\|_{L^3_x} = 0$.

$$u(t_0+\tau) - u(t_0) = \big(\Gamma(t_0+\tau-t_1) - \Gamma(t_0-t_1)\big) * u_0 + \int_{t_1}^{t_0} \big(K(t_0+\tau-s) - K(t_0-s)\big) * f(s) ds$$

$$+ \int_{t_0}^{t_0+\tau} K(t_0+\tau-s) * f(s) ds, \qquad (63)$$

where $f(s) = u(s)\otimes u(s) + \frac{1}{2}|u(s)|^2 I$. We can rewrite (63) as

$$u(t_0+\tau)-u(t_0) = \big(\Gamma(t_0+\tau-t_1)-\Gamma(t_0-t_1)\big)*u_0 + \int_{t_1}^{t_0}\big(\Gamma(t_0+\tau-s)-\Gamma(t_0-s)\big)*\partial_x f(s)\,ds + \int_{t_0}^{t_0+\tau}\Gamma(t_0+\tau-s)*\partial_x f(s)\,ds$$

$$\sim \underbrace{\big(\Gamma(t_0+\tau-t_1)-\Gamma(t_0-t_1)\big)*u_0}_{\{1\}} + C\underbrace{\int_{t_1}^{t_0}\big(\Gamma(t_0+\tau-s)-\Gamma(t_0-s)\big)*\big(u(s)\otimes\partial_x u(s)\big)ds}_{\{2\}}$$

$$+C\underbrace{\int_{t_0}^{t_0+\tau}\Gamma(t_0+\tau-s)*\big(u(s)\otimes\partial_x u(s)\big)ds}_{\{3\}}. \qquad (64)$$

We have

$$\|\{1\}\|_{L^3_x} \leq \|\Gamma(t_0+\tau-t_1)-\Gamma(t_0-t_1)\|_{L^1_x}\|u_0\|_{L^3_x}, \qquad (65)$$

$$\|\{2\}\|_{L^3_x} \overset{\text{Young}}{\leq} C\int_{t_1}^{t_0}\|\Gamma(t_0+\tau-s)-\Gamma(t_0-s)\|_{L^{5/4}_x}\|u(s)\|_{L^3_x}\|\partial_x u(s)\|_{L^5_x}\,ds$$

$$\leq C\|u\|_{L^\infty_t L^3_x}\int_{t_1}^{t_0}\frac{\|\Gamma(t_0+\tau-s)-\Gamma(t_0-s)\|_{L^{5/4}}}{\sqrt{s-t_1}}\sqrt{s-t_1}\,\|\partial_x u(s)\|_{L^5_x}\,ds$$

$$\overset{\text{Holder}}{\leq} C\|u\|_{L^\infty_t L^3_x}\|\sqrt{s-t_1}\,\partial_x u\|_{L^5_{t,x}}\left(\int_{t_1}^{t_0}\frac{\|\Gamma(t_0+\tau-s)-\Gamma(t_0-s)\|^{5/4}_{L^{5/4}_x}}{(s-t_1)^{5/8}}ds\right)^{4/5}. \qquad (66)$$

$$\|\{3\}\|_{L^3_x} \overset{\text{Young}}{\leq} C\int_{t_0}^{t_0+\tau}\|\Gamma(t_0+\tau-s)\|_{L^{5/4}_x}\|u(s)\|_{L^3_x}\|\partial_x u(s)\|_{L^5_x}\,ds$$

$$\leq C\|u\|_{L^\infty_t L^3_x}\int_{t_0}^{t_0+\tau}\frac{\|\Gamma(t_0+\tau-s)\|_{L^{5/4}_x}}{\sqrt{s-t_1}}\sqrt{s-t_1}\,\|\partial_x u(s)\|_{L^5_x}\,ds$$

$$\overset{\text{Holder}}{\leq} C\|u\|_{L^\infty_t L^3_x}\|\sqrt{s-t_1}\,\partial_x u\|_{L^5_{t,x}}\left(\int_{t_0}^{t_0+\tau}\frac{\|\Gamma(t_0+\tau-s)\|^{5/4}_{L^{5/4}_x}}{(s-t_1)^{5/8}}ds\right)^{4/5}$$

$$\leq C\|u\|_{L^\infty_t L^3_x}\|\sqrt{s-t_1}\,\partial_x u\|_{L^5_{t,x}}(t_0-t_1)^{-1/2}\left(\int_{t_0}^{t_0+\tau}\|\Gamma(t_0+\tau-s)\|^{5/4}_{L^{5/4}_x}ds\right)^{4/5}. \qquad (67)$$

By the method in pages 25-26, Part 1- Mild solutions, we have

$$\lim_{\tau\to 0^+} \|\Gamma(t_0+\tau-t_1)-\Gamma(t_0-t_1)\|_{L^1_x} = 0,$$

$$\lim_{\tau\to 0^+} \|\Gamma(t_0+\tau-t_1)-\Gamma(t_0-t_1)\|_{L^{5/4}_x} = 0.$$

Thus, (65) implies $\|\{1\}\|_{L^3_x} \to 0$ as $\tau \to 0^+$. Thanks to (61),

$$\|\Gamma(t_0+\tau-s)-\Gamma(t_0-s)\|_{L^{5/4}_x} \le \|\Gamma(t_0+\tau-s)\|_{L^{5/4}_x} + \|\Gamma(t_0-s)\|_{L^{5/4}_x}$$

$$\le \frac{C}{(t_0+\tau-s)^{3/10}} + \frac{C}{(t_0-s)^{3/10}} \le \frac{C}{(t_0-s)^{3/10}}.$$

Thus, $$\frac{\|\Gamma(t_0+\tau-s)-\Gamma(t_0-s)\|^{5/4}_{L^{5/4}_x}}{(s-t_1)^{5/8}} \le \frac{C}{(t_0-s)^{3/8}(s-t_1)^{5/8}} \quad \forall \tau>0,$$

which is an integrable function on (t_0, t_1). By Lebesgue's Dominated Convergence theorem, RHS(66)$\to 0$ as $\tau\to 0^+$. Thus, $\|\{2\}\|_{L^3_x} \to 0$ as $\tau\to 0^+$.

$$\int_{t_0}^{t_0+\tau} \|\Gamma(t_0+\tau-s)\|^{5/4}_{L^{5/4}_x}\, ds \overset{(61)}{\le} \int_{t_0}^{t_0+\tau} \frac{C}{(t_0+\tau-s)^{3/8}} ds = \int_0^\tau \frac{C}{r^{3/8}} dr = C\tau^{5/8}.$$

Thus, (67) implies that $\|\{3\}\|_{L^3_x} \to 0$ as $\tau \to 0^+$. Therefore, (64) implies $\|u(t_0+\tau)-u(t_0)\|_{L^3_x} \to 0$ as $\tau\to 0^+$.

So far, we have finished showing that if $\|\Gamma(t-t_1)*u_0\|_{Y_{t_1,t_2}} < \frac{1}{4A_6}$ (see page 25) then the Cauchy problem

$$\begin{cases} u_{it} + \dfrac{\partial}{\partial x_j}\left(u_i u_j + \dfrac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0, & t_1<t<t_2 \\ u(x,t_1) = u_0 \end{cases}$$

has a mild solution $u \in C_t L^3_x(\mathbb{R}^3 \times (t_1, t_2], \mathbb{R}^3)$. Now we consider the Cauchy problem

$$\begin{cases} u_{it} + \dfrac{\partial}{\partial x_j}\left(u_i u_j + \dfrac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0 \quad, \ t > 0 \\ u(\cdot, 0) = u_0 . \end{cases} \qquad (\mathrm{II})$$

We showed earlier that if $\|\Gamma(t) * u_0\|_{Y_{0,T_1}} < \dfrac{1}{4A_6}$ then (II) has a mild solution $u \in C_t L^3_x(\mathbb{R}^3 \times (0, T_1]) \cap L^5_{t,x}(\mathbb{R}^3 \times (0, T_1), \mathbb{R}^3)$. We repeat this procedure: if $\|\Gamma(t - T_1) * u(T_1)\|_{Y_{T_1,T_2}} < \dfrac{1}{4A_6}$ then the problem

$$\begin{cases} u_{it} + \dfrac{\partial}{\partial x_j}\left(u_i u_j + \dfrac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0, \\ u(\cdot, T_1) = u(T_1) \end{cases}$$

has a mild solution $u \in C_t L^3_x(\mathbb{R}^3 \times (T_1, T_2]) \cap L^5_{t,x}(\mathbb{R}^3 \times (T_1, T_2))$. Continuing this procedure, we can show that Problem (II) has a mild solution on a maximal time interval $[0, T^*)$, where $T^* = \lim\limits_{k \to \infty} T_k \leq \infty$. Suppose by contradiction that $T^* < \infty$ and $\|u\|_{Y_{0,T^*}} < \infty$. Let $\tilde{C} > 0$ be the numeric constant given at (58). There is a number $r > 0$ such that $r + \tilde{C} r^2 < \dfrac{1}{4A_6}$. Because $\|u\|_{Y_{0,T^*}} = \left(\displaystyle\int_0^{T^*} \|u(t)\|^5_{L^5_x} dt\right)^{1/5} < \infty$, there exists $\varepsilon_1 > 0$ such that $\|u\|_{Y_{T^*-\varepsilon_1, T^*}} < r$.

We have $u(t) = \underbrace{\Gamma(t - (T^* - \varepsilon_1)) * u(T^* - \varepsilon_1)}_{U} + B(u, u)$ for $T^* - \varepsilon_1 < t < T^*$. Then

$$\|U\|_{Y_{T^*-\varepsilon_1, T^*}} = \|u - B(u,u)\|_{Y_{T^*-\varepsilon_1, T^*}} \leq \|u\|_{Y_{T^*-\varepsilon_1, T^*}} + \|B(u,u)\|_{Y_{T^*-\varepsilon_1, T^*}}$$

$$\leq \|u\|_{\mathcal{Y}_{T^*-\varepsilon_1,T^*}} + \tilde{C}\|u\|^2_{\mathcal{Y}_{T^*-\varepsilon_1,T^*}}$$

$$\leq r + \tilde{C} r^2$$

$$< \frac{1}{4A_6}.$$

Thus, $u \in C_t L^3_x(\mathbb{R}^3 \times (T^*-\varepsilon_1, T^*])$. In particular, $u(T^*) \in L^3_x$. There exists $T > T^*$ such that $\|\Gamma(t-T^*) * u(T^*)\|_{\mathcal{Y}_{T^*,T}} < \frac{1}{4A_6}$. We proved earlier that the problem

$$\begin{cases} u_{i,t} + \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0, \\ u(\cdot, T^*) = u(T^*) \end{cases}$$

has a mild solution $u \in C_t L^3_x(\mathbb{R}^3 \times (T^*, T]) \cap L^5_{t,x}(\mathbb{R}^3 \times (T^*, T))$. Thus, Problem (II) has a mild solution on $[0, T)$. This contradicts the maximality of T^*. Therefore, if $T^* < \infty$ then $\|u\|_{\mathcal{Y}_{0,T^*}} = \infty$. This is the theorem of Ladyzhenskaya-Prodi-Serrin mentioned in lecture 02/28/2014.