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Math 8590: Topics in PDE

Homework #2

Let $u_0 \in (L^2 \cap L^\infty)(\mathbb{R}^3, \mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in sense of distribution.

Consider the 3-dimensional Navier-Stokes equations

$$\begin{cases} u_t - \Delta u + (u\cdot\nabla) u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(\cdot,0) = u_0, \end{cases} \qquad (NSE)$$

where $u = u(x,t)$ and $p = p(x,t)$. Suppose that (NSE) has a mild solution u on $\mathbb{R}^3\times(0,T)$ for some $0<T<\infty$. For each $\varepsilon>0$, we want to find a number $R = R(\varepsilon, T, u_0) > 0$ such that $\int_{|x|>R} |u(x,t)|^2 dx \leqslant \varepsilon \quad \forall t\in(0,T)$.

We will do the following steps.

(a) Recall the definition of u and p in the mild-solution approach. Then show that they are a regular solution to (NSE) in $\mathbb{R}^3\times(0,T)$.

(b) Establish the energy identity. This identity was stated in class (March 7^{th}) but was not proved in detail.

(c) Choose R based on (NSE) and the energy identity.

In Part (b) and (c), the symbol A denotes positive numeric constants for which we do not specify the values. We will adopt such operations as

$A + A = A$, $2A = A$, $A^2 = A$, ...

(a) First, we recall the definition of the mild solution u and its regularity. Put $f(x,t) = -(u.\nabla)u = (-u_i u_{j,i})_{1\le j\le 3}$. Then (NSE) becomes

$$\begin{cases} u_t - \Delta u + \nabla p = f, \\ \quad \operatorname{div} u = 0, \\ \quad u(\cdot, 0) = u_0 . \end{cases} \qquad (I)$$

Put $X = \{ v \in L^2(\mathbb{R}^3, \mathbb{R}^3) : \operatorname{div} v = 0 \text{ in sense of distribution} \}$. Let $\mathbb{P}: L^2(\mathbb{R}^3,\mathbb{R}^3) \to \overline{X}$ be the orthogonal projection map. Then we can write

$$f = \mathbb{P}f + (Id - \mathbb{P})f .$$

A mild solution in $\mathbb{R}^3 \times (0,T)$ is defined to be a function $u : \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$ such that

(i) $u \in L^\infty((0,T_1), L^\infty(\mathbb{R}^3)) \quad \forall\, 0 < T_1 < T,$

(ii) $u(t) = \Gamma(t) * u_0 + \int_0^t - K'(t-s)\, u(s) \otimes u(s)\, ds \qquad$ a.e. $(x,t) \in \mathbb{R}^3 \times (0,T)$ (1)

where $\Gamma(z,t) = \dfrac{1}{(4\pi t)^{3/2}} \exp\left(- \dfrac{|z|^2}{4t} \right)$ is the heat kernel.

With the condition $u_0 \in L^\infty$, we showed in class (March 3rd) that

$$t^{\frac{m}{2}+\ell}\, \partial_t^\ell \nabla_x^m u \in L^\infty((0,T_1), L^\infty(\mathbb{R}^3)) \quad \forall m, \ell \geq 0,\ \forall\, 0 < T_1 < T. \qquad (2)$$

By Sobolev's imbedding theorems, $u \in C^\infty(\mathbb{R}^3 \times (0,T))$. This result is also proved in Lemarié-Rieusset, "Recent developments in the Navier-Stokes problem", 2002, Proposition 15.1. Then by the theory of heat equations, u satisfies the equations

$$\begin{cases} u_t - \Delta u = \mathbb{P}f & \forall (x,t) \in \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \forall (x,t) \in \mathbb{R}^3 \times (0,T), \\ u(\cdot,0) = u_0 . \end{cases} \qquad (\mathrm{II})$$

Because $u_0 \in L^2 \cap L^\infty$, we have $u_0 \in L^2 \cap L^3$. By Theorem 3, Kato "Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$", 1984, and the continuation method in time, we have

$$\begin{cases} u \in C([0,T_1], L^2_x) \\ \sqrt{t}\,\nabla u \in C([0,T_1], L^2_x) \end{cases} \qquad \forall\, 0 < T_1 < T. \qquad (3)$$

Both were stated in class (March 7^{th}), but only a part of the former was proved. Consequently,

$$u(t),\ \nabla u(t) \in L^2_x \qquad \forall t \in (0,T). \qquad (4)$$

On the other hand, (2) implies

$$u(t),\ \nabla u(t) \in L^\infty_x \qquad \forall t \in (0,T). \qquad (5)$$

By (4) and (5), $f(t) = (-u_i\, u_{j,i})_{1 \le j \le 3} \in L^1_x \cap L^\infty_x$ for all $t \in (0,T)$.

Therefore, $f(t) \in L^1_x \cap L^\infty_x \cap C^\infty_x$ for all $t \in (0,T)$.

Theorem 4, lecture notes by Gutierrez, "Solution to Poisson's equation", 2013, reads:

[Suppose $\rho: \mathbb{R}^n \to \mathbb{R}$ is a locally Hölder continuous function of order $\alpha \in (0,1]$, and $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then the Newtonian potential

$$w(x) = \int_{\mathbb{R}^n} G(x-y)\rho(y)\,dy$$

belongs to $C^2(\mathbb{R}^n)$ and satisfies $\Delta w = f$ in $\mathbb{R}^n$.]

This is a generalization of Lemmas 4.1 and 4.2, Gilbarg – Trudinger's book, pages 54–55. Recall that the fundamental solution of Laplace's equation in $\mathbb{R}^3$ is $G(z) = -\frac{1}{4\pi|z|}$. Applying the above theorem for $\rho = f$, we conclude that the function $F: \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$,

$$F(x,t) = \int_{\mathbb{R}^3} -\frac{f(y,t)}{4\pi|x-y|}\,dy$$

satisfies $F(t) \in C_x^2$ and $\Delta F(t) = f(t)$ in $\mathbb{R}^3$. In fact, $F(t) \in C_x^\infty$ because $f(t) \in C_x^\infty$. We have

$$f = \Delta F = \nabla(\operatorname{div} F) - \underbrace{\operatorname{curl}(\operatorname{curl} F)}_{\text{divergence free}}.$$

Thus, $\mathbb{P} f = -\operatorname{curl}(\operatorname{curl} F)$ and $(\mathbb{I} - \mathbb{P}) f = \nabla(\operatorname{div} F)$. Define a function $p: \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$,

$$p(x,t) = \operatorname{div} F = -\frac{1}{4\pi}\frac{\partial}{\partial x_j}\int_{\mathbb{R}^3}\frac{f_j(y,t)}{|x-y|}\,dy. \qquad (6)$$

Then $p(t) \in C_x^\infty$ and $\nabla p(t) = (\mathbb{I} - \mathbb{P}) f$. The system (II) implies

$$\begin{cases} u_t - \Delta u + \nabla p = \mathbb{P}f + (\mathbb{I}-\mathbb{P})f = f & \forall (x,t) \in \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \forall (x,t) \in \mathbb{R}^3 \times (0,T), \\ u(\cdot,0) = u_0 . \end{cases}$$

We conclude that (u,p), given by (1) and (6), are a mild and regular solution to (NSE).

(b) Next, we establish the energy identity by doing two steps.

(i) $\|p(t)\|_{L^2_x} \leqslant A \|u\|_{L^3_x} \|\nabla u\|_{L^2_x} \qquad \forall t \in (0,T),$ (7)

(ii) $\frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,s)|^2 dx ds = \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 dx \quad \forall t \in (0,T).$ (8)

In the rest of the write-up, the symbol A denotes positive numeric constants for which we do not specify the values. We will adopt such operations as $A+A=A$, $A^2=A, \ldots$

Proof of (7)

By Part (a), $f(t) \in L^1_x \cap L^\infty_x \cap C^\infty_x$ for every $t \in (0,T)$ and

$$p(x,t) = -\frac{1}{4\pi} \frac{\partial}{\partial x_j} \int_{\mathbb{R}^3} \frac{f_j(y,t)}{|x-y|} dy .$$

By Lemma 4.1, Gilbarg-Trudinger's book, we can put the differentiation inside the integral, namely

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$$p(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\partial}{\partial x_j}\left(\frac{1}{|x-y|}\right) f_j(y,t)\,dy$$

$$= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x_j - y_j}{|x-y|^3} f_j(y,t)\,dy.$$

Thus, $$|p(x,t)| \leq \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|f(y,t)|}{|x-y|^2}\,dy. \qquad (9)$$

Because $f(t) = (-u_i u_{j,i})_{1\leq j\leq 3}$, we have $|f(y,t)| \leq |u(y,t)|\,|\nabla u(y,t)|$.

By Hölder's inequality,

$$\|f(t)\|_{L_x^{6/5}} \leq \|u\|_{L_x^3} \|\nabla u\|_{L_x^2}. \qquad (10)$$

Recall the fractional interpolation inequality:

For $f \in L^p(\mathbb{R}^n)$ and $I_\kappa f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\kappa}}\,dy$, we have $\|I_\kappa f\|_{L^q} \leq C_p \|f\|_{L^p}$,

where $p > 1$, $0 < \kappa < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{\kappa}{n} > 0$

A proof of this inequality can be found in the book by Mizuta, "Potential Theory in Euclidean spaces", 1996, Theorem 2.1. Applying this inequality for $p = \frac{6}{5}$, $n = 3$, $\kappa = 1$, $q = 2$, we get

$$\left\| \int_{\mathbb{R}^3} \frac{|f(y,t)|}{|x-y|^2}\,dy \right\|_{L_x^2} \leq A\|f(t)\|_{L_x^{6/5}}.$$

Combining this inequality with (9), we get

$$\|p(t)\|_{L^2_x} \leqslant A\|f\|_{L^{6/5}_x} \overset{(10)}{\leqslant} A\|u\|_{L^3_x}\|\nabla u\|_{L^2_x} \quad \forall t\in(0,T).$$

By (4) and (5), $u(t)\in L^2_x \cap L^\infty_x$ and $\nabla u(t)\in L^2_x$ for all $t\in(0,T)$.

Hence, $p(t)\in L^2_x$.

Proof of (8)

We first recall some integrability properties of u and p. By the definition of p in Part (a), $\nabla p = (I-P)f \in L^2_x$. Hence $p(t)\in H^1_x$.

By (4) and (5), $u(t)\in H^1_x \cap H^\infty_x$ for all $t\in(0,T)$. By (2), $t u_t \in L^\infty([0,T_1], L^\infty_x)$ for all $0<T_1<T$. Thus,

$$u_t \in L^\infty([t_1,t_2], L^\infty_x) \quad \forall [t_1,t_2]\subset(0,T).$$

Multiplying both sides of the equation $u_t - \Delta u + (u\cdot\nabla)u + \nabla p = 0$ by u and taking the integral over $x\in\mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} u_t u\,dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} \underbrace{u_i u_{j,i} u_j}_{=\frac{1}{2}u_{i,i}u_j u_j = 0} dx - \int_{\mathbb{R}^3} p \underbrace{\nabla\cdot u}_{=0}\,dx = 0 \quad \forall t\in(0,T)$$

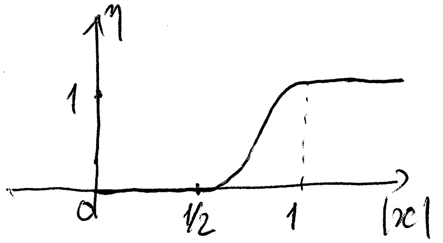
Hence, $\displaystyle \frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 dx + \int_{\mathbb{R}^3} |\nabla u(x,t)|^2 dx = 0 \quad \forall t\in(0,T).$ (11)

Note that $u\in C([0,T_1], L^2_x)$ for $0<T_1<T$ by (3). Integrating both sides of (11) over $[0,t]$, we get

$$\frac{1}{2}\int_{\mathbb{R}^3}|u(x,t)|^2dx-\frac{1}{2}\int_{\mathbb{R}^3}|u(x,0)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla u(x,s)|^2dx\,ds=0.$$

Therefore, $\displaystyle\int_{\mathbb{R}^3}|u(x,t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla u(x,s)|^2dx\,ds=\int_{\mathbb{R}^3}|u_0(x)|^2dx.$

(c) For each $\varepsilon>0$, we will choose a number $R=R(\varepsilon,T,u_0)>0$ such that $\displaystyle\int_{|x|>R}|u(x,t)|^2dx\le\varepsilon \quad \forall t\in(0,T).$



Define a map $\eta:\mathbb{R}^3\to\mathbb{R}$,

$$\eta(x)=\begin{cases}0 & \text{if } |x|\le\frac{1}{2},\\ 1 & \text{if } |x|\ge 1,\\ -16|x|^3+36|x|^2-24|x|+5 & \text{if } \frac{1}{2}\le|x|\le 1.\end{cases}$$

Then $\eta\in C^1(\mathbb{R}^3)$ and $0\le\eta(x)\le 1$. For each $R>0$, we define a map

$$\eta_R(x)=\eta\left(\frac{x}{R}\right)\quad\forall x\in\mathbb{R}^3.$$

Then $\eta_R\in C^1(\mathbb{R}^3)$, $0\le\eta_R(x)\le 1$ and $\eta_R(x)=\begin{cases}0 & \text{if } |x|\le R/2,\\ 1 & \text{if } |x|\ge R.\end{cases}$

Multiplying both sides of the equation $u_t-\Delta u+(u\cdot\nabla)u+\nabla p=0$ by $\eta_R u$ and taking the integral over $\mathbb{R}^3$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3}|u|^2\eta_R\,dx+\underbrace{\int_{\mathbb{R}^3}\nabla u\cdot\nabla(\eta_R u)\,dx}_{\{1\}}+\underbrace{\int_{\mathbb{R}^3}[(u\cdot\nabla)u]\eta_R u\,dx}_{\{2\}}-\underbrace{\int_{\mathbb{R}^3}p\,\nabla\cdot(u\eta_R)\,dx}_{\{3\}}=0. \quad (12)$$

We have
$$\{1\} = \int_{\mathbb{R}^3} |\nabla u|^2 \eta_R \, dx + \int_{\mathbb{R}^3} (\nabla u) u (\nabla \eta_R) \, dx, \tag{13}$$

$$\{2\} = \int_{\mathbb{R}^3} u_i u_{j,i} u_j \eta_R \, dx = \frac{1}{2} \int_{\mathbb{R}^3} u_i (u_j u_j)_{,i} \eta_R \, dx = -\frac{1}{2} \int_{\mathbb{R}^3} (u_j u_j)(u_i \eta_R)_{,i} \, dx$$
$$= -\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 (\underbrace{u_{i,i}}_{=0} \eta_R + u_i (\eta_R)_{,i}) \, dx$$
$$= -\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 u \nabla \eta_R \, dx, \tag{14}$$

$$\{3\} = \int_{\mathbb{R}^3} p \underbrace{(\nabla \cdot u)}_{=0} \eta_R \, dx + \int_{\mathbb{R}^3} p u \nabla \eta_R \, dx = \int_{\mathbb{R}^3} p u \nabla \eta_R \, dx. \tag{15}$$

Substituting (13), (14), (15) into (12), we get
$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 \eta_R \, dx = - \int_{\mathbb{R}^3} |\nabla u|^2 \eta_R \, dx - \int_{\mathbb{R}^3} (\nabla u) u (\nabla \eta_R) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 u \nabla \eta_R \, dx + \int_{\mathbb{R}^3} p u \nabla \eta_R \, dx.$$

Thus,
$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 \eta_R \, dx \leq \underbrace{- \int_{\mathbb{R}^3} (\nabla u) u (\nabla \eta_R) \, dx}_{\{4\}} + \underbrace{\frac{1}{2} \int_{\mathbb{R}^3} |u|^2 u \nabla \eta_R \, dx}_{\{5\}} + \underbrace{\int_{\mathbb{R}^3} p u (\nabla \eta_R) \, dx}_{\{6\}}. \tag{16}$$

We want to estimate $\{4\}, \{5\}, \{6\}$. By the definition of η_R,
$$\nabla \eta_R(x) = \frac{1}{R} \nabla \eta\left(\frac{x}{R}\right).$$

Thus,
$$|\nabla \eta_R| \leq \frac{1}{R} \max_{\mathbb{R}^3} |\nabla \eta| = \frac{A}{R}. \tag{17}$$

$$|\{4\}| \leq \int_{\mathbb{R}^3} |\nabla u| |u| |\nabla \eta_R| dx \overset{(17)}{\leq} \frac{A}{R} \int_{\mathbb{R}^3} |\nabla u| |u| dx$$

$$\overset{\text{Schwarz}}{\leq} \frac{A}{R} \|\nabla u\|_{L^2_x} \|u\|_{L^2_x}$$

$$\overset{(8)}{\leq} \frac{A}{R} \|\nabla u\|_{L^2_x} \|u_0\|_{L^2}. \qquad (18)$$

$$|\{5\}| \overset{(17)}{\leq} \frac{A}{R} \int_{\mathbb{R}^3} |u|^2 |u| dx = \frac{A}{R} \int_{\mathbb{R}^3} |u|^{3/2} |u|^{3/2} dx$$

$$\overset{\text{Hölder}}{\leq} \frac{A}{R} \left(\int_{\mathbb{R}^3} (|u|^{3/2})^4 dx \right)^{1/4} \left(\int_{\mathbb{R}^3} (|u|^{3/2})^{4/3} dx \right)^{3/4}$$

$$= \frac{A}{R} \|u\|^{3/2}_{L^6_x} \|u\|^{3/2}_{L^2_x}$$

$$\overset{(8)}{\leq} \frac{A}{R} \|u\|^{3/2}_{L^6_x} \|u_0\|^{3/2}_{L^2}. \qquad (19)$$

Recall the Gagliardo – Nirenberg – Sobolev inequality (Evans, "Partial Differential Equations", 2nd edition, p. 277):

$$\left[\begin{array}{ll} \|\varphi\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p,n) \|\nabla \varphi\|_{L^p(\mathbb{R}^n)} & \forall \varphi \in C^1_c(\mathbb{R}^n), \\ \text{where } 1 \leq p < n \text{ and } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} & \end{array} \right]$$

Since $C^1_c(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, the inequality also holds for $\varphi \in H^1(\mathbb{R}^n)$.

Applying this result for $n=3$, $p=2$, $p^*=6$, $\varphi = u(\cdot,t)$, we get $\|u\|_{L^6_x} \leq A \|\nabla u\|_{L^2_x}$. Then (19) implies

$$|\{5\}| \leq \frac{A}{R} \|u_0\|_{L^2}^{3/2} \|\nabla u\|_{L_x^2}^{3/2}. \qquad (20)$$

Note that we have implicitly derived

$$\|u\|_{L_x^3} \leq A \|u\|_{L_x^2}^{1/2} \|\nabla u\|_{L_x^2}^{1/2}. \qquad (21)$$

We have $|\{6\}| \overset{(17)}{\leq} \frac{A}{R} \int_{\mathbb{R}^3} |p||u|\,dx \overset{\text{Schwarz}}{\leq} \frac{A}{R} \|p\|_{L_x^2} \|u\|_{L_x^2}$

$$\overset{(7)}{\leq} \frac{A}{R} \|u\|_{L_x^3} \|\nabla u\|_{L_x^2} \|u\|_{L_x^2}$$

$$\overset{(21)}{\leq} \frac{A}{R} \|u\|_{L_x^2}^{3/2} \|\nabla u\|_{L_x^2}^{3/2}$$

$$\overset{(8)}{\leq} \frac{A}{R} \|u_0\|_{L^2}^{3/2} \|\nabla u\|_{L_x^2}^{3/2}. \qquad (22)$$

Substituting (18), (20), (22) into (16), we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 \eta_R \,dx \leq \frac{A}{R} \|u_0\|_{L^2} \|\nabla u\|_{L_x^2} + \frac{A}{R} \|u_0\|_{L^2}^{3/2} \|\nabla u\|_{L_x^2}^{3/2}.$$

Integrating both sides over $[0,t]$, we get

$$\int_{\mathbb{R}^3} |u(x,t)|^2 \eta_R(x)\,dx - \int_{\mathbb{R}^3} |u_0(x)|^2 \eta_R(x)\,dx \leq \frac{A}{R} \|u_0\|_{L^2} \int_0^t \|\nabla u(s)\|_{L_x^2}\,ds$$

$$+ \frac{A}{R} \|u_0\|_{L^2}^{3/2} \int_0^t \|\nabla u(s)\|_{L_x^2}^{3/2}\,ds$$

$$\overset{\text{Hölder}}{\leq} \frac{A}{R} \|u_0\|_{L^2} \sqrt{t} \left(\int_0^t \|\nabla u(s)\|_{L_x^2}^2\,ds \right)^{1/2} + \frac{A}{R} \|u_0\|_{L^2}^{3/2} t^{1/4} \left(\int_0^t \|\nabla u(s)\|_{L_x^2}^2\,ds \right)^{3/4}$$

$$\overset{(8)}{\leq} \frac{A}{R} \|u_0\|_{L^2} T^{1/2} \left(\frac{1}{\sqrt{2}} \|u_0\|_{L^2} \right) + \frac{A}{R} \|u_0\|_{L^2}^{3/2} T^{1/4} \left(\frac{1}{\sqrt{2}} \|u_0\|_{L^2} \right)^{3/2}$$

$$= \frac{A T^{1/2} \|u_0\|_{L^2}^2}{R} + \frac{A T^{1/4} \|u_0\|_{L^2}^3}{R}.$$

Hence,
$$\int_{\mathbb{R}^3} |u(x,t)|^2 \eta_R(x)\,dx \leq \int_{\mathbb{R}^3} |u_0(x)|^2 \eta_R(x)\,dx + \frac{A T^{1/2} \|u_0\|_{L^2}^2 + A T^{1/4} \|u_0\|_{L^2}^3}{R}. \quad (23)$$

For each $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon) > 0$ such that

$$\int_{|x| > \frac{R_0}{2}} |u_0(x)|^2\,dx < \frac{\varepsilon}{2}. \quad (24)$$

Choose $R = \max\left\{R_0, \frac{2}{\varepsilon}\left(A T^{1/2} \|u_0\|_{L^2}^2 + A T^{1/4} \|u_0\|_{L^2}^3\right)\right\}$.

Then
$$\begin{aligned}
\int_{|x| > R} |u(x,t)|^2\,dx &\leq \int_{\mathbb{R}^3} |u(x,t)|^2 \eta_R(x)\,dx \\
&\overset{(23)}{\leq} \int_{\mathbb{R}^3} |u_0(x)|^2 \eta_R(x)\,dx + \frac{A T^{1/2} \|u_0\|_{L^2}^2 + A T^{1/4} \|u_0\|_{L^2}^3}{R} \\
&\leq \int_{|x| > \frac{R}{2}} |u_0(x)|^2\,dx + \frac{\varepsilon}{2} \\
&\overset{(24)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$