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Math 8590: Topics in PDE
Homework #3.

Consider the complex Ginzburg-Landau equations

$$\begin{cases} i\partial_t u + (1-i\varepsilon)\Delta u + |u|^2 u = 0, & (x,t)\in \mathbb{R}^3\times(0,\infty) \\ u(x,0) = u_0(x), & \end{cases} \qquad (CGL)$$

where $u = u(x,t) = u_1(x,t) + i u_2(x,t)$ and ε is a positive constant.

We will do the following tasks.

(i) Show that a smooth solution u to (CGL) which satisfies certain integrability conditions also satisfies an identity similar to the energy identity for the Navier-Stokes equations.

(ii) Define a weak solution for Problem (CGL).

Show the existence of a weak solution when $u_0 \in L^2(\mathbb{R}^3,\mathbb{C})$.

In the sequel, the symbol C is used to denote various numeric constants which we do not specify the values. Also, we use the notations $u(x,t)$ and $u(t)$ interchangeably with the convention that the former assumes complex values while the latter assumes values in a function space.

(i) We will show that a smooth solution u to Problem (CGL) which satisfies certain integrability conditions also satisfies an identity similar to the energy identity for the Navier-Stokes equations.

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By separating the real parts and the imaginary parts, we obtain 4 equations from (CGL):

$$\begin{cases} \partial_t u_1 = -\Delta(u_2 - \varepsilon u_1) - |u|^2 u_2, & (1) \\ \partial_t u_2 = \Delta(u_1 + \varepsilon u_2) + |u|^2 u_1, & (2) \\ u_1(x,0) = u_{01}(x), & (3) \\ u_2(x,0) = u_{02}(x). & (4) \end{cases}$$

Multiplying (1) by u_1, multiplying (2) by u_2 and adding up the results, we get

$$\underbrace{u_1 \partial_t u_1 + u_2 \partial_t u_2}_{= \frac{1}{2} \partial_t(|u|^2)} = -u_1 \Delta(u_2 - \varepsilon u_1) + u_2 \Delta(u_1 + \varepsilon u_2).$$

Integrating both sides with respect to $x \in \mathbb{R}^3$, we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} \int_{\mathbb{R}^3} |u(x,t)|^2 dx\right) &= -\int_{\mathbb{R}^3} u_1 \Delta(u_2 - \varepsilon u_1) dx + \int_{\mathbb{R}^3} u_2 \Delta(u_1 + \varepsilon u_2) dx \\ &= \int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla(u_2 - \varepsilon u_1) dx - \int_{\mathbb{R}^3} \nabla u_2 \cdot \nabla(u_1 + \varepsilon u_2) dx \\ &= -\varepsilon \int_{\mathbb{R}^3} |\nabla u_1|^2 dx - \varepsilon \int_{\mathbb{R}^3} |\nabla u_2|^2 dx \\ &= -\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2 dx. \end{aligned}$$

Integrating both sides over $[0,t]$, we get

$$\frac{1}{2} \int_{\mathbb{R}^3} |u(x,t)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} |u_0|^2 dx = -\varepsilon \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,s)|^2 dx\, ds$$

Note that the symbol ∇ always denotes the gradient with respect to variable x.

We get $$\int_{\mathbb{R}^3} |u(x,t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,s)|^2 dx\,ds = \int_{\mathbb{R}^3} |u_0|^2 dx. \qquad (5)$$

This is an identity similar to the energy identity for the Navier–Stokes equations.

(ii) We will, first, define a weak solution for Problem (CGL) and then show the existence of a weak solution when $u_0 \in L^2(\mathbb{R}^3, \mathbb{C})$.

Suppose that u is a regular solution to (CGL). We test (1) and (2) with each function $\varphi \in \mathcal{D}(\mathbb{R}^3)$ and get

$$\frac{d}{dt}\int_{\mathbb{R}^3} u_1(t)\varphi\,dx = \int_{\mathbb{R}^3} \nabla\varphi \cdot \nabla(u_2(t) - \varepsilon u_1(t))\,dx - \int_{\mathbb{R}^3} |u(t)|^2 u_2(t)\varphi\,dx, \qquad (6)$$

$$\frac{d}{dt}\int_{\mathbb{R}^3} u_2(t)\varphi\,dx = -\int_{\mathbb{R}^3} \nabla\varphi \cdot \nabla(u_1(t) + \varepsilon u_2(t))\,dx + \int_{\mathbb{R}^3} |u(t)|^2 u_1(t)\varphi\,dx. \qquad (7)$$

We then test (6) and (7) with each function $\psi \in \mathcal{D}([0,\infty))$ and get

$$\left(\int_{\mathbb{R}^3} u_{01}\varphi\,dx\right)\psi(0) - \int_0^\infty\int_{\mathbb{R}^3} u_1(x,t)\varphi(x)\psi'(t)\,dt = \int_0^\infty\int_{\mathbb{R}^3} \nabla\varphi(x)\cdot\nabla\big(u_2(x,t) - \varepsilon u_1(x,t)\big)\psi(t)\,dx\,dt$$
$$- \int_0^\infty\int_{\mathbb{R}^3} |u(t)|^2 u_2(t)\varphi(x)\psi(t)\,dx\,dt. \qquad (8)$$

$$\left(\int_{\mathbb{R}^3} u_{02}\varphi\,dx\right)\psi(0) - \int_0^\infty\int_{\mathbb{R}^3} u_2(x,t)\varphi(x)\psi'(t)\,dt = -\int_0^\infty\int_{\mathbb{R}^3} \nabla\varphi(x)\cdot\nabla\big(u_1(x,t) + \varepsilon u_2(x,t)\big)\psi(t)\,dx\,dt$$
$$+ \int_0^\infty\int_{\mathbb{R}^3} |u(t)|^2 u_1(t)\varphi(x)\psi(t)\,dx\,dt. \qquad (9)$$

We say that a function $u = u_1 + iu_2$ is a <u>weak solution</u> to the Problem (CGL) if $u \in (L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1)(\mathbb{R}^3 \times (0,\infty), \mathbb{C})$ and satisfies (8) and (9) for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$, $\psi \in \mathcal{D}([0,\infty))$.

This type of weak solution was introduced by Hopf (1951). It is also possible to define a weak solution in sense of Leray (1934):

$$\int_{\mathbb{R}^3} u_1(t)\varphi(t)\,dx - \int_{\mathbb{R}^3} u_{01}\varphi(0)\,dx = \int_0^\infty\int_{\mathbb{R}^3} \nabla\varphi(s)\cdot\nabla(u_2(s)-\varepsilon u_1(s))\,dx\,ds - \int_0^\infty\int_{\mathbb{R}^3} |u(s)|^2 u_2(s)\varphi(s)\,dx\,ds,$$

$$\int_{\mathbb{R}^3} u_2(t)\varphi(t)\,dx - \int_{\mathbb{R}^3} u_{02}\varphi(0)\,dx = -\int_0^\infty\int_{\mathbb{R}^3} \nabla\varphi(s)\cdot\nabla(u_1(s)+\varepsilon u_2(s))\,dx\,ds + \int_0^\infty\int_{\mathbb{R}^3} |u(s)|^2 u_1(s)\varphi(s)\,dx\,ds,$$

$$\forall\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^3).$$

Leray called this type of solutions (to the Navier-Stokes equations) turbulent solutions. Perhaps, the existence of such a solution can also be proved by Leray's method, i.e. by examining the limit as $\lambda\to 0^+$ of the mild solution $(u)_\lambda$ to the problem

$$\begin{cases} i\partial_t(u)_\lambda + (1-i\varepsilon)\Delta(u)_\lambda + |\eta_\lambda * (u)_\lambda|^2 (u)_\lambda = 0, \\ (u)_\lambda(\cdot,0) = u_0, \end{cases} \qquad (I)$$

where $(\eta_\lambda)_{\lambda>0}$ is an approximate identity. A good reference for properties of the mild solution to (I) is Liu-Jia, 2003.

For now, however, we choose Hopf's definition for weak solutions. A weak solution can be obtained by Galerkin approximation. First, we construct an approximate sequence of a weak solution. Recall that $H^1(\mathbb{R}^3)$ is a Hilbert space with the inner product

$$(f,g) := \int_{\mathbb{R}^3} fg\,dx + \int_{\mathbb{R}^3} \nabla f\cdot\nabla g\,dx \qquad \forall f,g \in H^1(\mathbb{R}^3).$$

Because $H^1(\mathbb{R}^3)$ is separable and that $\mathcal{D}(\mathbb{R}^3)$ is a dense subspace of $H^1(\mathbb{R}^3)$, there exists a countable linearly independent subset $\{\varphi_1, \varphi_2, \dots\}$ of $\mathcal{D}(\mathbb{R}^3)$ such that the linear span of $\{\varphi_1, \varphi_2, \dots\}$ is dense in $H^1(\mathbb{R}^3)$. By the Gram-Schmidt orthogonalization process, we can assume that $\{\varphi_1, \varphi_2, \dots\}$ is an orthonormal set with respect to the inner product in $L^2(\mathbb{R}^3)$, i.e.

$$\int_{\mathbb{R}^3} \varphi_k(x)\varphi_j(x)\,dx = \delta_{kj} \qquad \forall k, j \in \mathbb{N}.$$

Because the linear span of $\{\varphi_1, \varphi_2, \dots\}$ is dense in $(H^1(\mathbb{R}^3), \|.\|_{H^1})$, it is also dense in $(H^1(\mathbb{R}^3), \|.\|_{L^2})$. We know that $(H^1(\mathbb{R}^3), \|.\|_{L^2})$ is dense in $L^2(\mathbb{R}^3)$. Hence, the linear span of $\{\varphi_1, \varphi_2, \dots\}$ is dense in $L^2(\mathbb{R}^3)$. For each $m \in \mathbb{N}$, we put $V_m = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. Write

$$u_{01} = \sum_{k=1}^{\infty} \alpha_k \varphi_k \quad \text{and} \quad u_{02} = \sum_{k=1}^{\infty} \beta_k \varphi_k,$$

where α_k and β_k are real constants. We will solve for $u^{(m)} = u^{(m)}(x,t)$ from the equations

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{\mathbb{R}^3} u_1^{(m)}(t)\varphi\,dx = \int_{\mathbb{R}^3} \nabla\varphi.\nabla\left(u_2^{(m)}(t) - \varepsilon u_1^{(m)}(t)\right)dx - \int_{\mathbb{R}^3} |u^{(m)}(t)|^2 u_2^{(m)}(t)\varphi\,dx, & \forall \varphi \in V_m \\ \dfrac{d}{dt}\displaystyle\int_{\mathbb{R}^3} u_2^{(m)}(t)\varphi\,dx = -\int_{\mathbb{R}^3} \nabla\varphi.\nabla\left(u_1^{(m)}(t) + \varepsilon u_2^{(m)}(t)\right)dx + \int_{\mathbb{R}^3} |u^{(m)}(t)|^2 u_1^{(m)}(t)\varphi\,dx, & (\mathrm{II}) \\ u_1^{(m)}(0) = \displaystyle\sum_{k=1}^{m} \alpha_k\varphi_k, \quad u_2^{(m)}(0) = \sum_{k=1}^{m} \beta_k\varphi_k. \end{cases}$$

(II) is satisfied for every $\varphi \in V_m$ if and only if it is satisfied for all $\varphi = \varphi_1, \dots, \varphi_m$. Write $u_1^{(m)}(x,t) = \sum_{k=1}^{m} c_{mk}(t)\varphi_k(x)$ and $u_2^{(m)}(x,t) = \sum_{k=1}^{m} d_{mk}(t)\varphi_k(x)$, where

c_{mk} and d_{mk} are real-valued functions. Then (II) becomes

$$\begin{cases} c_{mj}'(t) = F_{1j}\left(c_{m1}(t),\dots,c_{mm}(t), d_{m1}(t),\dots,d_{mm}(t)\right), \\ d_{mj}'(t) = F_{2j}\left(c_{m1}(t),\dots,c_{mm}(t), d_{m1}(t),\dots,d_{mm}(t)\right), \quad \forall 1\le j\le m \\ c_{mj}(0) = \alpha_j,\ d_{mj}(0) = \beta_j, \end{cases} \qquad \text{(III)}$$

where F_{1j} and F_{2j} are real-valued polynomials of degree ≤ 3. Put

$$X = X(t) = \begin{pmatrix} c_{m1}(t) \\ \vdots \\ c_{mm}(t) \\ d_{m1}(t) \\ \vdots \\ d_{mm}(t) \end{pmatrix}, \quad X_0 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \quad \text{and} \quad F(X) = \begin{pmatrix} F_{11}(X) \\ \vdots \\ F_{1m}(X) \\ F_{21}(X) \\ \vdots \\ F_{2m}(X) \end{pmatrix}.$$

Then $F : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is a vector-valued polynomial of degree ≤ 3. Problem (III) becomes a Cauchy problem

$$\begin{cases} X'(t) = F(X(t)), \\ X(0) = X_0. \end{cases} \qquad \text{(IV)}$$

By the theory of ODE, (IV) has a unique solution on a short-time interval. On the maximal time-interval of existence, (II) gives us

$$\int_{\mathbb{R}^3} \partial_t(u_1^{(m)})\,\varphi_j\,dx = \int_{\mathbb{R}^3} \nabla\varphi_j\cdot\nabla\left(u_2^{(m)}(t) - \varepsilon u_1^{(m)}(t)\right)dx - \int_{\mathbb{R}^3} |u^{(m)}(t)|^2 u_2^{(m)}(t)\,\varphi_j\,dx, \qquad (10)$$

$$\int_{\mathbb{R}^3} \partial_t(u_2^{(m)})\,\varphi_j\,dx = -\int_{\mathbb{R}^3} \nabla\varphi_j\cdot\nabla\left(u_1^{(m)}(t) + \varepsilon u_2^{(m)}(t)\right)dx + \int_{\mathbb{R}^3} |u^{(m)}(t)|^2 u_1^{(m)}(t)\,\varphi_j\,dx. \qquad (11)$$

Multiplying (10) by $c_{mj}(t)$, multiplying (11) by $d_{mj}(t)$ and adding up, we get

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^3} |u^{(m)}(t)|^2 dx\right) = -\varepsilon \int_{\mathbb{R}^3} |\nabla u^{(m)}(t)|^2 dx.$$

Integrating both sides over $[0,t]$, we get

$$\int_{\mathbb{R}^3} |u^{(m)}(t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^3} |\nabla u^{(m)}(s)|^2 dx\, ds = \int_{\mathbb{R}^3} |u^{(m)}(0)|^2 dx.$$

Because $\|u^{(m)}(0)\|_{L^2} \le \|u_0\|_{L^2}$, we get the energy inequality

$$\int_{\mathbb{R}^3} |u^{(m)}(t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^3} |\nabla u^{(m)}(s)|^2 dx\, ds \le \int_{\mathbb{R}^3} |u_0|^2 dx \quad \forall m \in \mathbb{N}. \quad (12)$$

Consequently, $|X(t)|^2 = \sum_{k=1}^{m} \left(c_{mk}(t)^2 + d_{mk}(t)^2 \right) = \int_{\mathbb{R}^3} |u^{(m)}(t)|^2 dx \le \|u_0\|^2_{L^2(\mathbb{R}^3, \mathbb{C})}$.

Thus, $|X(t)| \le \|u_0\|_{L^2}$ whenever t belongs to the maximal interval of existence.

We will show that (IV) has a global solution $X \in C^1([0,\infty), \mathbb{R}^{2m})$. Recall the Picard-Lindelöf's theorem (see Coddington-Levinson "Ordinary Differential Equations", 1955, page 12):

> Let $a, b, \tau \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $a, b > 0$. Put $R = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : |t-\tau| \le a, |x-\xi| \le b\}$.
> Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function on R. Put
>
> $$M = \max_R |f| \quad \text{and} \quad \alpha = \min\left\{a, \frac{b}{M}\right\}.$$
>
> Then the ODE $\begin{cases} x'(t) = f(t, x(t)), & \forall t \in (\tau-\alpha, \tau+\alpha), \\ x(\tau) = \xi, \end{cases}$
>
> has a unique solution $x \in C^1([\tau-\alpha, \tau+\alpha], \mathbb{R}^n)$.

Let $[0, T_1]$ be a time-interval of existence. Suppose that (IV) has a unique solution on $[0, T_k]$. Put

$$R = \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 : |t - T_k| \le 1, |x - X(T_k)| \le 1\}$$
$$\subset \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 : |t - T_k| \le 1, |x| \le \|u_0\|_{L^2} + 1\}.$$

Then $M := \max_R |F| \le \max_{|Y| \le \|u_0\|_{L^2}+1} |F(Y)|$.

Put $\alpha = \min\left\{1, \frac{1}{M}\right\} \geq \min\left\{1, \frac{1}{\max\{|F(Y)| : |Y| \le \|u_0\|_{L^2} + 1\}}\right\} > 0.$

By Picard-Linderlöf's theorem, Problem (IV) has a unique solution $X \in C^1([0, T_{k+1}], \mathbb{R}^{2m})$ where $T_{k+1} = T_k + \alpha$. The sequence $(T_k)_{k \in \mathbb{N}}$ is an increasing arithmetic sequence. Since $\lim T_k = \infty$, Problem (IV) has a global solution $X \in C^1([0,\infty), \mathbb{R}^{2m})$. Moreover, $X \in C^\infty([0,\infty), \mathbb{R}^{2m})$ because $F \in C^\infty(\mathbb{R}^{2m}, \mathbb{R}^{2m})$. Therefore, Problem (II) has a smooth solution $u^{(m)}$ which satisfies the energy inequality (12). Consequently,

$$\|u^{(m)}(t)\|_{L^2(\mathbb{R}^3, \mathbb{C})} \le \|u_0\|_{L^2(\mathbb{R}^3, \mathbb{C})} \quad \forall m \in \mathbb{N}, \forall t \geq 0, \tag{13}$$

$$\|\nabla u^{(m)}\|_{L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^4)} \le (2\varepsilon)^{-1/2} \|u_0\|_{L^2(\mathbb{R}^3, \mathbb{C})}, \quad \forall m \in \mathbb{N}. \tag{14}$$

By (14), the sequence $(\nabla u^{(m)})$ is bounded in $L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^4)$, which is a reflexive space. By Kakutani's theorem, there exists a convergent subsequence $(\nabla u^{(m')})$ in the weak topology of $L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^4)$. Taking this subsequence instead of $(\nabla u^{(m)})$ into our consideration, we can assume that $(\nabla u^{(m)})$ converges in the weak topology of $L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^4)$. Write

$$\nabla u^{(m)} = \nabla u_1^{(m)} + i\nabla u_2^{(m)} \rightharpoonup v_1 + i v_2 = v \in L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^4). \qquad (15)$$

By Fatou's lemma,

$$\int_0^\infty \liminf_{m\to\infty} \left(\int_{\mathbb{R}^3} |\nabla u^{(m)}(x,t)|^2 dx \right) dt \leq \liminf_{m\to\infty} \int_0^\infty \int_{\mathbb{R}^3} |\nabla u^{(m)}(x,t)|^2 dx\, dt \overset{(14)}{\leq} (2\varepsilon)^{-1} \|u_0\|_{L^2}^2$$

Thus, there exists a set $A \subset (0,\infty)$ such that $(0,\infty) \setminus A$ has measure zero and $\liminf_{m\to\infty} \int_{\mathbb{R}^3} |\nabla u^{(m)}(x,t)|^2 dx < \infty$ for all $t \in A$. Hence, for each $t \in A$ there exists a subsequence $(\nabla u^{(m')})$ such that

$$\int_{\mathbb{R}^3} |\nabla u^{(m')}(x,t)|^2 dx < M(t) < \infty, \ \forall m'.$$

This inequality and (13) imply that the sequence $(u^{(m')}(t))$ is bounded in $H^1(\mathbb{R}^3, \mathbb{C})$. For each ball $B_k \subset \mathbb{R}^3$ centered at the origin with radius $k \in \mathbb{N}$, we have the compact imbedding $H^1(\mathbb{R}^3, \mathbb{C}) \hookrightarrow L^2(B_k, \mathbb{C})$. Thus, there exists a subsequence of $(u^{(m')}(t))$ that converges in $L^2(B_k, \mathbb{C})$. By Cantor's diagonal method, there exists a subsequence $(u^{(m'')}(t))$ of $(u^{(m')}(t))$ that converges in $L^2(B_k, \mathbb{C})$ for every $k \in \mathbb{N}$.

Because $(0,\infty) \setminus A$ is of measure zero, there exists a countable subset A' of A such that A' is dense in $(0,\infty)$. By Cantor's diagonal method, there exists a subsequence $(u^{(m''')})$ of $(u^{(m)})$ such that $(u^{(m''')}(t))$ converges in $L^2(B_k, \mathbb{C})$ for every $k \in \mathbb{N}$ and $t \in A'$. Taking this subsequence instead of $(u^{(m)})$ into consideration, we can assume that the sequence $(u^{(m)}(t))$ converges in

$L^2(B_k, \mathbb{C})$ for every $k \in \mathbb{N}$ and $t \in A'$. We say $(u^{(m)}(t))$ converges in $L^2_{loc}(\mathbb{R}^3, \mathbb{C})$ for every $t \in A'$.

We now show that $(u^{(m)}(t))$ Converges in $L^2_{loc}(\mathbb{R}^3, \mathbb{C})$ for every $t \in A$. Take $t_0 \in A$. There exists a sequence (t_n) in A' such that $t_n \to t_0$. For each $\delta > 0$ and $j \in \mathbb{N}$, we look for $N = N(\delta, j) \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}^3} (u^{(m)}(t_n) - u^{(m)}(t_0)) \varphi_j \, dx \right| < \delta \quad \forall n > N, \ \forall m \geq j. \tag{16}$$

By the first equation of (II), we have

$$\int_{\mathbb{R}^3} (u_1^{(m)}(t_n) - u_1^{(m)}(t_0)) \varphi_j \, dx = \underbrace{\int_{t_0}^{t_n} \int_{\mathbb{R}^3} \nabla \varphi_j \cdot \nabla (u_2^{(m)}(t) - \varepsilon u_1^{(m)}(t)) \, dx \, dt}_{\{1\}}$$
$$- \underbrace{\int_{t_0}^{t_n} \int_{\mathbb{R}^3} |u^{(m)}(t)|^2 u_2^{(m)}(t) \varphi_j \, dx \, dt}_{\{2\}}. \tag{17}$$

By Hölder's inequality,

$$|\{1\}| \leq \left| \int_{t_0}^{t_n} \|\nabla \varphi_j\|_{L^2} \|\nabla (u_2^{(m)}(t) - \varepsilon u_1^{(m)}(t))\|_{L^2_x} \, dt \right|$$
$$\leq \|\nabla \varphi_j\|_{L^2} \left| \int_{t_0}^{t_n} (1+\varepsilon) \|\nabla u^{(m)}(t)\|_{L^2_x} \, dt \right|$$
$$\leq (1+\varepsilon) \|\nabla \varphi_j\|_{L^2} |t_n - t_0|^{1/2} \left(\int_{t_0}^{t_1} \|\nabla u^{(m)}(t)\|^2_{L^2_x} \, dt \right)^{1/2}$$
$$\overset{(14)}{\leq} (2\varepsilon)^{-1/2} (1+\varepsilon) \|u_0\|_{L^2} \|\nabla \varphi_j\|_{L^2} |t_n - t_0|^{1/2}. \tag{18}$$

We have

$$|\{2\}| \leq \|\varphi_j\|_{L^\infty} \left| \int_{t_0}^{t_n} \int_{\mathbb{R}^3} |u^{(m)}(x,t)|^3 \, dx \, dt \right| = \|\varphi_j\|_{L^\infty} \left| \int_{t_0}^{t_n} \|u^{(m)}(t)\|^3_{L^3_x} \, dt \right|. \tag{19}$$

By Hölder's inequality,

$$\|u^{(m)}(t)\|_{L^3_x} \leq \|u^{(m)}(t)\|^{1/2}_{L^2_x} \|u^{(m)}(t)\|^{1/2}_{L^6_x} \overset{(13)}{\leq} \|u_0\|^{1/2}_{L^2} \|u^{(m)}(t)\|^{1/2}_{L^6_x}.$$

Recall the Gagliardo-Nirenberg-Sobolev inequality (Evans, "Partial Differential Equations", 2^{nd} edition, page 277):

$$\left[\begin{array}{l} \|\psi\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p,n) \|\nabla\psi\|_{L^p(\mathbb{R}^n)}, \quad \forall \psi \in C^1_c(\mathbb{R}^n), \\ \text{where } 1 \leq p < n \text{ and } \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n}. \end{array}\right]$$

Because $u^{(m)}(t) \in \mathcal{D}(\mathbb{R}^3, \mathbb{C}^2)$, we can apply this inequality for "$\psi = u_1^{(m)}(t)$", "$u_2^{(m)}(t)$" and $n=3$, $p=2$, $p^*=6$. Then $\|u^{(m)}(t)\|_{L^6_x} \leq C\|\nabla u^{(m)}(t)\|_{L^2_x}$. Hence,

$$\|u^{(m)}(t)\|_{L^3_x} \leq C\|u_0\|^{1/2}_{L^2} \|\nabla u^{(m)}(t)\|^{1/2}_{L^2_x}.$$

With this inequality, we have

$$|\{2\}| \overset{(19)}{\leq} \|\varphi_j\|_{L^\infty} \left| \int_{t_0}^{t_n} C\|u_0\|^{3/2}_{L^2} \|\nabla u^{(m)}(t)\|^{3/2}_{L^2_x}\, dt \right|$$

$$= C\|u_0\|^{3/2}_{L^2} \|\varphi_j\|_{L^\infty} \left| \int_{t_0}^{t_n} \|\nabla u^{(m)}(t)\|^{3/2}_{L^2_x}\, dt \right|$$

$$\overset{\text{Hölder}}{\leq} C\|u_0\|^{3/2}_{L^2} \|\varphi_j\|_{L^\infty} |t_n - t_0|^{1/2} \left| \int_{t_0}^{t_n} \|\nabla u^{(m)}(t)\|^2_{L^2_x} dt \right|^{3/4}$$

$$\overset{(14)}{\leq} C(2\varepsilon)^{-3/4} \|u_0\|^3_{L^2} \|\varphi_j\|_{L^\infty} |t_n - t_0|^{1/4}. \qquad (20)$$

Substituting (18) and (20) into (17), we get

$$\left| \int_{\mathbb{R}^3} \left(u_1^{(m)}(t_n) - u_1^{(m)}(t_0)\right) \varphi_j\, dx \right| \leq (2\varepsilon)^{-1/2} \|u_0\|_{L^2} \|\nabla\varphi_j\|_{L^2} |t_n - t_0|^{1/2}$$
$$+ (2\varepsilon)^{-3/4} \|u_0\|^3_{L^2} \|\varphi_j\|_{L^\infty} |t_n - t_0|^{1/4}.$$

Thus, there exists $N = N(\delta, j) \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}^3} \left(u_1^{(m)}(t_n) - u_1^{(m)}(t_0)\right)\varphi_j \, dx \right| < \frac{\delta}{\sqrt{2}} \quad \forall n > N, \ \forall m \geq j.$$

By the same method of estimation, we get a similar inequality for $u_2^{(m)}$. Thus, (16) is proved. We have

$$\left| \int_{\mathbb{R}^3} \left(u^{(m)}(t_0) - u^{(k)}(t_0)\right)\varphi_j \, dx \right| \leq \left| \int_{\mathbb{R}^3} \left(u^{(m)}(t_0) - u^{(m)}(t_n)\right)\varphi_j \, dx \right|$$

$$+ \left| \int_{\mathbb{R}^3} \left(u^{(m)}(t_n) - u^{(k)}(t_n)\right)\varphi_j \, dx \right|$$

$$+ \left| \int_{\mathbb{R}^3} \left(u^{(k)}(t_n) - u^{(k)}(t_0)\right)\varphi_j \, dx \right|. \qquad (21)$$

For each $\delta > 0$, $j \in \mathbb{N}$, we showed earlier that

$$\left| \int_{\mathbb{R}^3} \left(u^{(m)}(t_0) - u^{(m)}(t_n)\right)\varphi_j \, dx \right|, \ \left| \int_{\mathbb{R}^3} \left(u^{(k)}(t_n) - u^{(k)}(t_0)\right)\varphi_j \, dx \right| < \frac{\delta}{3}$$

for all $n > N\left(\frac{\delta}{3}, j\right)$ and $m, k \geq j$. Choose some integer $n_0 > N\left(\frac{\delta}{3}, j\right)$. By (21),

$$\left| \int_{\mathbb{R}^3} \left(u^{(m)}(t_0) - u^{(k)}(t_0)\right)\varphi_j \, dx \right| < \frac{2\delta}{3} + \left| \int_{\mathbb{R}^3} \left(u^{(m)}(t_{n_0}) - u^{(k)}(t_{n_0})\right)\varphi_j \, dx \right| \quad \forall m, k \geq j. \qquad (22)$$

Because $t_{n_0} \in A'$, the sequence $(u^{(m)}(t_{n_0}))$ converges in $L^2_{loc}(\mathbb{R}^3, \mathbb{C})$. Thus, the sequence $\left(\int_{\mathbb{R}^3} u^{(m)}(t_{n_0})\varphi_j \, dx \right)_{m \in \mathbb{N}}$ converges. There exists $N' = N'(\delta, j) \in \mathbb{N}$

such that
$$\left| \int_{\mathbb{R}^3} \left(u^{(m)}(t_{n_0}) - u^{(k)}(t_{n_0})\right)\varphi_j \, dx \right| < \frac{\delta}{3} \quad \forall m, k > N'. \qquad (23)$$

By (22) and (23), $\left|\int_{\mathbb{R}^3} (u^{(m)}(t_0) - u^{(k)}(t_0))\varphi_j\, dx\right| < \delta$ for all $m, k > \max\{N', j\}$.

Therefore, the sequence $\left(\int_{\mathbb{R}^3} u^{(m)}(t_0)\varphi_j\, dx\right)_{m\in\mathbb{N}}$ converges in $\mathbb{R}$. Denote

$$T(\varphi_j) = \lim_{m\to\infty} \int_{\mathbb{R}^3} u^{(m)}(t_0)\varphi_j\, dx.$$

Then T is a linear map on the linear span of $\{\varphi_1, \varphi_2, \dots\}$. Because

$$\left|\int_{\mathbb{R}^3} u^{(m)}(t_0)\varphi\, dx\right| \le \|u^{(m)}(t_0)\|_{L^2_x} \|\varphi\|_{L^2} \overset{(13)}{\le} \|u_0\|_{L^2}\|\varphi\|_{L^2} \quad \forall m \in \mathbb{N},$$

we have $|T(\varphi)| \le \|u_0\|_{L^2}\|\varphi\|_{L^2}$ for all $\varphi \in \operatorname{span}\{\varphi_1, \varphi_2, \dots\}$. Because this set is dense in $L^2(\mathbb{R}^3)$, T can extend to a linear continuous functional on $L^2(\mathbb{R}^3)$. By Riesz's representation theorem, there exists $u(t_0) \in L^2(\mathbb{R}^3, \mathbb{C})$ such that

$$T(\varphi) = \int_{\mathbb{R}^3} u(t_0)\varphi\, dx \qquad \forall \varphi \in L^2(\mathbb{R}^3).$$

We have proved that $u^{(m)}(t_0) \rightharpoonup u(t_0)$ in $L^2(\mathbb{R}^3, \mathbb{C})$. For each $k \in \mathbb{N}$, let $(u^{(m')}(t_0))$ be a subsequence such that

$$\lim_{m'\to\infty} \|u^{(m')}(t_0)\|_{L^2(B_k,\mathbb{C})} = \limsup_{m\to\infty} \|u^{(m)}(t_0)\|_{L^2(B_k,\mathbb{C})} \overset{(13)}{\le} \|u_0\|_{L^2}.$$

$, \mathbb{C})$.

Because $t_0 \in A$, there exists a subsequence $(u^{(m'')}(t_0))$ of $(u^{(m')}(t_0))$ which converges in $L^2(B_k, \mathbb{C})$. Note that we still have $u^{(m'')}(t_0) \rightharpoonup u(t_0)$ in $L^2(B_k$

Thus, $u^{(m'')}(t_0) \to u(t_0)$ in $L^2(B_k, \mathbb{C})$. Consequently,

$$\|u(t_0)\|_{L^2(B_k,\mathbb{C})} = \lim_{m''\to\infty} \|u^{(m'')}(t_0)\|_{L^2(B_k,\mathbb{C})}$$

$$= \lim_{m'\to\infty} \|u^{(m')}(t_0)\|_{L^2(B_k,\mathbb{C})} = \limsup_{m\to\infty} \|u^{(m)}(t_0)\|_{L^2(B_k,\mathbb{C})}$$

Therefore, $u^{(m)}(t_0) \to u(t_0)$ in $L^2(B_R, \mathbb{C})$. In other words, we have showed that $u^{(m)}(t) \to u(t)$ in $L^2_{loc}(\mathbb{R}^3; \mathbb{C})$ for each $t \in A$.

With the function v given by (15), we will show that $v = \nabla u$ (the gradient with respect to the variable x) in sense of distribution. Write $u = u_1 + iu_2$. We need to show $v_1 = \nabla u_1$ and $v_2 = \nabla u_2$. Take any $\psi \in \mathcal{D}(\mathbb{R}^3 \times (0,\infty))$. Suppose $\operatorname{supp}\psi \subset B_R \times (t_1, t_2)$. Then

$$\left| \int_0^\infty \int_{\mathbb{R}^3} u_1^{(m)}(x,t) \nabla\psi(x,t)\,dx\,dt - \int_0^\infty \int_{\mathbb{R}^3} u_1(x,t)\nabla\psi(x,t)\,dx\,dt \right|$$

$$\leq \int_{t_1}^{t_2} \int_{B_R} |u_1^{(m)}(t) - u_1(t)|\,|\nabla\psi(x,t)|\,dx\,dt$$

$$\overset{\text{Schwarz}}{\leq} \|\nabla\psi\|_{L^2(\mathbb{R}^3\times(0,\infty))} \left(\int_{t_1}^{t_2} \int_{B_R} |u_1^{(m)}(t) - u_1(t)|^2\,dx\,dt \right)^{1/2}$$

$$= \|\nabla\psi\|_{L^2(\mathbb{R}^3\times(0,\infty))} \left(\int_{t_1}^{t_2} \|u_1^{(m)}(t) - u_1(t)\|^2_{L^2(B_R)}\,dt \right)^{1/2}. \qquad (24)$$

We have $\|u_1^{(m)}(t) - u_1(t)\|_{L^2(B_R)} \to 0$ as $m \to \infty$ and

$$\|u_1^{(m)}(t) - u_1(t)\|_{L^2(B_{R})} \leq \|u_1^{(m)}(t)\|_{L^2(B_{R})} + \|u_1(t)\|_{L^2(B_{R})} \overset{(13)}{\leq} 2\|u_0\|_{L^2}.$$

By Lebesgue's Dominated Convergence theorem, RHS (24) $\to 0$ as $m \to \infty$. Hence,

$$\lim_{m\to\infty} \int_0^\infty \int_{\mathbb{R}^3} u_1^{(m)}(x,t)\nabla\psi(x,t)\,dx\,dt = \int_0^\infty \int_{\mathbb{R}^3} u_1(x,t)\nabla\psi(x,t)\,dx\,dt. \qquad (25)$$

On the other hand,

$$\lim_{m\to\infty} \int_0^\infty \int_{\mathbb{R}^3} u_1^{(m)}(x,t)\nabla\psi(x,t)\,dx\,dt = -\lim_{m\to\infty} \int_0^\infty \int_{\mathbb{R}^3} \nabla u_1^{(m)}(x,t)\,\psi(x,t)\,dx\,dt$$

$$\overset{(15)}{=\!=} -\int_0^\infty \int_{\mathbb{R}^3} v_1(x,t)\psi(x,t)\,dx\,dt. \qquad (26)$$

By (25) and (26), we have $v_1 = \nabla u_1$ in sense of distribution in $\mathbb{R}^3\times(0,\infty)$. Simimilarly, $v = \nabla u_2$ in sense of distribution. We have showed that $\nabla u = v \in L^2(\mathbb{R}^3\times(0,\infty), \mathbb{R}^4)$.

Next, we show that $u \in (L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x)(\mathbb{R}^3\times(0,\infty), \mathbb{C})$. Because $u^{(m)}(t) \to u(t)$ in $L^2_{loc}(\mathbb{R}^3, \mathbb{C})$ for all $t \in A$, we have

$$\int_{B_R} |u(t)|^2 dx = \lim_{m\to\infty} \int_{B_R} |u^{(m)}(t)|^2 dx \overset{(13)}{\leq} \|u_0\|^2_{L^2} \quad \forall R>0.$$

Thus, $$\int_{\mathbb{R}^3} |u(t)|^2 dx = \lim_{R\to\infty} \int_{B_R} |u(t)|^2 dx \leq \|u_0\|^2_{L^2}. \qquad (27)$$

Hence, $u \in L^\infty_t L^2_x(\mathbb{R}^3\times(0,\infty), \mathbb{C})$. By (15), we have $\nabla u^{(m)} \rightharpoonup v = \nabla u$ in $L^2(\mathbb{R}^3\times(0,\infty), \mathbb{R}^4)$. Thus,

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 dx dt \leq \liminf_{m\to\infty} \int_0^\infty \int_{\mathbb{R}^3} |\nabla u^{(m)}|^2 dx dt \overset{(14)}{\leq} (\alpha-\varepsilon)^{-1} \|u_0\|^2_{L^2}. \qquad (28)$$

Hence, $u \in L^2_t \dot{H}^1_x(\mathbb{R}^3\times(0,\infty), \mathbb{C})$. Therefore, $u \in (L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x)(\mathbb{R}^3\times(0,\infty), \mathbb{C})$.

Now we show that u satisfies (8) and (9). For each $\varphi \in V_m$ and $\psi \in \mathcal{D}([0,\infty))$, we multiply both sides of the first equation of (III) by $\psi(t)$ and integrate over $t \in [0,\infty)$. We get

$$\underbrace{\left(\int_{\mathbb{R}^3} u_1^{(m)}(x,0)\varphi(x)dx\right)\psi(0)}_{\{3\}} - \underbrace{\int_0^\infty\int_{\mathbb{R}^3} u_1^{(m)}(x,t)\varphi(x)\psi'(t)dxdt}_{\{4\}} =$$

$$\underbrace{\int_0^\infty\int_{\mathbb{R}^3} \nabla\varphi(x)\cdot\nabla\left(u_2^{(m)}(x,t) - \varepsilon u_1^{(m)}(x,t)\right)\psi(t)dxdt}_{\{5\}} - \underbrace{\int_0^\infty\int_{\mathbb{R}^3} |u^{(m)}(t)|^2 u_2^{(m)}(t)\varphi(x)\psi(t)dxdt}_{\{6\}}. \qquad (29)$$

By the third equation of (II), $\| u_1^{(m)}(0) - u_{01}\|_{L^2(\mathbb{R}^3)} \to 0$ as $m \to \infty$.

Thus, $\lim\limits_{m\to\infty}\{3\} = \left(\int_{\mathbb{R}^3} u_{01}(x)\varphi(x)\,dx\right)\Psi(0)$. $\qquad$ (30)

Because $u_1^{(m)}(t) \to u_1(t)$ in $L^2_{loc}(\mathbb{R}^3)$ for all $t \in A$ and $\|u_1^{(m)}(t) - u_1(t)\|_{L^2} \le 2\|u_0\|_{L^2}$,

by Lebesgue's Dominated Convergence theorem, we get

$$\lim_{m\to\infty}\{4\} = \int_0^\infty \int_{\mathbb{R}^3} u_1(x,t)\varphi(x)\Psi'(t)\,dx\,dt. \qquad (31)$$

Because $\nabla u^{(m)} \to \nabla u$ in $L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^4)$,

$$\lim_{m\to\infty}\{5\} = \int_0^\infty \int_{\mathbb{R}^3} \nabla\varphi(x)\cdot\nabla\big(u_2(x,t) - \varepsilon u_1(x,t)\big)\Psi(t)\,dx\,dt. \qquad (32)$$

Suppose that $\operatorname{supp}\varphi \subset B_R \subset \mathbb{R}^3$ and $\operatorname{supp}\Psi \subset [t_1, t_2] \subset [0,\infty)$. By Hölder's

inequality,

$$\int_{B_R} |u^{(m)}(t)|^{19/6}\,dx \le \left(\int_{B_R} |u^{(m)}(t)|^2\,dx\right)^{\frac{17}{24}} \left(\int_{B_R} |u^{(m)}(t)|^6\,dx\right)^{\frac{7}{24}}$$

$$= \|u^{(m)}(t)\|_{L^2(B_R)}^{17/12}\, \|u^{(m)}(t)\|_{L^6(B_R)}^{7/4}$$

$$\overset{(13)}{\le} \|u_0\|_{L^2}^{17/12}\, \|u^{(m)}(t)\|_{L^6(B_R)}^{7/4}. \qquad (33)$$

By Gagliardo-Nirenberg-Sobolev's theorem, $\|u^{(m)}(t)\|_{L^6(B_R)} \le C\|\nabla u^{(m)}(t)\|_{L^2(\mathbb{R}^3)}$.

Then (3) implies

$$\int_{B_R} |u^{(m)}(t)|^{19/6}\,dx \le C\|u_0\|_{L^2}^{\frac{17}{12}}\, \|\nabla u^{(m)}(t)\|_{L^2(\mathbb{R}^3)}^{7/4}.$$

Hence, $\displaystyle\int_{t_1}^{t_2}\int_{B_R} |u^{(m)}(t)|^{19/6}\,dx\,dt \le C\|u_0\|_{L^2}^{17/12} \int_{t_1}^{t_2} \|\nabla u^{(m)}(t)\|_{L^2(\mathbb{R}^3)}^{7/4}\,dt$

$$\overset{\text{Hölder}}{\leq} C\|u_0\|_{L^2}^{17/12}(t_2-t_1)^{1/8}\left(\int_{t_1}^{t_2}\|\nabla u^{(m)}(t)\|^2_{L^2(\mathbb{R}^3)}\,dt\right)^{7/8}$$

$$\overset{(14)}{\leq} C\|u_0\|^{19/6}(t_2-t_1)^{1/8}.$$

Thus, the sequence $(u^{(m)})$ is bounded in $L^{19/6}(\mathbb{R}^3\times(0,\infty),\mathbb{C})$. We know that

$$\int_{B_R}|u^{(m)}(t)-u(t)|^2dx \longrightarrow 0 \text{ as } m\to\infty, \quad \forall t\in A.$$

Moreover, $\displaystyle\int_{B_R}|u^{(m)}(t)-u(t)|^2dx \leq \int_{B_R}|u^{(m)}(t)|^2dx+\int_{B_R}|u(t)|^2dx \overset{(13),(27)}{\leq} 2\|u_0\|_{L^2}$

$\forall m\in\mathbb{N}, \forall t\in(0,\infty)$.

By Lebesgue's Dominated convergence theorem,

$$\lim_{m\to\infty}\int_{t_1}^{t_2}\int_{B_R}|u^{(m)}(t)-u(t)|^2dx\,dt = 0.$$

This means $u^{(m)}\to u$ in $L^2(B_R\times(t_1,t_2),\mathbb{C})$. By Theorem 4.9, Brezis "Functional Analysis, Sobolev Spaces and Partial Differential equations", 2011, page 94, there exists a subsequence $(u^{(m')})$ of $(u^{(m)})$ which converges almost everywhere in $B_R\times(t_1,t_2)$ to u. Take this subsequence instead of $(u^{(m)})$ into consideration, we can assume that $(u^{(m)})$ converges to u almost everywhere in $B_R\times(t_1,t_2)$.

So far, ~~so~~ we have

$$\begin{cases} u^{(m)} \text{ is bounded in } L^{19/6}(B_R\times(t_1,t_2),\mathbb{C}), \\ u^{(m)}\to u \quad \text{a.e. in } B_R\times(t_1,t_2). \end{cases}$$

The following result is in Brezis, page 123, which can be proved by Vitali's convergence theorem:

[Let $1<p<\infty$ and Ω be a subset of $\mathbb{R}^N$ of finite measure. Let (f_n) be a sequence in $L^p(\Omega)$ such that

(i) (f_n) is bounded in $L^p(\Omega)$,

(ii) $f_n \to f$ a.e. in Ω.

Then $f_n \to f$ in $L^q(\Omega)$ for every $q \in [1,p)$.]

We apply this result for $\Omega = B_R \times (t_1,t_2) \subset \mathbb{R}^4$,

$$f_m(x,t) = |u^{(m)}(x,t)|^2 u_2^{(m)}(x,t),$$

$$f(x,t) = |u(x,t)|^2 u_2(x,t),$$

$$p = \frac{19}{18}.$$

We get $|u|^2 u_2 \in L^1(B_R \times (t_1,t_2))$ and $|u^{(m)}|^2 u_2^{(m)} \longrightarrow |u|^2 u_2$ in $L^1(B_R \times (t_1,t_2))$.

Therefore, $$\lim_{m\to\infty} \{6\} = \int_0^\infty \int_{\mathbb{R}^3} |u(t)|^2 u_2(t)\, \varphi(x)\, \psi(t)\, dx\, dt. \quad (34)$$

Replacing (30), (31), (32), (34) into (29), we get

$$\left(\int_{\mathbb{R}^3} u_{01}(x)\varphi(x)dx\right)\psi(0) - \int_0^\infty \int_{\mathbb{R}^3} u_1(x,t)\varphi(x)\psi'(t)dx\,dt =$$

$$= \int_0^\infty \int_{\mathbb{R}^3} \nabla\varphi(x)\cdot\nabla\big(u_2(x,t) - \varepsilon u_1(x,t)\big)\psi(t)dx\,dt - \int_0^\infty\int_{\mathbb{R}^3} |u(t)|^2 u_2(t)\varphi(x)\psi(t)dx\,dt \quad (35)$$

for all $\psi \in \mathcal{D}([0,\infty))$, $\varphi \in \operatorname{span}\{\varphi_1, \varphi_2, \ldots\}$.

Because $u \in \left(L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1\right)(\mathbb{R}^3 \times (0,\infty), \mathbb{C})$ and that $\operatorname{span}\{\varphi_1,\varphi_2,\ldots\}$ is dense in $\left(\mathcal{D}(\mathbb{R}^3), \|\cdot\|_{H^1(\mathbb{R}^3)}\right)$, the equation (35) is also true for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$. Hence, we achieve (8). By the same method, we achieve (9). Therefore, u is a weak solution for the problem (CGL).