Math 8590 **Spring 2014**

Homework assigment 1

due March 26, 2014

Consider the following equations in $\mathbf{R}^3 \times (t_1, t_2)$:

1. *The Burgers equation:*

   This is an equation for a vector field $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$:

$$u_t + u\nabla u - \Delta u = 0\,. \tag{1}$$

2. *The semi-linear heat equation with cubic non-linearity:*

   This is an equation for a scalar real-valued function $u(x,t)$:

$$u_t - \Delta u = \kappa u^3\,, \tag{2}$$

   where κ can be either 1 or -1.

3. *A model equation:*

   This is an equation for a vector valued function $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$:

$$u_{it} + \frac{\partial}{\partial x_j}\left(u_i u_j + \frac{1}{2}\delta_{ij}|u|^2\right) - \Delta u_i = 0\,, \quad i = 1,2,3, \text{ summation over } j \text{ understood} \tag{3}$$

Choose one of these equations, define the notion of mild solutions of the Cauchy problem similarly to what we did in class for the Navier-Stokes, and outline a proof of a local-in-time existence result for the Cauchy problem. The choice of the function spaces is up to you. For example, you can work with $u_0 \in L^\infty$ (the sub-critical setting) or with $u_0 \in L^n$ (the critical setting).

As an optional part, you can investigate whether each of the equation has good "conserved quantities", which could be used to turn the local existence result into a global existence result.

Hints: The model equation has the non-linearity in the "divergence form", and therefore it is closest to the Navier-Stokes cases which we did in class. In examples 1 and 2 the function space setup might be slightly different from what we did in class for NSE, so from this point of view these examples might be somewhat more difficult.

Concerning the conserved quantities: an energy-type estimate similar to what we have for Navier-Stokes is one thing to try, but there are other possibilities. For example, in connection with the Burgers equation, think of the maximum principle. For the semi-linear heat equation one should distinguish the case $\kappa = 1$ from the case $\kappa = 1$.

The global well-posedness problem is quite well-understood for examples 1 and 2, whereas it is open for example 3. (It should be an easier problem than Navier-Stokes.)