Math 8590 **Spring 2014**

Homework assignment 3

due May 14, 2014

Important: please send your completed assignment by e-mail[1]

We will consider *complex-valued* functions $u = u(x,t)$ on $\mathbf{R}^3 \times (0,\infty)$. We use the usual notation $\overline{u}$ for complex conjugate: if $u = u_1 + iu_2$, then $\overline{u} = u_1 - iu_2$. We will write $|u|^2 = u\overline{u}$. Let $\varepsilon > 0$ and let us consider the *Complex Ginzburg-Landau equation*

$$iu_t + (1 - i\varepsilon)\Delta u + |u|^2 u = 0\,. \tag{1}$$

We will consider the Cauchy problem for this equation in $\mathbf{R}^3 \times (0,\infty)$: given an initial condition $u_0 = u_0(x)$, find a solution $u(x,t)$ of the equation with $u(x,0) = u_0(x)$.
(One can also consider the same problem on for solutions which are, say, 1−periodic in space, i. e. $u(x+k,t) = u(x,t)$ for each $k \in \mathbf{Z^3}$. This amounts to solving the equation on the torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z^3}$. You can work in that setting, if you prefer. The advantage is that one can use Fourier series.)

(i) Show that the solutions of the equation satisfy an energy identity similar to the energy identity for the Navier-Stokes equations.

(ii) Outline a proof of existence of weak solutions of the Cauchy problem for any initial condition $u_0 \in L^2(\mathbf{R}^3)$.

(iii) Show that for a smooth $u_0 \in L^2 \cap L^3$, any two smooth solutions of the Cauchy problem which belong to the energy space $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ have to coincide.[2]

(iv) (optional) Try to formulate and prove a weak-strong uniqueness theorem.

[1] A scan of a hand-written text is fine

[2] One can prove a stronegr uniqueness result for smooth solutions, but we are not aiming for the maximal generality.