Part 1 : Mild solutions

The notion of mild solutions comes from the observation that the Navier-Stokes equations can be transformed into a form similar to a heat equation. For heat equations, we know the explicit formula, which involves the heat source and the initial condition, for the solution. We the use this formula to define mild solutions for the Navier-Stokes equations. Our analysis for mild solutions is guided by our knowledge of heat equation's solutions. Such analysis is called <u>perturbation analysis</u>. In this write-up, we discuss the following issues regarding to mild solutions based on Leray's paper (1934), Kato's paper (1984) and the series of lectures by Professor Vladimir Sverak in the course Topics in PDE, Spring 2014.

- Definition
- Local-in-time existence and uniqueness
- Regularity
- Energy Identity
- Sufficient conditions for global-in-time existence.
- Characterizations of finite time blowup.

Some parts of the proofs which were already included in the homework solutions of the course are often not repeated. In the sequel, we use the symbol C to denote various positive numeric constants which we do not specify their values. We adopt such operations as $C^2 = C$, $2C = C$, $C + C = C, \dots$

Let $u_0 \in (L^2 \cap L^\infty)(\mathbb{R}^3, \mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in sense of distribution, and $0 < T \leq \infty$. Consider the 3D Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3 \end{cases} \qquad \text{(NSE)}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$. The gradient and Laplacian are with respect to the spatial variables.

$\boxed{1}$ Definition

With the notation $u \otimes u := (u_i u_j)_{1 \leq i,j \leq 3}$, we have $(u \cdot \nabla) u = \operatorname{div}(u \otimes u)$.

Put $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) = -\operatorname{div}(u \otimes u)$. Then (NSE) becomes

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3. \end{cases} \qquad (I)$$

Put $X = \{ v \in L^2(\mathbb{R}^3, \mathbb{R}^3) : \operatorname{div} v = 0 \text{ in sense of distribution} \}$.

Let $P : L^2(\mathbb{R}^3, \mathbb{R}^3) \to \overline{X}$ be the orthonormal projection map. Write

$$f = \underbrace{Pf}_{\substack{\text{divergence} \\ \text{free}}} + \underbrace{(Id - P) f}_{\substack{\text{gradient of} \\ \text{some function}}}.$$

If f is a regular function and decays rapidly as $x \to \infty$ (for each fixed t), then the map $F: \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$,

$$F(x,t) = \int_{\mathbb{R}^3} -\frac{f(y,t)}{4\pi|x-y|}\,dy$$

satisfies $F(t) \in C_x^\infty$ and $\Delta F(t) = f(t)$ in $\mathbb{R}^3$. We have

$$\mathbb{P}f = -\mathrm{curl}(\mathrm{curl}\, F) \quad \text{and} \quad (\mathrm{Id} - \mathbb{P})f = \nabla(\mathrm{div}\, F).$$

We choose the pressure $p(x,t) = \mathrm{div}\, F = -\frac{1}{4\pi}\frac{\partial}{\partial y_j}\int_{\mathbb{R}^3}\frac{f_j(x,t)}{|x-y|}\,dy.$ (1)

Then (I) becomes

$$\begin{cases} \partial_t u - \Delta u = \mathbb{P}f & \text{in } \mathbb{R}^3 \times (0,T), \\ \mathrm{div}\, u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3. \end{cases} \qquad (\mathrm{II})$$

Now (II) looks like a heat equation if we ignore the dependancy between f and u. By Duhamel's principle,

$$u(x,t) = \Gamma(t) * u_0 + \int_0^t \Gamma(s) * (\mathbb{P}f)(t-s)\,ds, \qquad (2)$$

where $\Gamma(x,t) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{|x|^2}{4t}\right)$.

In non-rigorous terms, we call a function u satisfying (2) a mild solution to (NSE). Whenever u is given, p is obtained by (1). Now we want to write

$$\int_0^t \Gamma(s) * (\mathbb{P}f)(t-s)\,ds = \int_0^t K(s) * f(t-s)\,ds.$$

This representation helps us define the spaces where mild solutions live. By that, the definition of mild solutions can be made rigorous. Recall that $G(x) = \frac{-1}{4\pi|x|}$ is the fundamental solution to Laplace's equation in 3D.

Then $F = G * f$. We have

$$\operatorname{div} F = \operatorname{div}(G * f) = (G * f_j)_{,j} = G_{,j} * f_j .$$

Thus, $(\mathbb{I}d - \mathbb{P})f = \nabla(\operatorname{div} F) = (G_{,ij} * f_j)_{1\le i \le 3}$. Then

$$(\mathbb{P}f)_i = f_i - [(\mathbb{I}d - \mathbb{P})f]_i = f_i - G_{,ij} * f_j .$$

Thus, $\Gamma(s) * (\mathbb{P}f)_i(t-s) = \Gamma(s) * f_i(t-s) - \Gamma(s) * G_{,ij} * f_j(t-s)$

$$= \left(\delta_{ij}\Gamma(s) - \Gamma(s) * G_{,ij}\right) * f_j(t-s) \qquad (3)$$

Put $\Phi(x,t) = (\Gamma(t) * G)(x)$. Then

$$\Delta\Phi(x,t) = (\Gamma(t) * \Delta G)(x) = \Gamma(t), \text{ and } \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \Gamma(t) * G_{,ij} .$$

Then (3) can be written as

$$\Gamma(s) * (\mathbb{P}f)_i(t-s) = \left(\delta_{ij}\Delta\Phi(s) - \frac{\partial^2\Phi}{\partial x_i \partial x_j}(s)\right) * f_j(t-s).$$

Put $K_{ij}(x,t) = \delta_{ij}\Delta\Phi(x,t) - \frac{\partial^2\Phi}{\partial x_i\partial x_j}(x,t)$, and $K = (K_{ij})_{1\le i,j\le 3}$. (4)

Then $\Gamma(s) * (\mathbb{P}f)(t-s) = K(s) * f(t-s)$. We can rewrite (2) as

$$\begin{aligned} u(x,t) &= \Gamma(t) * u_0 + \int_0^t K(s) * f(t-s)\,ds \\ &= \Gamma(t) * u_0 + \int_0^t K(t-s) * f(s)\,ds \\ &= \Gamma(t) * u_0 - \int_0^t K(t-s) * \operatorname{div}(u(s)\otimes u(s))\,ds. \end{aligned}$$

Componentwise, the above formula reads

$$u_i(x,t) = \Gamma(t) * u_{0i} - \int_0^t K_{ij}(t-s) * (u_j(s)u_\ell(s))_{,\ell}\, ds$$

$$= \Gamma(t) * u_{0i} - \int_0^t K_{ij,\ell}(t-s) * (u_j(s)u_\ell(s))\, ds.$$

Put $K'_{ij\ell}(x,t) = -K_{ij,\ell}(x,t)$ and $K' = (K'_{ij\ell})_{1\le i,j\le 3}$. Then we can write

$$u(x,t) = \Gamma(t) * u_0 + \int_0^t K'(t-s) * (u(s)\otimes u(s))\, ds. \qquad (5)$$

Here K' plays the role of a "Navier-Stokes" kernel. Similar to the heat kernel, its decays as $x\to\infty$ reveals something about the regularity of u. We have

$$\Phi(x,t) = \Gamma(t) * G = -\frac{1}{4\pi(4\pi t)^{3/2}} \int_{\mathbb{R}^3} \exp\left(-\frac{|y|^2}{4t}\right) \frac{1}{|x-y|}\, dy$$

$$\overset{z=y/\sqrt{t}}{=\!=} -\frac{1}{(4\pi)^{5/2}} \int_{\mathbb{R}^3} \exp\left(-\frac{|z|^2}{4}\right) \frac{1}{|x-z\sqrt{t}|}\, dz$$

$$= -Ct^{-1/2} \int_{\mathbb{R}^3} \exp\left(-\frac{|z|^2}{4}\right) \frac{1}{\left|\frac{x}{\sqrt{t}} - z\right|}\, dz$$

$$= -C\, t^{-1/2} F\left(\frac{x}{\sqrt{t}}\right), \qquad (6)$$

where $$F(x) = \int_{\mathbb{R}^3} \frac{\exp\left(-\frac{|z|^2}{4}\right)}{|x-z|}\, dz. \qquad (7)$$

Because F is the convolution of a smooth rapidly decaying function and an $L^1_{loc}(\mathbb{R}^3)$ function, $F: \mathbb{R}^3 \to \mathbb{R}$ is smooth. Hence, K' is smooth in $\mathbb{R}^3\times(0,T)$.

To examine the decay of K' as $x\to\infty$, it is necessary to examine the decay of $\dfrac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_\ell}$. By (6),

$$\frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_\ell}(x,t) = -C t^{-2} \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_\ell}\left(\frac{x}{\sqrt{t}}\right). \qquad (8)$$

We'll examine the decay of $\dfrac{\partial^2 F}{\partial x_i \partial x_j}$. The ~~the~~ decay of $\dfrac{\partial^3 F}{\partial x_i \partial x_j \partial x_\ell}$ can be done in a ~~mi~~ similar manner. By (7),

$$\frac{\partial F}{\partial x_i}(x) = \int_{\mathbb{R}^3} -\frac{x_i - z_i}{|x-z|^3} \exp\left(-\frac{|z|^2}{4}\right) dz \sim \tilde{F}(x) = \int_{\mathbb{R}^3} \frac{\exp\left(-\frac{|z|^2}{4}\right)}{|x-z|^2} dz.$$

The second derivatives of F are of the same order as the first derivatives of $\tilde{F}$. We cannot, however, write

$$\frac{\partial \tilde{F}}{\partial x_i}(x) = \int_{\mathbb{R}^3} \frac{\partial}{\partial x_i}\left(\frac{1}{|x-z|^2}\right) \exp\left(-\frac{|z|^2}{4}\right) dz$$

because the function $x \mapsto |x|^{-3}$ is not locally integrable in $\mathbb{R}^3$. We have

$$\tilde{F}(x) = \underbrace{\int_{|x-z|<1} \frac{1}{|x-z|^2} \exp\left(-\frac{|z|^2}{4}\right) dz}_{\{1\}} + \underbrace{\int_{|x-z|>1} \frac{1}{|x-z|^2} \exp\left(-\frac{|z|^2}{4}\right) dz}_{\{2\}}. \qquad (9)$$

$$\{1\} = \int_{|z|<1} \frac{\exp\left(-\frac{|x-z|^2}{4}\right)}{|z|^2} dz,$$

$$\frac{\partial \{1\}}{\partial x_i} = \int_{|z|<1} -\frac{x_i - z_i}{2} \frac{\exp\left(-\frac{|x-z|^2}{4}\right)}{|z|^2} dz.$$

Thus, $$\left|\frac{\partial \{1\}}{\partial x_i}\right| \le \int_{|z|<1} \frac{\frac{1}{2}|x-z| \exp\left(-\frac{|x-z|^2}{4}\right)}{|z|^2} dz$$

$$\leq \underbrace{\left(\int_{|z|<1} \frac{dz}{|z|^2} \right)}_{\text{Const}} \underbrace{\sup\left\{ \frac{1}{2}|y| \exp\left(-\frac{|y|^2}{4}\right) : |y-x|<1 \right\}}_{\to 0 \text{ exponentially as } x\to\infty}.$$

Thus, $\frac{\partial\{1\}}{\partial x_i}$ decays exponentially as $x\to\infty$.

Let e_i be the unit vector in the i'th direction in $\mathbb{R}^3$. We have

$$\frac{\{2\}(x+se_i)-\{2\}(x)}{s} = \frac{1}{s}\left(\int_{\mathbb{R}^3\setminus B_1(x+se_i)} \frac{1}{|x+se_i-z|^2}\exp\left(-\frac{|z|^2}{4}\right)dz - \int_{\mathbb{R}^3\setminus B_1(x)} \frac{1}{|x-z|^2}\exp\left(-\frac{|z|^2}{4}\right)dz \right)$$

$$= \underbrace{\frac{1}{s}\int_{\mathbb{R}^3\setminus B_1(x)} \left(\frac{1}{|x+se_i-z|^2} - \frac{1}{|x-z|^2}\right)\exp\left(-\frac{|z|^2}{4}\right)dz}_{\{3\}}$$

$$\underbrace{+\frac{1}{s}\left(\int_{B_1(x)\setminus B_1(x+se_i)} - \int_{B_1(x+se_i)\setminus B_1(x)} \right) \frac{1}{|x+se_i-z|^2}\exp\left(-\frac{|z|^2}{4}\right)dz}_{\{4\}}.$$

For $0<|s|<\frac{1}{4}$, $|x+se_i-z|>\frac{1}{4}$ for all $z\in(B_1(x)\setminus B_1(x+se_i))\cup(B_1(x+se_i)\setminus B_1(x))$

Thus,

$$|\{4\}| \leq \frac{1}{s}\left| \int_{B_1(x)\setminus B_1(x+se_i)} - \int_{B_1(x+se_i)\setminus B_1(x)} \right| \underbrace{4\exp\left(-\frac{|z|^2}{4}\right)}_{\leq 4} dz$$

$$\leq \frac{4}{s}\left(\underbrace{|B_1(x)\setminus B_1(x+se_i)|}_{\sim s^3} + \underbrace{|B_1(x+se_i)\setminus B_1(x)|}_{\sim s^3} \right)$$

$$\to 0 \quad \text{as} \quad s\to 0.$$

On the other hand, $\{3\} \longrightarrow \int_{\mathbb{R}^3\setminus B_1(x)} \frac{\partial}{\partial x_i}\left(\frac{1}{|x-z|^2}\right)\exp\left(-\frac{|z|^2}{4}\right)dz$ as $s\to 0$.

Therefore, $\dfrac{\partial\{2\}}{\partial x_i} = \displaystyle\int_{\mathbb{R}^3\setminus B_1(x)} \frac{\partial}{\partial x_i}\left(\frac{1}{|x-z|^2}\right)\exp\left(-\frac{|z|^2}{4}\right)dz.$

Then (9) implies

$$\frac{\partial\tilde{F}}{\partial x_i} = \underbrace{\frac{\partial\{1\}}{\partial x_i}}_{\substack{\text{decays exponentially}\\ \text{as } x\to\infty}} + \int_{\mathbb{R}^3\setminus B_1(x)} \frac{\partial}{\partial x_i}\left(\frac{1}{|x-z|^2}\right)\exp\left(-\frac{|z|^2}{4}\right)dz \qquad (10)$$

Thus, $\dfrac{\partial\tilde{F}}{\partial x_i} \sim \displaystyle\int_{\mathbb{R}^3\setminus B_1(x)} \frac{1}{|x-z|^3}\exp\left(-\frac{|z|^2}{4}\right)dz$ as $x\to\infty$. (11)

For $z\in\mathbb{R}^3\setminus B_1(x)$, we have

$$\frac{|x|^3}{|x-z|^3} \le \frac{4(|x-z|^3+|z|^3)}{|x-z|^3} = 4\left(1+\frac{|z|^3}{|x-z|^3}\right) \le 4(1+|z|^3).$$

Multiplying both sides of (11) by $|x|^3$ and using the above inequality, we get

$$\begin{aligned}|x|^3\frac{\partial\tilde{F}}{\partial x_i}(x) &\sim \int_{\mathbb{R}^3\setminus B_1(x)} 4(1+|z|^3)\exp\left(-\frac{|z|^2}{4}\right)dz\\ &\le \int_{\mathbb{R}^3} 4(1+|z|^3)\exp\left(-\frac{|z|^2}{4}\right)dz\\ &= C\end{aligned}$$

Thus, $\left|\dfrac{\partial\tilde{F}}{\partial x_i}(x)\right| \le C|x|^{-3}$ for all $x\in\mathbb{R}^3\setminus\{0\}$. Hence,

$$\left|\frac{\partial^2 F}{\partial x_i\partial x_j}(x)\right| \le C|x|^{-3} \quad \forall x\in\mathbb{R}^3\setminus\{0\}.$$

By differentiating both sides of (10), we get

$$\left|\frac{\partial^3 F}{\partial x_i\partial x_j\partial x_\ell}(x)\right| \le C|x|^{-4} \quad \forall x\in\mathbb{R}^3\setminus\{0\}. \qquad (12)$$

By (8) and (12), we get

$$|K'(x,t)| \leq C t^{-2} H\left(\frac{x}{\sqrt{t}}\right), \qquad (13)$$

where $H(x) \leq C|x|^{-4}$ as $x \to \infty$, and H is a smooth function in $\mathbb{R}^3$.

Now we return to define the mild solutions to (NSE). In case $u_0 \in (L^2 \cap L^\infty)(\mathbb{R}^3;\mathbb{R}^3)$, there are several options for the solution space $\mathcal{X}$. Consider 2 options:

<u>Subcritical setting</u>: $\mathcal{X} = L^\infty_{t,x}(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$.

A function $u \in \mathcal{X}$ is called a mild solution to (NSE) if it satisfies

$$u(x,t) = T(t) * u_0 + \int_0^t K'(t-s) * (u(s) \otimes u(s))\, ds. \qquad (14)$$

The pressure is then given by

$$p(x,t) = \frac{1}{4\pi} \frac{\partial}{\partial x_j} \int_{\mathbb{R}^3} \frac{(u_j(s)\, u_\ell(s))_{,\ell}}{|x-y|}\, dy. \qquad (15)$$

<u>Critical setting</u>: $\mathcal{X} = L^5_{t,x}(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$.

The formulae defining u and p are still (14) and (15).

In the next section, Local-in-time existence, we'll show that (14) is satisfied in sense of an equation in $\mathcal{X}$ if T is sufficiently small. In the section following it, Regularity, we'll show that (14) is satisfied pointwise and u is smooth. Consequently, (15) is a well-defined formula for the pressure.

$\boxed{2}$ Local-in-time existence and uniqueness

In this section, we show the short-time existence of mild solutions in the critical and subcritical settings. In each setting, we show that the short-time existence can be applied repeatedly, so ~~that~~ called the <u>Continuation method</u>, so that we get a mild solution on a maximal time-interval.

* <u>Subcritical setting</u>: $\mathcal{X} = L^\infty_{t,x}(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$.

Our proof is similar to Homework #1 (the model equation), Topics in PDE, Spring 2014. $\mathcal{X}$ is a Banach space with norm

$$\|f\|_{\mathcal{X}} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{R}^3\times(0,T)} |f(x,t)|.$$

Define a bilinear map $B: \mathcal{X}\times\mathcal{X} \to \mathcal{X}$, $B(u,v)_i(x,t) = \int_0^t K'_{ije}(t-s) * (u_j(s) v_e(s))\,ds.$ (16)

By the previous section, a function $u \in \mathcal{X}$ satisfying the equation

$$u(t) = \Gamma(t) * u_0 + B(u,u)(x,t) \qquad (17)$$

is called a mild solution to Problem (NSE). But first, we need to show that B is well-defined, i.e. to show that $B(u,v) \in \mathcal{X}$. Let us evaluate the quantity $\|K'(t)\|_{L^a_x}$ based on (13). Because $H: \mathbb{R}^3 \to \mathbb{R}$ is smooth and $H(x) \sim |x|^{-4}$ as $x \to \infty$, $H \in L^a(\mathbb{R}^3)$ for every $1 \le a \le \infty$. For $1 \le a < \infty$,

$$\cancel{\|K'(t)\|^a_{L^a_x} = \int_{\mathbb{R}^3} |K'(t)|^a dx \overset{(13)}{\le} C t^{-3a/2} \int_{\mathbb{R}^3} H\left(\tfrac{x}{\sqrt{t}}\right)^a dx}$$

$$\cancel{\overset{y = x/\sqrt{t}}{=\!=} C t^{-3a/2} \int_{\mathbb{R}^3} H(y)^a\, t^{3/2}\, dy}$$

$$\cancel{= C t^{-\frac{3}{2}(a-1)} \|H\|^a_{L^a}.}$$

$$\|K'(t)\|_{L^a_x}^a = \int_{\mathbb{R}^3} |K'(t)|^a dx \overset{(13)}{\leq} C t^{-2a} \int_{\mathbb{R}^3} H\left(\frac{x}{\sqrt{t}}\right)^a dx$$

$$\underline{\underline{y = x/\sqrt{t}}} \; C t^{-2a} \int_{\mathbb{R}^3} H(y)^a t^{3/2} dy$$

$$= C t^{-a/2} t^{-\frac{3}{2}(a-1)} \|H\|_{L^a}^a .$$

Thus, $\|K'(t)\|_{L^a_x} \leq C t^{-\frac{1}{2} - \frac{3}{2a'}} \|H\|_{L^a}$, (18)

where $\frac{1}{a} + \frac{1}{a'} = 1$. We see that (18) is also true for $a = \infty$. For $a = 1$,

(18) gives $\|K'(t)\|_{L^1_x} \leq C t^{-1/2}$. (19)

From (16), $|B(u,v)(x,t)| \leq \int_0^t |K'(t-s) * (u(s) \otimes v(s))| ds$

$$\leq \int_0^t \|K'(t-s)\|_{L^1_x} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} ds$$

$$\overset{(19)}{\leq} C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \int_0^t \frac{ds}{\sqrt{t-s}}$$

$$= C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \sqrt{t}. \qquad (20)$$

Thus, $\|B(u,v)\|_{\mathcal{X}} \leq C\sqrt{T} \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \quad \forall u, v \in \mathcal{X}$. (21)

Thus, B is well-defined. Moreover, B is a continuous bilinear map. The rest of the proof of local-in-time existence is completely the same as Part (b), Subcritical setting, of Homework #1, Topics in PDE, 2014, from page 4 to 8. We obtain the existence of a solution $u \in \mathcal{X}$ to (17) if T is small enough. Specifically, the existence is obtained if $\sqrt{T} \|u_0\|_{L^\infty} < C$. For $0 \leq t_1 < t_2 < \infty$, put $\mathcal{X}_{t_1, t_2} = L^\infty_{t,x}(\mathbb{R}^3 \times (t_1, t_2))$.

According to the homework solution, if $\sqrt{t_2-t_1}\,\|u(\cdot,t_1)\|_{L^\infty_x} < C$ then there exists a unique mild solution $u \in \mathcal{X}_{t_1,t_2}$. Moreover, $u \in C_t L^\infty_x(\mathbb{R}^3 \times (t_1,t_2])$.

Thus, thanks to the continuation method, the mild solutions exist and are unique on a maximal time-interval $[0,T^*)$. Moreover, $u \in C_t L^\infty_x(\mathbb{R}^3 \times (0,T^*))$.

* <u>Critical setting</u>: $\mathcal{Y} = L^5_{t,x}(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$

Our proof is similar to Homework # 1 (the model equation), Topics in PDE, Spring 2014. $\mathcal{Y}$ is a Banach space with norm

$$\|f\|_{\mathcal{Y}} = \left(\int_0^T \int_{\mathbb{R}^3} |f(x,t)|^5 \, dx\, dt \right)^{1/5}.$$

Define a bilinear map $B : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$,

$$B(u,v)_i(x,t) = \int_0^t K'_{ij\ell}(t-s) * (u_j(s)\, v_\ell(s))\, ds. \qquad (22)$$

By Section $\boxed{1}$, a function $u \in \mathcal{Y}$ satisfying the equation

$$u(t) = \Gamma(t) * u_0 + B(u,u)(x,t) \qquad (23)$$

is called a mild solution to problem (NSE). The rest of the proof of local-in-time existence is completely the same as Part (a) and (b), Critical setting, Homework #1, Topics in PDE, 2014, pages 18-33. Perhaps the only point to doublecheck is that $\|K'(t)\|_{L^{5/4}_x} \le C t^{-4/5}$. But this follows immediately from (18) where $a = 5/4$. ~~For $0 \le t_1 < t_2 < \infty$, put~~

$$\cancel{\mathcal{Y}_{t_1,t_2} = L^5_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3).}$$

We obtain the existence of a solution $u \in \mathcal{Y}$ to (23) if T is small enough. Specifically, the existence is obtained if $\|\Gamma(t) * u_0\|_{\mathcal{Y}_{0,T}} < C$. For $0 \le t_1 < t_2 < \infty$, put $\mathcal{Y}_{t_1,t_2} = L^5_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$. According to the Homework solution, if $\|\Gamma(t-t_1) * u(\cdot,t_1)\|_{\mathcal{Y}_{t_1,t_2}} < C$ then there exists a unique mild solution $u \in \mathcal{Y}_{t_1,t_2}$. Moreover, $u \in C_t L^3_x(\mathbb{R}^3 \times (t_1,t_2])$. Thus, by the continuation method, the mild solutions exist and are unique on a maximal time-interval $[0,T^{**})$. Moreover, $u \in C_t L^3_x(\mathbb{R}^3 \times (0,T^{**}), \mathbb{R}^3) \cap L^5_{t,x}(\mathbb{R}^3 \times (0,T^{**}), \mathbb{R}^3)$. In the Homework solution, we also verified the Ladyzhenskaya-Prodi-Serrin's theorem saying if $T^{**} < \infty$ then $\|u\|_{\mathcal{Y}_{0,T^{**}}} = \infty$. A remarkable result in the critical setting is that if $\|u_0\|_{L^3} < C$ for some constant C then (23) has a global-in-time solution.

$\boxed{3}$ Regularity

* <u>Subcritical setting</u>: $\mathcal{X} = L^\infty_{t,x}(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$

We will show that $u \in C^\infty(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$. Thanks to the Sobolev imbedding theorems, it suffices to show that

$$t^{\frac{m}{2}+l} \partial_t^l \partial_x^m u \in L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3) \qquad \forall\, 0 < T_1 < T. \qquad (24)$$

By Section $\boxed{2}$, Subcritical setting, we have

(i) $u \in C_t L^\infty_x(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$.

(ii) If $\sqrt{t_2 - t_1}\, \|u(\cdot,t_1)\|_{L^\infty_x} < C$ then the mild solution to (17) exists in $\mathcal{X}_{t_1,t_2} = L^\infty_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$.

Fix $T_1 \in (0,T)$. Put $M = \sup_{t\in[0,T_1]} \|u(t)\|_{L^\infty_x} < \infty$. By dividing the interval $(0,T_1)$ into subinterval of length less than $\left(\frac{C}{M}\right)^2$ if necessary, we can assume $\sqrt{T_1}\, M < C$. First, we'll show that

$$t^{1/2}\partial_x u \in L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3). \qquad (25)$$

We have $u(t) = \Gamma(t) * u_0 + B(u,u)$. By Section [2], Subcritical setting,

$$\|B(u,v)\|_{\mathcal{X}_{0,T_1}} \le C\sqrt{T_1}\, \|u\|_{\mathcal{X}_{0,T_1}} \|v\|_{\mathcal{X}_{0,T_1}} \quad \forall u,v \in \mathcal{X}_{0,T_1}$$

For each $i=1,2,3$ and $h \in (-1,1)\setminus\{0\}$, denote $\Delta_i^h u(x,t) = \dfrac{u(x+he_i,t)-u(x,t)}{h}$.

Then $\Delta_i^h u(x,t) \cong (\Delta_i^h \Gamma(t)) * u_0 + B(\Delta_i^h u, u) + B(u, \Delta_i^h u)$. $\qquad (26)$

We have $|B(\Delta_i^h u, u)(x,t)| \le C\sqrt{T_1}\, \|\Delta_i^h u\|_{\mathcal{X}_{0,T_1}} \|u\|_{\mathcal{X}_{0,T_1}} \le \frac{1}{4}\|\Delta_i^h u\|_{\mathcal{X}_{0,T_1}} \quad \forall t \in (0,T_1)$

By (26),

$$\begin{aligned} |\Delta_i^h u(x,t)| &\le |\Delta_i^h \Gamma(t) * u_0| + |B(\Delta_i^h u, u)| + |B(u, \Delta_i^h u)| \\ &\le \|\Delta_i^h \Gamma(t)\|_{L^1_x} \|u_0\|_{L^\infty} + \frac{1}{4}\|\Delta_i^h u\|_{\mathcal{X}_{0,T_1}} + \frac{1}{4}\|\Delta_i^h u\|_{\mathcal{X}_{0,T_1}} \\ &\le \|\partial_x \Gamma(t)\|_{L^1_x} \|u_0\|_{L^\infty} + \frac{1}{2}\|\Delta_i^h u\|_{\mathcal{X}_{0,T_1}}. \qquad (27) \end{aligned}$$

We have $\partial_x \Gamma(t) = \dfrac{-Cx}{t^{5/2}} \exp\left(-\dfrac{|x|^2}{4t}\right)$, and

$$\|\partial_x \Gamma(t)\|_{L^1_x} = \int_{\mathbb{R}^3} \frac{C|x|}{t^{5/2}} \exp\left(-\frac{|x|^2}{4t}\right) dx \xrightarrow{y = x/\sqrt{t}} \int_{\mathbb{R}^3} \frac{C|y|}{t^2} \exp(-|y|^2)\, t^{3/2}\, dy = Ct^{-1/2}.$$

Thus, (27) implies

$$|\Delta_i^h u(x,t)| \le Ct^{-1/2}\|u_0\|_{L^\infty} + \frac{1}{2}\|\Delta_i^h u\|_{\mathcal{X}_{0,T_1}} \quad \forall t \in (0,T_1).$$

Multiplying both sides by $t^{1/2}$, we get

$$t^{1/2}|\Delta_i^h u(x,t)| \le C\|u_0\|_{L^\infty} + \frac{1}{2} t^{1/2}\|\Delta_i^h u\|_{\mathcal{X}_{0,T_1}} \quad \forall t \in (0,T_1).$$

Thus, $\| t^{1/2}\Delta_i^h u\|_{\mathcal{X}_{0,T_1}} \leq C\|u_0\|_{L^\infty} + \frac{1}{2}\| t^{1/2}\Delta_i^h u\|_{\mathcal{X}_{0,T_1}}$. Hence,

$$\|\cancel{t^{1/2}}\Delta_i^h (t^{1/2}u)\|_{\mathcal{X}_{0,T_1}} \leq 2C\|u_0\|_{L^\infty} \qquad \forall h \in (-1,1)\setminus\{0\}.$$

This means $t^{1/2}\partial_x u \in L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$. We now show by induction in $m\in\mathbb{N}$ that $t^{m/2}\partial_x^m u \in L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$. (28)

(28) is true for $m=1$. Suppose that (28) is true for some $m\geq 1$. Differentiating m times the equation $u(x,t) = \Gamma(t) * u_0 + B(u,u)$, we get

$$\partial_x^m u(x,t) = (\partial_x^m \Gamma(t)) * u_0 + \sum_{k=0}^m \binom{m}{k} B(\partial_x^k u, \partial_x^{m-k} u). \qquad (29)$$

Thus,

$$\Delta_i^h(\partial_x^m u)(x,t) = (\Delta_i^h(\partial_x^m\Gamma)(t)) * u_0 + \sum_{k=0}^m \binom{m}{k}\big[B(\Delta_i^h(\partial_x^k u), \partial_x^{m-k}u) + B(\partial_x^k u, \Delta_i^h(\partial_x^{m-k}u))\big].$$

Hence,

$$\begin{aligned}
t^{(m+1)/2}|\Delta_i^h(\partial_x^m u)(x,t)| &\leq t^{(m+1)/2}|(\Delta_i^h(\partial_x^m\Gamma)(t)) * u_0| \\
&\quad + \sum_{k=0}^m \binom{m}{k}\Big(|B(t^{\frac{k+1}{2}}\Delta_i^h(\partial_x^k u), t^{\frac{m-k}{2}}\partial_x^{m-k}u)| + \\
&\quad + |B(t^{\frac{k}{2}}\partial_x^k u, t^{\frac{m-k}{2}}\Delta_i^h(\partial_x^{m-k}u))|\Big) \\
&\leq t^{(m+1)/2}\|\Delta_i^h(\partial_x^m\Gamma)(t)\|_{L^1_x}\|u_0\|_{L^\infty} + \sum_{k=0}^m\binom{m}{k} C\|t^{\frac{k+1}{2}}\Delta_i^h(\partial_x^k u)\|_{\mathcal{X}_{0,T_1}}\|t^{\frac{m-k}{2}}\partial_x^{m-k}u\|_{\mathcal{X}_{0,T_1}} \\
&\leq t^{(m+1)/2}\|(\partial_x^{m+1}\Gamma)(t)\|_{L^1_x}\|u_0\|_{L^\infty} + \underbrace{\sum_{k=0}^{m-1}\binom{m}{k}C\|t^{(k+1)/2}\Delta_i^h(\partial_x^k u)\|_{\mathcal{X}_{0,T_1}}\|t^{(m-k)/2}\partial_x^{m-k}u\|_{\mathcal{X}_{0,T_1}}}_{=M_1<\infty} \\
&\quad + C\|t^{\frac{m+1}{2}}\Delta_i^h(\partial_x^m u)\|_{\mathcal{X}_{0,T_1}}\|u\|_{\mathcal{X}_{0,T_1}}. \\
&\leq t^{(m+1)/2}\|(\partial_x^{m+1}\Gamma)(t)\|_{L^1_x}\|u_0\|_{L^\infty} + M_1 + \frac{1}{2}\|t^{(m+1)/2}\Delta_i^h(\partial_x^m u)\|_{\mathcal{X}_{0,T_1}}. \qquad (30)
\end{aligned}$$

By induction in m, we can show that

$$(\partial_x^m\Gamma)(t) = \frac{P_m(x/\sqrt{t})}{t^{(m+3)/2}}\exp\left(-\frac{|x|^2}{4t}\right), \qquad (31)$$

where P_m is a homogeneous polynomial of degree m. Then

$$\|\partial_x^m \Gamma(t)\|_{L^1_x} \le \int_{\mathbb{R}^3} \frac{|P_m(x/\sqrt{t})|}{t^{(m+3)/2}} \exp\left(-\frac{|x|^2}{4t}\right) dx$$

$$\underset{\overline{\overline{\quad}}}{y=x/\sqrt{t}} \int_{\mathbb{R}^3} \frac{|P_m(y)|}{t^{(m+3)/2}} \exp\left(-\frac{|y|^2}{4}\right) t^{3/2} dy$$

$$= t^{-m/2} \int_{\mathbb{R}^3} |P_m(y)| \exp\left(-\frac{|y|^2}{4}\right) dy$$

$$= C(m)\, t^{-m/2}.$$

From now on, the notation $C(m)$ is used to denote various quantities which depend on m. We will adopt such notations as $C(m)+1 = C(m)$, $2C(m)=C(m)$, $C(m)^2 = C(m)$, ... From (30), we have

$$t^{\frac{m+1}{2}} |\Delta_i^h (\partial_x^m u)(x,t)| \le C(m)\|u_0\|_{L^\infty} + M_1 + \frac{1}{2}\|t^{\frac{m+1}{2}} \Delta_i^h (\partial_x^m u)\|_{X_{0,T_1}}.$$

Thus, $\|t^{\frac{m+1}{2}} \Delta_i^h (\partial_x^m u)\|_{X_{0,T_1}} \le 2(C(m)\|u_0\|_{L^\infty} + M) \quad \forall h \in (-1,1)\setminus\{0\}.$

Thus, $t^{\frac{m+1}{2}} \partial_x^{m+1} u \in L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1), \mathbb{R}^3)$. This means (28) is also true for $m+1$. Thus, it is true for all nonnegative integers m.

We'll show by induction on $l \ge 0$ that

$$t^{l+\frac{m}{2}} \partial_t^l \partial_x^m u \in L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1), \mathbb{R}^3) \quad \forall m \ge 0. \qquad (32)$$

(32) is true for $l=0$. Suppose that (32) is true for some $l\ge 0$. We'll show that it is true for $l+1$. From (29) we have

$$\partial_x^m u(x,t) = (\partial_x^m \Gamma)(t) * u_0 + \sum_{k=0}^{m} \binom{m}{k} \int_0^t K'(t-s) * (\partial_x^k u(s) \otimes \partial_x^{m-k} u(s))\, ds$$

$$= (\partial_x^m \Gamma)(t) * u_0 + \sum_{k=0}^{m} \binom{m}{k} \int_0^{t/2} K'(t-s) * \left(\partial_x^k u(s) \otimes \partial_x^{m-k} u(s)\right) ds$$

$$+ \sum_{k=0}^{m} \binom{m}{k} \int_0^{t/2} K'(s) * \left(\partial_x^k u(t-s) \otimes \partial_x^{m-k} u(t-s)\right) ds$$

$$= (\partial_x^m \Gamma)(t) * u_0 + \int_0^{t/2} K'(t-s) * \partial_x^m \left(u(s) \otimes u(s)\right) ds + \sum_{k=0}^{m} \binom{m}{k} \int_0^{t/2} K'(s) * \left(\partial_x^k u(t-s) \otimes \partial_x^{m-k} u(t-s)\right) ds$$

$$= (\partial_x^m \Gamma)(t) * u_0 + \int_0^{t/2} \partial_x^m K'(t-s) * \left(u(s) \otimes u(s)\right) ds + \sum_{k=0}^{m} \binom{m}{k} \int_0^{t/2} K'(s) * \left(\partial_x^k u(t-s) \otimes \partial_x^{m-k} u(t-s)\right) ds. \quad (33)$$

We'll work with the case $l=0$ only. The case $l>0$ can be done in the same way although the expressions look cumbersome. For $h \in (-1,1)\setminus\{0\}$ and function $v : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, we denote
$$\delta^h v(x,t) = \frac{v(x,t+h) - v(x,t)}{h}.$$

Applying δ^h to both sides of (33), we get

$$\delta^h(\partial_x^m u)(x,t) \approx \underbrace{\left((\partial_t \partial_x^m)\Gamma(t)\right) * u_0}_{\{1\}} + \underbrace{\partial_x^m K'\left(\tfrac{t}{2}\right) * \left(u\left(\tfrac{t}{2}\right) \otimes u\left(\tfrac{t}{2}\right)\right)}_{\{2\}} + \underbrace{\int_0^{t/2} \partial_t \partial_x^m K'(t-s) * \left(u(s) \otimes u(s)\right) ds}_{\{3\}}$$

$$+ \underbrace{\sum_{k=0}^{m} \binom{m}{k} K'\left(\tfrac{t}{2}\right) * \left(\partial_x^k u\left(\tfrac{t}{2}\right) \otimes \partial_x^{m-k} u\left(\tfrac{t}{2}\right)\right)}_{\{4\}} + \underbrace{\sum_{k=0}^{m} \binom{m}{k} \int_0^{t/2} K'(s) * \left(\delta^h \partial_x^k u(t-s) \otimes \partial_x^{m-k} u(t-s)\right) ds}_{\{5\}}$$

$$+ \underbrace{\sum_{k=0}^{m} \binom{m}{k} \int_0^{t/2} K'(s) * \left(\partial_x^k u(t-s) \otimes \delta^h \partial_x^{m-k} u(t-s)\right) ds}_{\{6\}}. \quad (34)$$

By (31), we have $\partial_t \partial_x^m \Gamma(x,t) = \frac{1}{t^{1+\frac{m+3}{2}}} Q\left(\frac{x}{\sqrt{t}}\right) \exp\left(-\frac{|x|^2}{4t}\right)$, where Q is a polynomial. Thus,

$$\|\partial_t^l \partial_x^m \Gamma(t)\|_{L^1_x} \overset{y = x/\sqrt{t}}{=\!=\!=} \int_{\mathbb{R}^3} \frac{1}{t^{1+\frac{m+3}{2}}}\, Q\left(\frac{\cancel{x}\, y}{\cancel{\sqrt{t}}}\right) \exp(-|y|^2)\, t^{3/2}\, dy = \frac{C}{t^{1+\frac{m}{2}}}.$$

Hence, $|\{1\}| = |\partial_t \partial_x^m \Gamma(t) * u_0| \le \|\partial_t^l \partial_x^m \Gamma(t)\|_{L^1_x} \|u_0\|_{L^\infty} \le \dfrac{C\|u_0\|_{L^\infty}}{t^{1+\frac{m}{2}}}$.

Thus, $\|t^{1+\frac{m}{2}}\{1\}\|_{\mathcal{X}_{0,T_1}} \le C\|u_0\|_{L^\infty} < \infty.$ (35)

By (13), $|\partial_x^m K'(x,t)| \le t^{-2-\frac{m}{2}} Q_m\left(\frac{x}{\sqrt{t}}\right)$, where $Q_m(x) \le C|x|^{-4-m}$ as $x \to \infty$.

Thus, $\|\partial_x^m K'(t)\|_{L^1_x} = \displaystyle\int_{\mathbb{R}^3} t^{-2-\frac{m}{2}} Q_m\left(\frac{x}{\sqrt{t}}\right) dx \overset{y = \frac{x}{\sqrt{t}}}{=\!=\!=} \int_{\mathbb{R}^3} t^{-2-\frac{m}{2}} Q_m(y)\, t^{3/2} dy$

$$= C(m)\, t^{-\frac{1}{2}-\frac{m}{2}}.$$

Hence, $|\{2\}| \le \|\partial_x^m K'(t)\|_{L^1_x} \left\|u\left(\frac{t}{2}\right)\right\|^2_{L^\infty_x} \le \dfrac{C(m)\,\|u\|^2_{\mathcal{X}_{0,T_1}}}{t^{1/2+m/2}}$.

Thus, $\|t^{1+\frac{m}{2}}\{2\}\|_{\mathcal{X}_{0,T_1}} \le C(m)\sqrt{T_1}\,\|u\|^2_{\mathcal{X}_{0,T_1}} < \infty.$ (36)

We have $|\partial_t \partial_x^m K'(x,t)| \le t^{-3-\frac{m}{2}} R_m\left(\frac{x}{\sqrt{t}}\right)$, where $|R_m(x)| \le C|x|^{-3-m}$ as $x \to \infty$.

Thus, $\|\partial_t \partial_x^m K'(t)\|_{L^1_x} \le \displaystyle\int_{\mathbb{R}^3} t^{-3-\frac{m}{2}} R_m\left(\frac{x}{\sqrt{t}}\right) dx \overset{y = \frac{x}{\sqrt{t}}}{=\!=\!=} t^{-3/2-m/2} \int_{\mathbb{R}^3} R_m(y)\, dy$

$$= t^{-3/2-m/2}\, C(m).$$

Then
$$|\{3\}| \le \int_0^{t/2} \|\partial_t \partial_x^m K'(t-s)\|_{L^1_x} \|u(s)\|^2_{L^\infty_x}\, ds$$

$$\le C(m)\,\|u\|^2_{\mathcal{X}_{0,T_1}} \int_0^{t/2} (t-s)^{-3/2-m/2}\, ds$$

$$\le C(m)\,\|u\|^2_{\mathcal{X}_{0,T_1}} \int_0^{t/2} \left(\frac{t}{2}\right)^{-3/2-m/2} ds$$

$$= C(m)\,\|u\|^2_{\mathcal{X}_{0,T_1}}\, t^{-1/2-m/2}.$$

Hence, $\|t^{1+\frac{m}{2}}\{3\}\|_{\mathcal{X}_{0,T_1}} \le C(m)\sqrt{T_1}\,\|u\|^2_{\mathcal{X}_{0,T_1}}.$ (37)

Because $\{4\}=\{2\}$, we have

$$\|t^{1+\frac{m}{2}}\{4\}\|_{\mathcal{X}_{0,T_1}} = \|t^{1+\frac{m}{2}}\{2\}\|_{\mathcal{X}_{0,T_1}} \leq C(m)\sqrt{T_1}\,\|u\|^2_{\mathcal{X}_{0,T_1}} < \infty. \qquad (38)$$

- For $0\leq k<m$,

$$\{5\} \approx \int_0^{t/2} K'(s) * (\partial_t \partial_x^k u(t-s) \otimes \partial_x^{m-k} u(t-s))\,ds \quad \text{(by the inductive hypothesis).}$$

Then

$$|\{5\}| \leq \int_0^{t/2} \underbrace{\|K'(s)\|_{L^1_x}}_{\overset{(18)}{\leq} \frac{C}{\sqrt{s}}} \underbrace{\|\partial_t \partial_x^k u(t-s)\|_{L^\infty_x}}_{\leq \frac{C(k)}{(t-s)^{1+\frac{k}{2}}}} \underbrace{\|\partial_x^{m-k} u(t-s)\|_{L^\infty_x}}_{\leq \frac{C(k)}{(t-s)^{\frac{m-k}{2}}}}\,ds$$

$$\leq \int_0^{t/2} \frac{C(k)}{\sqrt{s}\,(t-s)^{1+\frac{m}{2}}}\,ds \leq C(k)\,t^{-1-\frac{m}{2}} \int_0^{t/2} \frac{ds}{\sqrt{s}} = C(k)\,t^{-\frac{1}{2}-\frac{m}{2}}.$$

Thus, $\|t^{1+\frac{m}{2}}\{5\}\|_{\mathcal{X}_{0,T_1}} \leq C(k)\sqrt{T_1}$. $\qquad (39)$

- For $k=m$,

$$\{5\} = \int_0^{t/2} K'(s) * (\delta^h \partial_x^m u(t-s) \otimes u(t-s))\,ds.$$

Then

$$|\{5\}| \leq \int_0^{t/2} \|K'(s)\|_{L^1_x} \|\delta^h \partial_x^m u(t-s)\|_{L^\infty_x} \|u(t-s)\|_{L^\infty_x}\,ds$$

$$\leq \left(\int_0^{t/2} \frac{C}{\sqrt{s}}\,ds\right) \|u\|_{\mathcal{X}_{0,T_1}} \|\delta^h \partial_x^m u\|_{\mathcal{X}_{0,T_1}}$$

$$= C\sqrt{T_1}\,\|u\|_{\mathcal{X}_{0,T_1}} \|\delta^h \partial_x^m u\|_{\mathcal{X}_{0,T_1}}.$$

Thus, $\|t^{1+\frac{m}{2}}\{5\}\|_{\mathcal{X}_{0,T_1}} \leq C\sqrt{T_1}\,\|u\|_{\mathcal{X}_{0,T_1}} \|t^{1+\frac{m}{2}}\delta^h \partial_x^m u\|_{\mathcal{X}_{0,T_1}}$ $\qquad (40)$

The estimates for $\{6\}$ are the same as those for $\{5\}$. Substituting (35)-(40) into (34), we get

$$t^{1+\frac{m}{2}} |\delta^h(\partial_x^m u)(x,t)| \leq C\|u_0\|_{L^\infty} + C(m)\sqrt{T_1}\,\|u\|^2_{\mathcal{X}_{0,T_1}} + C(m)\sqrt{T_1} +$$
$$+ C\sqrt{T_1}\,\|u\|_{\mathcal{X}_{0,T_1}} \|t^{1+\frac{m}{2}}\partial_x^m u\|_{\mathcal{X}_{0,T_1}}. \qquad (41)$$

For sufficiently small T_1, we have $C\sqrt{T_1}\|u\|_{\mathcal{X}_{0,T_1}} < \frac{1}{2}$. Then (41) implies

$$\|t^{1+\frac{m}{2}}\delta^h \partial_x^m u\|_{\mathcal{X}_{0,T_1}} \leq 2\left(C\|u_0\|_{L^\infty} + C(m)\sqrt{T_1}\|u\|^2_{\mathcal{X}_{0,T_1}}\right), \quad \forall h.$$

Thus, $\partial_t \partial_x^m u$ exists for $0<t<T_1$. However, $\|t^{1+\frac{m}{2}}\partial_t\partial_x^m u\|_{\mathcal{X}_{0,T_1}} < \infty$.

Now let us discuss the regularity of u as $t \to 0^+$. Recall that u satisfies

$$u(x,t) = \Gamma(t) * u_0 + \int_0^t \underbrace{K'(s) * (u(t-s) \otimes u(t-s))}_{f(x,t,s)}\, ds \qquad \forall t \in (0,T).$$

Take any $T_1 \in (0,T)$. We know that $u \in L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3) = \mathcal{X}_{0,T_1}$. Thus,

$$|f(x,t,s)| \leq \|K'(s)\|_{L^1_x}\|u\|^2_{\mathcal{X}_{0,T_1}} \qquad \forall\ 0<s\leq t<T_1$$

$$\overset{(17)}{\leq} \frac{C\|u\|^2_{\mathcal{X}_{0,T_1}}}{\sqrt{s}} \qquad \forall\ 0<s\leq t<T_1.$$

Thus, $$\left|\int_0^t f(x,t,s)\,ds\right| \leq \int_0^t \frac{C\|u\|^2_{\mathcal{X}_{0,T_1}}}{\sqrt{s}}\,ds = C\|u\|^2_{\mathcal{X}_{0,T_1}}\sqrt{t} \longrightarrow 0 \text{ as } t \to 0^+.$$

By Theorem 1, page 63, Evans "Partial Differential Equations", if $u_0 \in C(\mathbb{R}^3,\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3,\mathbb{R}^3)$ then $(\Gamma(t)*u_0)(x) \longrightarrow u_0(x_0)$ as $(x,t)\to(x_0,0^+)$ for every $x_0 \in \mathbb{R}^3$. In such a case, $u \in C(\mathbb{R}^3\times[0,T),\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3\times(0,T),\mathbb{R}^3)$

* Critical setting: $\mathcal{Y} = L^5_{t,x}(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$.

The paper by Dong-Du "On the local smoothness of solutions of the Navier-Stokes equations", 2007, showed that

$$t^{l+\frac{m}{2}}\partial_t^l \partial_x^m u \in L^5_{t,x}(\mathbb{R}^d \times (0,T_1)) \qquad \forall\ 0<T_1<T. \qquad (42)$$

This result together with the Sobolev imbedding theorems implies the local smoothness of u: $\quad u \in C^\infty(\mathbb{R}^3\times(0,T),\mathbb{R}^3)$.

Unlike the subcritical setting, it is not clear how to show that u is continuous at $t=0^+$. The trouble is of the nonlinear term

$$B(u,u)(x,t) = \int_0^t K'(s) * (u(t-s) \otimes u(t-s))\, ds.$$

We only know that $u \in L^5_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)$ for all $T_1 \in (0,T)$ and $u \in C_t L^3(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$. Then $u(t-s) \otimes u(t-s) \in L^{5/2}_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)$. Then

$$|B(u,u)(x,t)| \leq \int_0^t \underbrace{\|K'(s)\|_{L^{5/3}_x}}_{\overset{(18)}{\leq} C s^{-1/10}} \underbrace{\|u(t-s)\|^2_{L^5_x}}_{\in L^{5/2}_s} ds.$$

We have failed to show that $B(u,u)(x,t) \to 0$ as $t \to 0^+$.

$\boxed{4}$ Energy identity

In Section $\boxed{2}$, subcritical setting, we proved the local-in-time existence and uniqueness of a mild solution $u \in L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)$, $T_1 \in (0,T)$. Although we assumed $u_0 \in L^2 \cap L^\infty$, we only used the assumption $u_0 \in L^\infty$. Recall that in order to use the continuation method to get the maximal time-interval of existence, we had to show that $u \in C_t L^\infty(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$. Thus, we seemed to lose the hypothesis $u_0 \in L^2$ after the first stage of existence. It is not clear how to show that $u(t) \in L^2_x$ from the identity

$$u(t) = \Gamma(t) * u_0 + \int_0^t K'(t-s) * (u(s) \otimes u(s))\, ds$$

and the assumption $u_0 \in L^\infty \cap L^2$. As we shall see, the property $u \in L^\infty_t L^2_x$ is essential to achieve the energy identity.

The idea to get the energy identity from (NSE) is as follows. Multiplying both sides of (NSE) by u and taking the integral both sides over $x \in \mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} u_t u \, dx - \int_{\mathbb{R}^3} u \Delta u \, dx + \int_{\mathbb{R}^3} [(u\cdot\nabla)u] u \, dx + \int_{\mathbb{R}^3} (\nabla p)\cdot u \, dx = 0. \qquad (43)$$

Ideally,

$$\int_{\mathbb{R}^3} u_t u \, dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 dx, \qquad (44)$$

$$\int_{\mathbb{R}^3} u \Delta u \, dx = -\int_{\mathbb{R}^3} |\nabla u|^2 dx, \qquad (45)$$

$$\int_{\mathbb{R}^3} [(u\cdot\nabla)u] u \, dx = \int_{\mathbb{R}^3} \left(\frac{|u|^2}{2}\right)_{,i} u_i \, dx = -\int_{\mathbb{R}^3} \frac{|u|^2}{2} \underbrace{u_{i,i}}_{=0} \, dx = 0, \qquad (46)$$

$$\int_{\mathbb{R}^3} (\nabla p). u \, dx = -\int_{\mathbb{R}^3} p \underbrace{(\nabla\cdot u)}_{=0} \, dx = 0. \qquad (47)$$

Then (43) becomes $\displaystyle \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx = 0.$

Integrating both sides over $t \in [t_1, t_2] \subset (0,\infty)$, we get

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(t_2)|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} |u(t_1)|^2 dx + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\nabla u|^2 dx\,dt = 0 \qquad (47')$$

Suppose that

$$\| u(t) - u_0 \|_{L^2_x} \to 0 \quad \text{as } t \to 0^+. \qquad (48)$$

Then (47') gives $\displaystyle \frac{1}{2}\int_{\mathbb{R}^3} |u(t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,s)|^2 dx\,ds = \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx. \qquad (49)$

This is called the <u>energy identity</u>. We need to justify the identities (44)–(48).

Note that a consequence of the energy identity is that $u \in L^\infty_t L^2_x$.

However, it is not clear how to prove it from the subcritical setting in Section $\boxed{2}$. We first adjust the solution space so that this property is included. Then we show that this solution actually coincides the solution $u \in L^\infty_{t,x}$ in subcritical setting in Section $\boxed{2}$.

For $0 \le t_1 < t_2 < \infty$, we put $Z_{t_1,t_2} = (L^\infty_{t,x} \cap L^\infty_t L^2_x)(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$. Then Z_{t_1,t_2} is a Banach space with norm

$$\|f\|_{Z_{t_1,t_2}} = \|f\|_{L^\infty_{t,x}} + \|f\|_{L^\infty_t L^2_x} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{R}^3\times(t_1,t_2)} |f| + \operatorname*{ess\,sup}_{t\in(t_1,t_2)} \|f(t)\|_{L^2_x}.$$

With $u_0 \in L^2 \cap L^\infty$, we have

$$\begin{aligned}\|\Gamma(t) * u_0\|_{Z_{t_1,t_2}} &= \|\Gamma(t) * u_0\|_{L^\infty_{t,x}} + \|\Gamma(t) * u_0\|_{L^\infty_t L^2_x} \\ &\le \|u_0\|_{L^\infty} + \|u_0\|_{L^2_x} \\ &= \|u_0\|_{Z_{t_1,t_2}}.\end{aligned}$$

As in Section $\boxed{2}$, subcritical setting,

$$\|B(u,v)\|_{L^\infty_{t,x}} \le C\sqrt{t_2 - t_1}\, \|u\|_{L^\infty_{t,x}} \|v\|_{L^\infty_{t,x}} \qquad (50)$$

$$\forall u,v \in L^\infty_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3).$$

Recall the Young inequality for convolution:

$$\left[\begin{array}{l} \|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}, \\ \text{where } f \in L^p(\mathbb{R}^n),\ g \in L^q(\mathbb{R}^n),\ \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \end{array}\right]$$

Applying this inequality for $f(x) = K'(x, t-s)$, $g(x) = u(s) \otimes v(s)$, $q = r = 2$, $p = 1$, we get

$$\| K'(t-s) * (u(s) \otimes v(s)) \|_{L^2_x} \le \| K'(t-s) \|_{L^1_x} \| u(s) \otimes v(s) \|_{L^2_x}$$
$$\overset{(18)}{\le} \frac{C}{\sqrt{t-s}} \| u(s) \|_{L^2_x} \| v(s) \|_{L^\infty_x}$$
$$\le \frac{C}{\sqrt{t-s}} \| u \|_{Z_{t_1,t_2}} \| v \|_{Z_{t_1,t_2}}. \qquad (51)$$

Thus, $\| B(u,v) \|_{L^2_x} = \left\| \int_{t_1}^t K'(t-s) * (u(s) \otimes u(s))\, ds \right\|_{L^2_x}$

$$\le \int_{t_1}^t \| K'(t-s) * (u(s) \otimes u(s)) \|_{L^2_x}\, ds$$
$$\overset{(51)}{\le} \left(\int_{t_1}^t \frac{C}{\sqrt{t-s}}\, ds \right) \| u \|_{Z_{t_1,t_2}} \| v \|_{Z_{t_1,t_2}}$$
$$\le C \sqrt{t_2 - t_1}\, \| u \|_{Z_{t_1,t_2}} \| v \|_{Z_{t_1,t_2}}.$$

Thus, $\| B(u,v) \|_{L^\infty_t L^2_x} \le C \sqrt{t_2 - t_1} \| u \|_{Z_{t_1,t_2}} \| v \|_{Z_{t_1,t_2}}$. Together with (50), this estimate implies $\| B(u,v) \|_{Z_{t_1,t_2}} \le C \sqrt{t_2 - t_1} \| u \|_{Z_{t_1,t_2}} \| v \|_{Z_{t_1,t_2}}$. Then we achieve local-in-time existence of solution $u \in Z_{t_1,t_2}$ to the equation

$$u(t) = \Gamma(t-t_1) * u_0 + B(u,u).$$

The continuation method can be used to get the maximal time-interval of existence if we can show $u \in (C_t L^\infty_x \cap C_t L^2_x)(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$. By the method in Homework #1, Topics in PDE, Spring 2014, we can show $u \in C_t L^\infty_x$. By the same method, we can show $u \in C_t L^2_x$. We make it clear as follows. For $t_1 < t < t+\tau \le t_2$,

$$B(u,u)(x,t+\tau) - B(u,u)(x,t) = \int_{t_1}^t (K'(t+\tau-s) - K'(t-s)) * (u(s) \otimes u(s))\, ds$$
$$+ \int_t^{t+\tau} K'(t+\tau-s) * (u(s) \otimes u(s))\, ds.$$

Hence, $\|B(u,u)(t+\tau)-B(u,u)(t)\|_{L^2_x} \leq \int_{t_1}^{t} \|K'(t+\tau-s)-K'(t-s)\|_{L^1_x} \|u(s)\otimes u(s)\|_{L^2_x} ds$

$$+\int_t^{t+\tau} \|K'(t+\tau-s)\|_{L^1_x} \|u(s)\otimes u(s)\|_{L^2_x} ds$$

$$\leq \Big(\underbrace{\int_{t_1}^{t} \|K'(t+\tau-s)-K'(t-s)\|_{L^1_x}}_{\{1\}} + \underbrace{\int_0^{\tau} \|K'(s)\|_{L^1_x} ds}_{\{2\}}\Big) \|u\|^2_{Z_{t_1,t_2}}.$$

By (18), $\{2\} = \int_0^{\tau} \frac{C}{\sqrt{s}} ds = C\sqrt{\tau} \to 0$ as $\tau \to 0$.

$$\int_0^T \int_{\mathbb{R}^3} |K'(x,s)|\, dx ds = \int_0^T \frac{C}{\sqrt{s}} ds = C\sqrt{T} < \infty.$$

Thus, $K' \in L^1(\mathbb{R}^3\times(0,T),\mathbb{R}^3)$. Hence, $\{1\} \to 0$ as $\tau \to 0$. We get

$$\|B(u,u)(t+\tau)-B(u,u)(t)\|_{L^2_x} \to 0 \quad \text{as} \quad \tau \to 0^+ \tag{52}$$

On the other hand,

$$\|\Gamma(t+\tau-t_1)*u_0 - \Gamma(t-t_1)*u_0\|_{L^2_x} = \|(\Gamma(t+\tau-t_1)-\Gamma(t-t_1))*u_0\|_{L^2_x}$$
$$\leq \|\Gamma(t+\tau-t_1)-\Gamma(t-t_1)\|_{L^1_x} \|u_0\|_{L^2}.$$

Put $s = t-t_1 > 0$. We'll show that $\|\Gamma(s+\tau)-\Gamma(s)\|_{L^1_x} \to 0$ as $\tau \to 0$. We have

$$\|\Gamma(s+\tau)-\Gamma(s)\|_{L^1_x} = \underbrace{\int_{|x|<\sqrt{6s}+1} |\Gamma(s+\tau)-\Gamma(s)|\, dx}_{\{3\}} + \underbrace{\int_{|x|>\sqrt{6s}+1} |\Gamma(s+\tau)-\Gamma(s)|\, dx}_{\{4\}}.$$

We have $\int_{\mathbb{R}^3} |\Gamma(s)|^2 dx = \frac{1}{(4\pi s)^3} \int_{\mathbb{R}^3} \exp\left(-\frac{|x|^2}{2s}\right) ds$

$$\overset{y=x/\sqrt{s}}{=\!=} \frac{1}{(4\pi s)^3} s^{3/2} \int_{\mathbb{R}^3} \exp\left(-\frac{|y|^2}{2}\right) dy = \frac{C}{s^{3/2}}. \tag{53}$$

Put $f_\tau(x) = |\Gamma(x,s+\tau)-\Gamma(x,s)|$. Then

$$\int_{|x|<\sqrt{6s}+1} |f_\tau(x)|^2 dx \leq 2\int_{|x|<\sqrt{6s}+1} |\Gamma(x,s+\tau)|^2 dx + 2\int_{|x|<\sqrt{6s}+1} |\Gamma(x,s)|^2 dx$$

$$\overset{(53)}{\leq} \frac{C}{(s+\tau)^{3/2}} + \frac{C}{s^{3/2}} < \frac{2C}{s^{3/2}}, \quad \forall \tau > 0.$$

Thus, (f_τ) is a bounded family in $L^2(B_{\sqrt{6s}+1}(0))$. The following result is in Brezis "Functional Analysis, Sobolev spaces and PDE", 2011, p. 123, which can be proved by Vitali's convergence theorem:

> Let $1<p<\infty$ and Ω be a subset of $\mathbb{R}^N$ with finite measure. Let (f_n) be a sequence in $L^p(\Omega)$ such that
>
> (i) (f_n) is bounded in $L^p(\Omega)$,
>
> (ii) $f_n \to f$ a.e. in Ω.
>
> Then $f_n \to f$ in $L^q(\Omega)$ for every $q \in [1,p)$.

We have $\lim_{\tau\to 0} f_\tau(x) = 0$ for all $x \in \mathbb{R}^3$. Therefore, $f_\tau \to 0$ in $L^1(B_{\sqrt{6s}+1}(0))$ as $\tau \to 0^+$. Thus, $\{3\} \to 0$ as $\tau \to 0^+$.

$$\frac{\partial \Gamma}{\partial s}(x,s) = C s^{-5/2} \exp\left(-\frac{|x|^2}{4s}\right)\left(\frac{|x|^2}{4s} - \frac{3}{2}\right) > 0 \quad \forall |x| > \sqrt{6s}+1.$$

We have $|\Gamma(x,s+\tau) - \Gamma(x,s)| = \Gamma(x,s+\tau) - \Gamma(x,s) \leq \underbrace{\Gamma(x,s+1)}_{\in L^1_x}$ $\quad \forall |x| > \sqrt{6s}+1$ $\quad \forall \tau \in (0,1)$.

Thus, by Lebesgue's Dominated convergence theorem,

$$\{4\} = \int_{|x|>\sqrt{6s}+1} |\Gamma(s+\tau) - \Gamma(s)| dx \to 0 \quad \text{as } \tau \to 0^+.$$

Hence, $\|\Gamma(s+\tau) - \Gamma(s)\|_{L^1_x} \to 0$ as $\tau \to 0^+$ and thus

$$\|\Gamma(t+\tau-t_1) * u_0 - \Gamma(t-t_1) * u_0\|_{L^2_x} \to 0 \quad \text{as } \tau \to 0^+. \qquad (54)$$

By (52) and (54), $u \in C_t L^2_x$.

The regularity properties of u achieved in Section $\boxed{2}$ remain valid in this new setting:
$$t^{\ell+\frac{m}{2}} \partial_t^\ell \partial_x^m u \in L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3) \quad \forall T_1 \in (0,T^*). \qquad (55)$$
We will show by induction in $m \geq 0$ that
$$t^{\frac{m}{2}} \partial_x^m u \in L^\infty_t L^2_x(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3) \quad \forall T_1 \in (0,T^*). \qquad (56)$$
This is true for $m=0$. Suppose that (56) is true for ~~some m~~ all integers less than some $m \geq 1$. From the identity
$$\begin{aligned} u(t) &= \Gamma(t)*u_0 + \int_0^t K'(t-s)*(u(s)\otimes u(s))\,ds, \\ &= \Gamma(t)*u_0 + \int_0^{t/2} K'(t-s)*(u(s)\otimes u(s))\,ds + \int_{t/2}^t K'(t-s)*(u(s)\otimes u(s))\,ds, \end{aligned}$$
we get
$$\begin{aligned} \partial_x^m u(t) = (\partial_x^m \Gamma(t))*u_0 &+ \int_0^{t/2} \partial_x^m K'(t-s)*(u(s)\otimes u(s))\,ds \\ &+ \int_{t/2}^t K'(t-s)*(\partial_x^k u(s) \otimes \partial_x^{m-k} u(s))\,ds. \end{aligned}$$
Thus,
$$\begin{aligned} \|\partial_x^m u(t)\|_{L^2_x} \leq \underbrace{\|\partial_x^m \Gamma(t)\|_{L^1_x}}_{\leq C(m)\, t^{-m/2} \text{ by page 16}} \|u_0\|_{L^2} &+ \int_0^{t/2} \underbrace{\|\partial_x^m K'(t-s)\|_{L^1_x}}_{\leq C(m)(t-s)^{-\frac{1}{2}-\frac{m}{2}} \text{ by page 18}} \|u(s)\otimes u(s)\|_{L^2_x}\,ds \\ &+ \sum_{k=0}^m \binom{m}{k} \int_{t/2}^t \underbrace{\|K'(t-s)\|_{L^1_x}}_{= \frac{C}{\sqrt{t-s}} \text{ by (18)}} \|\partial_x^k u(s) \otimes \partial_x^{m-k} u(s)\|\,ds. \end{aligned}$$

Hence,

$$\|\partial_x^m u(t)\|_{L^2_x} \leq \frac{C(m)}{t^{m/2}}\|u_0\|_{L^2} + \underbrace{\int_0^{t/2} \frac{C(m)}{(t-s)^{\frac{1}{2}+\frac{m}{2}}} \|u(s)\otimes u(s)\|_{L^2_x} ds}_{\{5\}}$$

$$+\sum_{k=0}^{m} \binom{m}{k} \int_{t/2}^{t} \frac{c}{\sqrt{t-s}} \underbrace{\|\partial_x^k u(s)\otimes \partial_x^{m-k} u(s)\|_{L^2_x}}_{\{6\}} ds. \quad (57)$$

We have $\{5\} \leq \int_0^{t/2} \frac{C(m)}{(t/2)^{\frac{1}{2}+\frac{m}{2}}} \|u\|^2_{Z_{a T_1}} ds = t^{-m/2} C(m) \|u\|^2_{Z_{aT_1}}$. (58)

For $0 \leq k \leq \frac{m}{2}$, we have

$$\{6\} \leq \|\partial_x^k u(s)\|_{L^2_x} \|\partial_x^{m-k} u(s)\|_{L^\infty_x}$$

$$= s^{-\frac{m}{2}} \left(s^{\frac{k}{2}}\|\partial_x^k u(s)\|_{L^2_x}\right)\left(s^{\frac{m-k}{2}}\|\partial_x^{m-k} u(s)\|_{L^\infty_x}\right)$$

$$\leq \left(\frac{t}{2}\right)^{-\frac{m}{2}} \underbrace{\|s^{\frac{k}{2}}\partial_x^k u(s)\|_{L^\infty_s L^2_x}}_{<\infty \text{ by induction}} \underbrace{\|s^{\frac{m-k}{2}}\partial_x^{m-k} u(s)\|_{L^\infty_{s,x}}}_{<\infty \text{ by } (55)}. \quad (59)$$

For $\frac{m}{2} \leq k < m$, similarly we get

$$\{6\} \leq \left(\frac{t}{2}\right)^{-\frac{m}{2}} \underbrace{\|s^{\frac{k}{2}}\partial_x^k u(s)\|_{L^\infty_{s,x}}}_{<\infty \text{ by } (55)} \underbrace{\|s^{\frac{m-k}{2}}\partial_x^{m-k} u(s)\|_{L^\infty_s L^2_x}}_{<\infty \text{ by induction}}. \quad (60)$$

Replacing (58), (59), (60) into (57), we get

$$\|\partial_x^m u(t)\|_{L^2_x} \leq \frac{C(m)}{t^{m/2}}\|u_0\|_{L^2} + \frac{C(m)}{t^{m/2}}\|u\|^2_{Z_{aT_1}} + C(m)\, t^{\frac{1}{2}-\frac{m}{2}} \times$$

$$\times \max_{0\leq k\leq \frac{m}{2}} \left\{ \|s^{\frac{k}{2}}\partial_x^k u(s)\|_{L^\infty_s L^2_x(\mathbb{R}^3\times(0,T_1)),\mathbb{R}^3)} \right\}$$

$$\times \max_{\frac{m}{2}<k\leq m} \left\{ \|s^{\frac{k}{2}}\partial_x^k u(s)\|_{L^\infty_{s,x}(\mathbb{R}^3\times(0,T_1)),\mathbb{R}^3)} \right\}.$$

Therefore, $t^{\frac{m}{2}}\partial_x^m u(t) \in L_t^\infty L_x^2(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$ for all $T_1\in(0,T^*)$. We have finished the proof of (56). By (55) and (56),

$$u(t)\in W^{m,2}(\mathbb{R}^3,\mathbb{R}^3)\cap W^{m,\infty}(\mathbb{R}^3,\mathbb{R}^3) \quad \forall m\geq 0,\ \forall t\in(0,T^*). \qquad (61)$$

With (61), the identities (45) and (46) are justified.

By (1),
$$p(t)\sim C\int_{\mathbb{R}^3}\frac{\partial_x u(y,t)\otimes u(y,t)}{|x-y|^2}\,dy \qquad (62)$$

$$\nabla p(t)\sim C\int_{\mathbb{R}^3}\frac{\partial_x^2 u(y,t)\otimes u(y,t)}{|x-y|^2}\,dy + C\int_{\mathbb{R}^3}\frac{\partial_x u(y,t)\otimes\partial_x u(y,t)}{|x-y|^2}\,dy \qquad (63)$$

We have
$$\underbrace{\partial_x u(t)}_{\in L_y^3}\otimes\underbrace{u(t)}_{\in L_y^2}\in L_y^{6/5},$$
$$\underbrace{\partial_x^2 u(t)}_{\in L_y^3}\otimes\underbrace{u(t)}_{\in L_y^2}\in L_y^{6/5},$$
$$\underbrace{\partial_x u(t)}_{\in L_y^3}\otimes\underbrace{\partial_x u(t)}_{\in L_y^2}\in L_y^{6/5}.$$

Recall the fractional interpolation (Theorem 4.18, p. 229, Bennett–Sharpley "Interpolation of Operators")

[For $f\in L^p(\mathbb{R}^n)$ and $I_k f(x)=\int_{\mathbb{R}^3}\frac{f(y)}{|x-y|^{n-k}}\,dy$, we have $\|I_k f\|_q\leq C_p\|f\|_p$ where $p>1$ and $\frac{1}{q}=\frac{1}{p}-\frac{k}{n}>0$.]

Applying this result for $k=1$, $n=3$, $p=6/5$, $q=2$, we have $p(t),\nabla p(t)\in L_x^2$. Thus, $p(t)\in H_x^1$. Moreover, thanks to (55), (62) and (63), we have $\sqrt{t}\,p,\ t\nabla p\in L_t^\infty L_x^2(\mathbb{R}^3\times(0,T_1))$. The identity (47) is satisfied.
(64)

Because u satisfies (NSE), $u_t = \Delta u - (u\cdot\nabla)u - \nabla p$. By (56) and (65), we have $u_t \in L^\infty_t L^2_x(\mathbb{R}^3\times(t_1,t_2),\mathbb{R}^3)$ for all $[t_1,t_2]\subset(0,T^*)$. Thus, $u_t u \in L^\infty_t L^1_x(\mathbb{R}^3\times(t_1,t_2),\mathbb{R}^3)$ for all $[t_1,t_2]\subset(0,T^*)$. This property, however, is not enough to justify (44). Instead, we have

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3} u_t u\,dx\,dt = \int_{\mathbb{R}^3}\int_{t_1}^{t_2} u_t u\,dt\,dx = \frac{1}{2}\int_{\mathbb{R}^3}\left(|u(x,t_2)|^2 - |u(x,t_1)|^2\right)dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^3}|u(t_2)|^2dx - \frac{1}{2}\int_{\mathbb{R}^3}|u(t_1)|^2dx,$$

which still gives us (47'). We have

$$\|B(u,u)(t)\|_{L^2_x} = \left\|\int_0^t K'(t-s)*(u(s)\otimes u(s))\,ds\right\|_{L^2_x}$$

$$\leq \int_0^t \|K'(t-s)\|_{L^1_x}\|u(s)\otimes u(s)\|_{L^2_x}\,ds$$

$$\overset{(48)}{\leq} \int_0^t \frac{C}{\sqrt{t-s}}\|u\|^2_{Z_{0,T_1}}\,ds = C\sqrt{t}\,\|u\|^2_{Z_{0,T_1}}.$$

Thus, $\|B(u,u)(t)\|_{L^2_x}\to 0$ as $t\to 0$. On the other hand,

$$\Gamma(t)*u_0 - u_0 = \int_{\mathbb{R}^3}\Gamma(y,t)\left(u_0(x-y) - u_0(x)\right)dy$$

$$= \int_{\mathbb{R}^3}\frac{C}{t^{3/2}}F\left(\frac{y}{\sqrt{t}}\right)\left(u_0(x-y) - u_0(x)\right)dy,$$

(where $F(z) = \exp(-|z|^2)$)

$$= C\int_{\mathbb{R}^3}F(z)\left(u_0(x-z\sqrt{t}) - u_0(x)\right)dz.$$

Thus, $\|\Gamma(t)*u_0 - u_0\|_{L^2_x} \leq C\int_{\mathbb{R}^3}|F(z)|\,\|u_0(x-z\sqrt{t}) - u_0(x)\|_{L^2_x}\,dz \to 0$ as $t\to 0^+$.

Hence, $\|u(t)-u_0\|_{L^2_x} \le \|\Gamma(t)*u_0-u_0\|_{L^2_x} + \|B(u,u)(t)\|_{L^2_x} \longrightarrow 0$ as $t\to 0^+$.

This justifies (48). Therefore, we get the energy identity (49).

Next, we show that the solution in the setting $L^\infty_{t,x}\cap L^\infty_t L^2_x$ coincides the one in setting $L^\infty_{t,x}$ which was obtained in Section $\boxed{2}$. Let u and v be the solutions in these settings respectively. Suppose that the maximal time-interval of existence for u and v are $(0,T^*)$ and $(0,T^{**})$ respectively. If $T^*<\infty$ then by the proof of local-in-time existence, we have

$$\lim_{T_1\to(T^*)^-}\left(\|u\|_{L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} + \|u\|_{L^\infty_t L^2_x(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)}\right) = \infty.$$

Because of the energy identity (49), $\|u(t)\|_{L^\infty_t L^2_x} \le \|u_0\|_{L^2}$. Thus, if $T^*<\infty$ then

$$\lim_{T_1\to(T^*)^-}\|u\|_{L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} = \infty.$$

If $T^{**}<\infty$ then $\lim_{T_1\to(T^{**})^-}\|v\|_{L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} = \infty$. Because of the uniqueness of the mild solution in setting $L^\infty_{t,x}$, we have $u=v$ on $(0,T^{**})$ and $T^{**}\le T^*$. Suppose by contradiction that $T^{**}<T^*$. Then

$$\lim_{T_1\to(T^{**})^-}\|u\|_{L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} = \lim_{T_1\to(T^{**})^-}\|v\|_{L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} = \infty.$$

This is a contradiction because $u\in L^\infty_{t,x}(\mathbb{R}^3\times(0,T^{**}),\mathbb{R}^3)$. Therefore, $T^{**}=T^*$ and $u\equiv v$.

$\boxed{5}$ Sufficient conditions for the global-in-time existence

* Subcritical setting: $u\in L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$ for all $T_1\in(0,T^*)$, where $(0,T^*)$ is the maximal time-interval of existence.

Put $V(t)=\|u(t)\|_{L^\infty_x}$, $W(t)=\|u(t)\|_{L^2_x}$ and $J(t)=\|\nabla u(t)\|_{L^2_x}$. We will derive two conditions each of which guarantees the global-in-time existence of a mild solution to (NSE). One condition is for $u_0 \in L^2 \cap L^\infty$, the other is for $u_0 \in H^1 \cap L^\infty$. Recall that u satisfies

$$u(t) = \underbrace{\Gamma(t) * u_0}_{\{1\}} + \int_0^t \underbrace{K'(t-s) * (u(s) \otimes u(s))}_{\{2\}} ds. \qquad (66)$$

There are two ways to estimate each of $\{1\}$ and $\{2\}$. We have

$$|\{1\}| \leq \underbrace{\|\Gamma(t)\|_{L^1_x}}_{=1} \|u_0\|_{L^\infty} = V(0) \qquad (67)$$

On the other hand,

$$|\{1\}|^2 = \left| \int_{\mathbb{R}^3} \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) u_0(y)\, dy \right|^2$$

$$= \frac{1}{(4\pi t)^3} \left| \int_{\mathbb{R}^3} |x-y| \exp\left(-\frac{|x-y|^2}{4t}\right) \frac{u_0(y)}{|x-y|}\, dy \right|^2$$

$$\overset{\text{Schwarz}}{\leq} \frac{1}{(4\pi t)^3} \underbrace{\int_{\mathbb{R}^3} |x-y|^2 \exp\left(-\frac{|x-y|^2}{2t}\right) dy}_{\overset{z=y/\sqrt{t}}{=\!=}\ t^{5/2} \int_{\mathbb{R}^3} |z|^2 \exp(-|z|^2)} \int_{\mathbb{R}^3} \frac{|u_0(y)|^2}{|x-y|^2}\, dy$$

$$= \frac{C}{t^{1/2}} \int_{\mathbb{R}^3} \frac{|u_0(x-y)|^2}{|y|^2}\, dy$$

$$\leq \frac{C}{t^{1/2}} \|\nabla u_0(x-.)\|_{L^2}^2 \quad \text{(Hardy's inequality, Evans "PDE", 2010, page 308)}$$

$$= \frac{C\|\nabla u_0\|_{L^2}^2}{t^{1/2}}.$$

Thus, $|\{1\}| \le \dfrac{C\|\nabla u_0\|_{L^2}}{t^{1/4}}$. $\quad (68)$

We have $|\{2\}| \le \|K'(t-s)\|_{L^\infty_x} \|u(s)\otimes u(s)\|_{L^1_x} \le C\|K'(t-s)\|_{L^\infty_x} \|u(s)\|^2_{L^2_x}$

$$\overset{(49)}{\le} C\|u_0\|^2_{L^2} \|K'(t-s)\|_{L^\infty_x}.$$

By (13), $\|K'(t-s)\|_{L^\infty_x} \leqslant \dfrac{C}{(t-s)^2}$. Thus,

$$|\{2\}| \le \frac{CW(0)^2}{(t-s)^2} \qquad (69)$$

On the other hand,

$$|\{2\}| \leqslant \|K'(t-s)\|_{L^1_x} \|u(s)\otimes u(s)\|_{L^\infty_x} \overset{(18)}{\leqslant} \frac{CV(s)^2}{(t-s)^{1/2}} \qquad (70)$$

Substituting (67)-(70) into (66), we get

$$|u(x,t)| \le \min\left\{V(0), \frac{CJ(0)}{t^{1/4}}\right\} + \int_0^t \min\left\{\frac{CV(s)^2}{(t-s)^{1/2}}, \frac{CW(0)^2}{(t-s)^2}\right\} ds.$$

We would like to distinguish the two constants C in the last terms by C_1 and C_2. Thus, we write

$$|V(t)| \le \min\left\{V(0), \frac{CJ(0)}{t^{1/4}}\right\} + \int_0^t \min\left\{\frac{C_1 V(s)^2}{(t-s)^{1/2}}, \frac{C_2 W(0)^2}{(t-s)^2}\right\} ds \quad \forall t\in[0,T^*) \qquad (71)$$

Suppose that there is a continuous function $\varphi: (0,T^*)\to \mathbb{R}$ such that $\varphi \in L^2((0,T^*))$, $\liminf_{t\to 0^+} \varphi(t) > V(0)$ and

$$\varphi(t) \geqslant \min\left\{V(0), \frac{CJ(0)}{t^{1/4}}\right\} + \int_0^t \min\left\{\frac{C_1\varphi(s)^2}{(t-s)^{1/2}}, \frac{C_2 W(0)^2}{(t-s)^2}\right\} ds. \qquad (72)$$

For $u_0 \not\equiv 0$, $V(t) < \varphi(t)$ for all $t\in(0,T^*)$. Indeed, suppose otherwise. Then there exists $t_0 \in (0,T^*)$, such that $V(t_0) \geqslant \varphi(t_0)$. By the continuity of φ for and V, t_0 can be chosen to be minimum. Then $V(t_0) = \varphi(t_0)$ and $\varphi(s) > V(s)$ for all $t \in (0, t_0)$. We have

$$\varphi(t_0) \geq \min\left\{V(0), \frac{CJ(0)}{t_0^{1/4}}\right\} + \int_0^{t_0} \min\left\{\frac{C_1 \varphi(s)^2}{(t_0-s)^{1/2}}, \frac{C_2 W(0)^2}{(t_0-s)^2}\right\} ds$$

$$\geq \min\left\{V(0), \frac{CJ(0)}{t_0^{1/4}}\right\} + \int_0^{t_0} \min\left\{\frac{C_1 V(s)^2}{(t_0-s)^{1/2}}, \frac{C_2 W(0)^2}{(t_0-s)^2}\right\} ds$$

$$\geq V(t_0).$$

This means the equalities must hold. This happens only if

$$\min\left\{\frac{C_1 \varphi(s)^2}{(t_0-s)^{1/2}}, \frac{C_2 W(0)^2}{(t_0-s)^2}\right\} = \frac{C_2 W(0)^2}{(t_0-s)^2} \quad \text{a.e. } s \in (0, t_0).$$

This is impossible because $\frac{C_2 W(0)^2}{(t_0-s)^2}$ is not an integrable function on $(0, t_0)$.

For $u_0 \equiv 0$, $V(t) = 0 \leq \varphi(t)$ for all $t \in (0, T^*)$. Therefore, we always have

$$V(t) \leq \varphi(t) \quad \forall t \in (0, T^*) \tag{73}$$

For the first choice of φ, we choose $\varphi(t) \equiv (1+A)V(0)$ where A is a possitive constant to be determined. Then (72) is satisfied if

$$A V(0) \geq \int_0^t \min\left\{\frac{C_1 (1+A)^2 V(0)^2}{s^{1/2}}, \frac{C_2 W(0)^2}{s^2}\right\} ds. \tag{74}$$

We have
$$\frac{C_2 W(0)^2}{s^2} \geq \frac{C_1 (1+A)^2 V(0)^2}{s^{1/2}} \iff s \leq s_0 = \left(\frac{2 C_2 W(0)^2}{C_1 (1+A)^2 V(0)^2}\right)^{2/3}.$$

Then
$$\text{RHS } (74) \leq \int_0^{s_0} \frac{C_1 (1+A)^2 V(0)^2}{s^{1/2}} ds + \int_{s_0}^{\infty} \frac{C_2 W(0)^2}{s^2} ds.$$

Then (74) is satisfied if
$$V(0)\, W(0)^2 \leq \frac{4}{27 C_1^2 C_2} \frac{A^3}{(1+A)^4} \tag{75}$$

The condition (75) is satisfied for some $A > 0$ if and only if

$$V(0)W(0)^2 \leq \frac{4}{27C_1^2C_2} \max_{A>0} \frac{A^3}{(1+A)^4} = \frac{1}{64C_1^2C_2}.$$

The maximum is attained at $A=3$. Thus,

$$V(0)\,W(0)^2 < C. \qquad (76)$$

In Section [2], subcritical setting, the problem

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(\cdot,t_1) = u(t_1) \end{cases}$$

has a mild solution $u \in L^\infty_{t,x}(\mathbb{R}^3\times(t_1,t_2),\mathbb{R}^3)$ if $\sqrt{t_2-t_1}\,\|u(t_1)\|_{L^\infty_x} < C$.

Consequently, if $T^*<\infty$ then $\|u(t)\|_{L^\infty_x}\to\infty$ as $t\to(T^*)^-$. If the condition (76) is satisfied then the constant function $\varphi(t)=4V(0)$ satisfies $\varphi(t)\geq V(t)$ for all $t\in(0,T^*)$; thus, $T^*=\infty$. Therefore, (NSE) has a global-in-time mild solution when (76) is satisfied. Leray called it the first case of regularity.

For the second choice of φ (this time assuming $u_0\in H^1\cap L^\infty$), we choose $\varphi(t)=\frac{AJ(0)}{t^{1/4}}$ where A is a positive constant to be determined. Then (72) is satisfied if

$$\frac{AJ(0)}{t^{1/4}} \geq \frac{CJ(0)}{t^{1/4}} + \int_0^t \min\left\{\frac{C_1A^2J(0)^2}{s^{1/2}(t-s)^{1/2}}, \frac{C_2W(0)^2}{(t-s)^2}\right\} ds,$$

which is satisfied if

$$\frac{AJ(0)}{t^{1/4}} \geq \frac{CJ(0)}{t^{1/4}} + \int_0^t \frac{C_1A^2J(0)^2}{s^{1/2}(t-s)^{1/2}}\,ds. \qquad (77)$$

We have $\displaystyle\int_0^t \frac{ds}{s^{1/2}(t-s)^{1/2}} = 2\int_0^{t/2}\frac{ds}{s^{1/2}(t-s)^{1/2}} \geq 2\sqrt{\frac{2}{t}}\int_0^{t/2}\frac{ds}{s^{1/2}} = 1.$

Thus, (77) is satisfied if $\frac{AJ(0)}{t^{1/4}} \geq \frac{CJ(0)}{t^{1/4}} + C_1 A^2 J(0)^2$, which is satisfied if $t \leq \left(\frac{A-C}{C_1 A^2}\right)^4 J(0)^{-4}$. (78)

Choose A to be the maximizer of $\left(\frac{A-C}{C_1 A^2}\right)^4$ in the interval (C,∞). We still denote the maximum value by C. Then (78) becomes $t \leq C J(0)^{-4}$.

Put $\tau = CJ(0)^{-4}$. Then

$$V(t) \leq \varphi(t) = \frac{AJ(0)}{t^{1/4}} < \infty \qquad \forall t \in (0,\tau]. \qquad (79)$$

Thus, (NSE) has a mild solution in the time-interval $[0,\tau]$. By (79), we have $V(\tau)W(\tau)^2 \leq \frac{AJ(0)}{\tau^{1/4}} W(0)^2 = CJ(0)^2 W(0)^2$.

Thus, if $J(0)^2 W(0)^2 < C$ then $V(\tau)W(\tau)^2 < C$; then by the first case of regularity, (NSE) has a mild solution in the time interval $[\tau,\infty)$; thus, (NSE) has a mild solution in the time interval $(0,\infty)$. In other words, the condition

$$J(0)^2 W(0)^2 < C \qquad (80)$$

is sufficient for (NSE) to have a global-in-time mild solution. It is called the second case of regularity. Together with the regularity of u in Section [3], subcritical setting, we know that if the first or second case of regularity happens, $u \in C^\infty(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^3) \cap C(\mathbb{R}^3 \times [0,\infty), \mathbb{R}^3)$.

* Critical setting: $u \in L^5_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)$ for all $T_1 \in (0,T^*)$, where $(0,T^*)$ is the maximal time-interval of existence.

Our method is exactly the same as that in Homework #1, Topics in PDE, Spring 2014, pages 24–25, in which we dealt with a model equation instead of the Navier–Stokes equations. Accordingly,

$$\|B(u,v)\|_{\mathcal{Y}_{t_1,t_2}} \leq C \|u\|_{\mathcal{Y}_{t_1,t_2}} \|v\|_{\mathcal{Y}_{t_1,t_2}} \quad \forall u,v \in \mathcal{Y}_{t_1,t_2}$$

where $\mathcal{Y}_{t_1,t_2} = L^5_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$. It is important to note that the continuity constant is just a numeric constant whereas it depends on time in subcritical setting:

$$\|B(u,v)\|_{\mathcal{X}_{t_1,t_2}} \leq C\sqrt{t_2-t_1}\, \|u\|_{\mathcal{X}_{t_1,t_2}} \|v\|_{\mathcal{X}_{t_1,t_2}} \quad \forall u,v \in \mathcal{X}_{t_1,t_2}$$

where $\mathcal{X}_{t_1,t_2} = L^\infty_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$. Consequently, if $\|u_0\|_{L^3} < C$ then (NSE) has a global-in-time mild solution. This solution is in $C^\infty(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^3)$ according to Section $\boxed{3}$, but we don't know if it is continuous up to time $t=0$.

$\boxed{6}$ Characterizations of finite time blowup

We will show two following properties of the solution u to (NSE) whose ~~minimal~~ maximal time-interval of existence is $(0,T^*)$ with $T^* < \infty$.

(i) If $u_0 \in L^2 \cap L^\infty$ then $\displaystyle \|u(t)\|_{L^\infty_x} \geq \frac{C}{(T^*-t)^{1/2}} \quad \forall t \in (0,T^*)$.

(ii) If $u_0 \in H^1 \cap L^\infty$ then $\displaystyle \|\nabla u(t)\|_{L^2_x} \geq \frac{C}{(T^*-t)^{1/4}} \quad \forall t \in (0,T^*)$.

Leray called them the first and second characterization of irregularities.

Proof of (i)

In Section [2], subcritical setting, the problem

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0 \\ u(\cdot, t_1) = u(t_1) \end{cases}$$

has a mild solution $u \in L^\infty_{t,x}(\mathbb{R}^3 \times (t_1,t_2), \mathbb{R}^3)$ if $\sqrt{t_2-t_1}\,\|u(t_1)\|_{L^\infty_x} < C$. Take any $s \in (0,T^*)$ and $\varepsilon > 0$. Because $(0,T^*)$ is the maximal interval of existence, the above problem cannot have a mild solution on the interval $(s, T^*+\varepsilon)$. Thus, the condition $\sqrt{T^*+\varepsilon-s}\,\|u(s)\|_{L^\infty_x} < C$ cannot be satisfied. Thus, $\sqrt{T^*+\varepsilon-s}\,\|u(s)\|_{L^\infty_x} \geq C$. Equivalently,

$$\|u(s)\|_{L^\infty_x} \geq \frac{C}{\sqrt{T^*+\varepsilon-s}}.$$

Because $\varepsilon > 0$ was taken arbitrarily, we get

$$\|u(s)\|_{L^\infty_x} \geq \frac{C}{\sqrt{T^*-s}} \quad \forall s \in (0,T^*).$$

Proof of (ii) Put $V(t) = \|u(t)\|_{L^\infty_x}$, $W(t) = \|u(t)\|_{L^2_x}$, $J(t) = \|\nabla u(t)\|_{L^2_x}$.
Take any $s \in (0,T^*)$ and $\varepsilon > 0$. Because $(0,T^*)$ is the maximal time-interval of existence, (NSE) cannot have a mild solution on $(s, T^*+\varepsilon)$. By (79), (NSE) has a mild solution on $(s, s+\tau)$ with $\tau = C J(s)^{-4}$. Thus, $\tau < T^*+\varepsilon-s$. Hence,

$$J(s) > \left(\frac{C}{T^*+\varepsilon-s}\right)^{1/4}.$$

Because this is true for all $s \in (0,T^*)$ and $\varepsilon > 0$, we have

$$J(s) \geq \frac{C}{(T^*-s)^{1/4}} \quad \forall s \in (0,T^*).$$