Part 2: Leray's weak solutions

By the perturbation analysis of heat equations, Leray in his paper in 1934 introduced the notion of mild solutions, which inherit many properties of heat equations' solutions. Due to the nonlinearity of the perturbating equation, the method only gives the existence of solutions over a short time period (so called the local-in-time existence). By some regularity properties, we can show that the mild solutions are unique on a short time period. Then thanks to the continuation method, a mild solution exists and is unique on a maximal time interval. It is a classical solution to the Navier-Stokes equations. The global-in-time existence is achieved if the initial data u_0 satisfies certain smallness condition, specifically one of the following:

(i) $\|u_0\|_{L^2}^2 \|u_0\|_{L^\infty} < C$.

(ii) $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < C$.

(iii) $\|u_0\|_{L^3} < C$.

As a convention, the symbol C denotes various positive numeric constants which we do not specify their values. We adopt such operations as $C^2 = C$, $2C = C$, $C + C = C$, ... All of these issues were proved in Part 1- Mild solutions.

Another type of solutions is weak solutions. They satisfy a weak form of the Navier-Stokes equations in which the demand for regularity is lessened. Because there are more than one way to define a weak form, for example in Leray (1934) and Hopf (1951), we would like to specify that our concern in this write-up is the Leray's weak form. He called its solutions <u>turbulent solutions</u>. Leray showed that there exists a global-in-time weak solution. It coincides the mild solution on the maximal time-interval on which the mild solution exists. This property is called the <u>weak-strong uniqueness</u>. However, it has not been showed since then whether Leray's weak solutions are unique. We will discuss the following issues regarding to Leray's weak solutions based on Leray's paper (1934) and the series of lectures by Professor Vladimir Sverak in the course Topics in PDE, Spring 2014.

- Definition
- Existence
- Weak-strong uniqueness
- The set of singular times
- Asymptotic behavior as time goes to infinity

Let $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in sense of distribution. Consider the 3D Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u + (u.\nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3, \end{cases} \qquad \text{(NSE)}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$. The gradient and Laplacian are with respect to spatial variables.

1 Definition

The idea to get a weak form for (NSE) is as follows. Put

$$\mathcal{N} = \{ \varphi \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^3) : \operatorname{div} \varphi(t) = 0 \ \ \forall t \geq 0 \}.$$

Multiplying both sides of the equation $\partial_t u - \Delta u + (u.\nabla)u + \nabla p = 0$ by $\varphi \in \mathcal{N}$ and taking the integral over $x \in \mathbb{R}^3$, we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} u(t)\varphi(t)\,dx - \int_{\mathbb{R}^3} u(t)\partial_t \varphi(t)\,dx + \int_{\mathbb{R}^3} \nabla u(t) . \nabla \varphi(t)\,dx + \int_{\mathbb{R}^3} u_{i,j}\, u_j\, \varphi_i\,dx$$
$$- \int_{\mathbb{R}^3} p(t) \underbrace{\nabla . \varphi(t)}_{=0}\,dx = 0.$$

Integrating bothe sides over $t \in [0,T]$, we get

$$\int_{\mathbb{R}^3} u(T)\varphi(T)\,dx - \int_0^T \int_{\mathbb{R}^3} u(t)\partial_t \varphi(t)\,dx\,dt + \int_0^T \int_{\mathbb{R}^3} \nabla u(t) . \nabla \varphi(t)\,dx\,dt$$
$$+ \int_0^T \int_{\mathbb{R}^3} u_{i,j}(t)\, u_j(t)\, \varphi_i(t)\,dx\,dt = \int_{\mathbb{R}^3} u_0\, \varphi(0)\,dx \quad \forall T \in (0,\infty).$$

We come to the definition: a function u is called a <u>Leray's weak solution</u>

to (NSE) if it satisfies

(i) $u \in (L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1)(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^3)$,

(ii) For each $\varphi \in \mathcal{N}$,

$$\int_{\mathbb{R}^3} u(t)\varphi(t)dx - \int_0^t \int_{\mathbb{R}^3} u(s)\partial_t \varphi(s)dxds + \int_0^t \int_{\mathbb{R}^3} \nabla u(s).\nabla \varphi(s)dxds$$

$$+ \int_0^t \int_{\mathbb{R}^3} u_j(s) u_{i,j}(s) \varphi_i(s) dxds = \int_{\mathbb{R}^3} u_0 \varphi(0)dx \quad \text{a.e. } t\in(0,\infty),$$

(iii) $\lim_{t\to 0^+} \|u(t) - u_0\|_{L_x^2}^2 = 0$.

By (i) we mean $\operatorname*{ess\,sup}_{t\in(0,\infty)} \|u(t)\|_{L_x^2}^2 + \int_0^\infty \int_{\mathbb{R}^3} |\nabla u(t)|^2 dxdt < \infty$.

By (iii) we mean: for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|u(t) - u_0\|_{L_x^2}^2 < \varepsilon \quad \text{a.e. } t \in (0,\delta).$$

$\boxed{2}$ Existence

Let $\eta \in \mathcal{D}(\mathbb{R}^3)$ be a function satisfying

- $\eta(x) \geq 0 \quad \forall x \in \mathbb{R}^3$,
- $\int_{\mathbb{R}^3} \eta\, dx = 1$.

For example, we can take $\eta(x) = \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$

For each $\varepsilon > 0$, put $\eta_\varepsilon(x) = \frac{1}{\varepsilon^3}\eta\left(\frac{x}{\varepsilon}\right)$. We know that $(\eta_\varepsilon)_{\varepsilon>0}$ is an approximate identity on $\mathbb{R}^3$. The existence of a Leray's weak solution follows from two steps.

<u>Step 1</u>: Show that the mollified version of (NSE), namely

$$\begin{cases} \partial_t u - \Delta u + ((u * \eta_\varepsilon)\cdot\nabla)u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u(x,0) = u_0(x) \end{cases} \qquad (NSE)_\varepsilon$$

has a global-in-time solution u_ε. It should be emphasized that u_ε is a classical solution to $(NSE)_\varepsilon$ and satisfies the energy identity

$$\frac{1}{2}\int_{\mathbb{R}^3} |u_\varepsilon(t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u_\varepsilon(s)|^2 dx ds = \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx.$$

<u>Step 2</u>: passing to the limit as $\varepsilon \to 0$. We show that there exists a sequence $(\varepsilon_n) \downarrow 0$ such that (u_{ε_n}) converges in certain space. The limit function turns out to be Leray's weak solution.

- <u>Making detail Step 1</u>:

We follow the method in Part 1 to show that $(NSE)_\varepsilon$ has a mild solution. Accordingly, the order of studying mild solutions is as follows:

- Definition
- Local-in-time existence and uniqueness (similar to Part 1, Section $\boxed{2}$)
- Regularity (similar to Part 1, Section $\boxed{3}$)
- Energy identity (similar to Part 1, Section $\boxed{4}$)
- Global-in-time existence.

Let us start with the local version of $(NSE)_\varepsilon$:

$$\begin{cases} \partial_t u - \Delta u + ((u * \eta_\varepsilon).\nabla) u + \nabla p = 0 & \text{in } \mathbb{R}^3 \times (t_1, t_2), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (t_1, t_2), \\ u(x, t_1) = u_0(x) & \text{in } \mathbb{R}^3. \end{cases} \qquad (I)$$

Note that we only assume $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$.

* Definition

The mild solution u to (I) is defined to be a function in $\mathcal{X}_{t_1, t_2} = L^\infty_t L^2_x(\mathbb{R}^3 \times (t_1, t_2), \mathbb{R}^3)$ satisfying

$$u(t) = \Gamma(t - t_1) * u_0 + B(u, u) \qquad (1)$$

where

$$B(u, v)(x, t) = \int_{t_1}^{t} K'(t-s) * ((u(s) * \eta_\varepsilon) \otimes v(s))\, ds. \qquad (2)$$

Here $K' : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^{27}$ is the same function as in Part 1: Mild Solutions. It is a smooth function with

$$|K'(x, t)| \leq C t^{-2} H\left(\frac{x}{\sqrt{t}}\right) \qquad (3)$$

(see Eq. (13), Part 1, page 9) where H is a smooth function in $\mathbb{R}^3$ and $H(x) \leq C|x|^{-4}$ as $x \to \infty$. As a consequence,

$$\|K'(t)\|_{L^a_x} \leq C t^{-2 + \frac{3}{2a}} \|H\|_{L^a} \quad \forall\, 1 \leq a \leq \infty \qquad (4)$$

(see Eq. (18), Part 1, page 11). When u is given, p is obtained by

$$p(x, t) = -\frac{1}{4\pi} \frac{\partial}{\partial x_j} \int_{\mathbb{R}^3} \frac{\operatorname{div}((u * \eta_\varepsilon) \otimes u)}{|x - y|}\, dy. \qquad (5)$$

* Local-in-time existence and uniqueness

We are going to show the following:

- $\mathcal{X}_{t_1,t_2}$ is a Banach space with norm $\|f\|_{\mathcal{X}_{t_1,t_2}} = \operatorname*{ess\,sup}_{t\in(t_1,t_2)} \|f(t)\|_{L^2_x}$.
- $B: \mathcal{X}_{t_1,t_2} \times \mathcal{X}_{t_1,t_2} \to \mathcal{X}_{t_1,t_2}$ is a well-defined, bilinear and continuous map.

First, we show that $\mathcal{X}_{t_1,t_2}$ is a Banach space. Recall the definition

$$\mathcal{X}_{t_1,t_2} = \left\{ f : \mathbb{R}^3 \times (t_1,t_2) \to \mathbb{R}^3 \;\middle|\; \begin{array}{l} f \text{ is measurable and the function } t\in(t_1,t_2) \mapsto \|f(t)\|_{L^2_x} \\ \text{is in } L^\infty((t_1,t_2)) \end{array} \right\}$$

Note that the same notation $\mathcal{X}_{t_1,t_2}$ was used in Part 1 to denote $L^\infty_{t,x}(\mathbb{R}^3\times(t_1,t_2),\mathbb{R}^3)$. In Part 2, however, we use the above definition for $\mathcal{X}_{t_1,t_2}$.

We can even show that $\mathcal{X}_{0,\infty}$ is a Banach space. Take $f\in\mathcal{X}_{0,\infty}$ such that $\operatorname*{ess\,sup}_{t\in(0,\infty)} \|f(t)\|_{L^2_x} = 0$. Then for $0<T<\infty$,

$$\int_0^T \int_{\mathbb{R}^3} |f(x,t)|^2 \,dx\,dt \le T \operatorname*{ess\,sup}_{t\in(0,\infty)} \|f(t)\|^2_{L^2_x} = 0.$$

Thus, $f=0$ a.e. in $\mathbb{R}^3\times(0,T)$. Because $\mathbb{R}^3\times(0,\infty) = \bigcup_{n=1}^\infty \mathbb{R}^3\times(0,n)$, we have $f=0$ a.e. in $\mathbb{R}^3\times(0,\infty)$. Also,

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} \|\lambda f(t)\|_{L^2_x} = |\lambda| \operatorname*{ess\,sup}_{t\in(0,\infty)} \|f(t)\|_{L^2_x} \qquad \forall \lambda\in\mathbb{R}.$$

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} \|f(t)+g(t)\|_{L^2_x} \le \operatorname*{ess\,sup}_{t\in(0,\infty)} \|f(t)\|_{L^2_x} + \operatorname*{ess\,sup}_{t\in(0,\infty)} \|g(t)\|_{L^2_x} \quad \forall f,g\in\mathcal{X}_{0,\infty}.$$

Hence, $\mathcal{X}_{0,\infty}$ is a normed space. Let (f_n) be a Cauchy sequence in $\mathcal{X}_{0,\infty}$. For each $\varepsilon>0$, there exists $N(\varepsilon)\in\mathbb{N}$ such that $\|f_m - f_n\|_{\mathcal{X}_{0,\infty}} < \varepsilon$ for all $m,n > N(\varepsilon)$.

For any $0<T<\infty$, $\int_0^T \int_{\mathbb{R}^3} |f_m(x,t)-f_n(x,t)|^2 dx\,dt \le T\|f_m - f_n\|_{\mathcal{X}_{0,\infty}} < T\varepsilon$.

Thus, (f_n) is a Cauchy sequence in $L^2_{t,x}(\mathbb{R}^3\times(0,T),\mathbb{R}^3)$, which is a Banach space. Hence, (f_n) converges in $L^2_{t,x}(\mathbb{R}^3\times(0,T),\mathbb{R}^3)$ to a function f. For a.e. $t\in(0,\infty)$,

$$\|f_m(t)-f(t)\|_{L^2_x} \le \|f_m(t)-f_n(t)\|_{L^2_x} + \|f_n(t)-f(t)\|_{L^2_x}$$
$$< \varepsilon + \|f_n(t)-f(t)\|_{L^2_x} \qquad \forall m,n > N(\varepsilon).$$

Let $n\to\infty$, we get $\|f_m(t)-f(t)\|_{L^2_x} < \varepsilon$ for all $m > N(\varepsilon)$. Therefore, $\|f_m - f\|_{\mathcal{X}_{0,\infty}}$ tends to 0 as $m\to\infty$.

Secondly, we show that B is a well-defined, bilinear and continuous map.

By (2), $$\|B(u,v)(t)\|_{L^2_x} \le \int_{t_1}^t \|K'(t-s)\,\cancel{\#}*((u(s)*\eta_\varepsilon)\otimes v(s))\|_{L^2_x}\, ds$$
$$\le \int_{t_1}^t \|K'(t-s)\|_{L^1_x}\, \|(u(s)*\eta_\varepsilon)\otimes v(s)\|_{L^2_x}\, ds$$
$$\overset{(4)}{\le} \int_{t_1}^t \frac{C}{\sqrt{t-s}}\, \|u(s)*\eta_\varepsilon\|_{L^\infty_x}\, \|v(s)\|_{L^2_x}\, ds$$
$$\overset{\text{Holder}}{\le} \int_{t_1}^t \frac{C}{\sqrt{t-s}}\, \|u(s)\|_{L^2_x}\, \|\eta_\varepsilon\|_{L^2}\, \|v(s)\|_{L^2_x}\, ds \qquad (6)$$

We have $\|\eta_\varepsilon\|_{L^2} = \varepsilon^{-3/2}\|\eta\|_{L^2}$. Thus, (6) implies

$$\|B(u,v)(t)\|_{L^2_x} \le \int_{t_1}^{t_2} \frac{C\varepsilon^{-3/2}}{\sqrt{t-s}}\, \|u\|_{\mathcal{X}_{t_1,t_2}} \|v\|_{\mathcal{X}_{t_1,t_2}}\, ds = C\varepsilon^{-3/2}\sqrt{t_2-t_1}\, \|u\|_{\mathcal{X}_{t_1,t_2}} \|v\|_{\mathcal{X}_{t_1,t_2}}.$$

Therefore,

$$\|B(u,v)\|_{\mathcal{X}_{t_1,t_2}} \le C\varepsilon^{-3/2}\sqrt{t_2-t_1}\, \|u\|_{\mathcal{X}_{t_1,t_2}} \|v\|_{\mathcal{X}_{t_1,t_2}}. \qquad (7)$$

Hence, B is a well-defined, bilinear and continuous map. We recall a useful lemma:

Let E be a Banach space and $B: E\times E \longrightarrow E$ be a linear map. Suppose that B is continuous, i.e. there exists a number $C>0$ such that

$$\|B(x,y)\|_E \leq C\|x\|_E \|y\|_E \qquad \forall x,y \in E.$$

Take $a\in E$. If $4C\|a\|_E < 1$ then the equation $x = a + B(x,x)$ has a solution in the ball $B_R = \{x : \|x\|_E < R\}$ with $R = \dfrac{1+\sqrt{1-4C\|a\|_E}}{2C}$.

Moreover, it is the unique solution in that ball and can be obtained by taking the limit of any sequence $\begin{cases} x_0 \in B_R, \\ x_{n+1} = a + B(x_n, x_n) \quad \forall n \geq 0. \end{cases}$

Applying this lemma, we know that if

$$C_\varepsilon^{-3/2}\sqrt{t_2-t_1}\ \|\Gamma(t) * u_0\|_{\mathcal{X}_{t_1,t_2}} < 1 \qquad (8)$$

then (1) has a solution $u \in \mathcal{X}_{t_1,t_2}$. Note that

$$\|\Gamma(t) * u_0\|_{L^2_x} \leq \underbrace{\|\Gamma(t)\|_{L^1_x}}_{=1} \|u_0\|_{L^2} = \|u_0\|_{L^2} = \|u(t_1)\|_{L^2_x}.$$

Thus, $\|\Gamma(t) * u_0\|_{\mathcal{X}_{t_1,t_2}} \leq \|u(t_1)\|_{L^2_x}$. Condition (8) is satisfied if

$$\sqrt{t_2-t_1}\ \|u(t_1)\|_{L^2_x} \leq C\varepsilon^{3/2} \qquad (9)$$

By the exact proof as in Part 1, pages 25-26, we have $u \in C_t L^2_x(\mathbb{R}^3 \times (t_1,t_2], \mathbb{R}^3)$. Therefore, we can use the continuation method to get the maximal time-interval of existence, say $(0,T^*)$, of a mild solution to $(NSE)_\varepsilon$.

* Regularity

We show that $t^{\ell+\frac{m}{2}}\partial_t^{\ell}\partial_x^m u \in L_t^\infty L_x^2(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$ $\forall m,\ell\geq 0$, $\forall T_1\in(0,T^*)$ (10)

and $\lim_{t\to 0^+}\|u(t)-u_0\|_{L_x^2}=0$. (11)

If (10) is proved then $u\in C^\infty(\mathbb{R}^3\times(0,T^*),\mathbb{R}^3)$ thanks to Sobolev's imbedding theorems. By (6) we have

$$\|B(u,u)(t)\|_{L_x^2}\leq C\varepsilon^{-3/2}\sqrt{t}\,\|u\|^2_{X_{0,T_1}} \qquad \forall t\in(0,T_1). \qquad (12)$$

$$\begin{aligned}\Gamma(t)*u_0-u_0 &= \int_{\mathbb{R}^3}\Gamma(y,t)\left(u_0(x-y)-u_0(x)\right)dy\\ &= \int_{\mathbb{R}^3}\frac{C}{t^{3/2}}F\left(\frac{y}{\sqrt{t}}\right)\left(u_0(x-y)-u_0(x)\right)dy\end{aligned}$$

(where $F(z)=\exp(-|z|^2)$)

$$= C\int_{\mathbb{R}^3}F(z)\left(u_0(x-z\sqrt{t})-u_0(x)\right)dz.$$

Thus, $\|F(t)*u_0-u_0\|_{L_x^2}\leq C\int_{\mathbb{R}^3}F(z)\|u_0(x-z\sqrt{t})-u_0(x)\|_{L_x^2}\,dz \to 0$ as $t\to 0^+$. (13)

By (12) and (13),

$$\|u(t)-u_0\|_{L_x^2}\leq\|\Gamma(t)*u_0-u_0\|_{L_x^2}+\|B(u,u)\|_{L_x^2}\to 0 \quad \text{as } t\to 0^+.$$

We have proved (11). Now fix $T_1\in(0,T)$. To prove (10), first we show by induction in $m\geq 0$ that

$$t^{\frac{m}{2}}\partial_x^m u\in L_t^\infty L_x^2(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3). \qquad (14)$$

(14) is true for $m=0$. Suppose that it is true for some $m\geq 0$. We know that $u\in C_t L_x^2(\mathbb{R}^3\times(0,T^*),\mathbb{R}^3)$. Thus, $M=\sup_{t\in[0,T_1]}\|u(t)\|_{L_x^2}<\infty$. By

dividing the interval $(0,T_1)$ into subintervals of length less than $\left(\frac{C\varepsilon^{3/2}}{M}\right)^2$ if necessary, we can assume $C\varepsilon^{-3/2}\sqrt{T_1}\,M < \frac{1}{2}$. Differentiating m times the equation $u(x,t) = \Gamma(t)*u_0 + B(u,u)$, we get

$$\partial_x^m u(x,t) = (\partial_x^m \Gamma(t))*u_0 + \sum_{k=0}^{m} \binom{m}{k} B(\partial_x^k u, \partial_x^{m-k} u). \qquad (15)$$

For each $i=1,2,3$ and $h \in (-1,1)\setminus\{0\}$ and function $f = f(x,t)$, denote

$$\Delta_i^h f(x,t) := \frac{f(x+he_i,t) - f(x,t)}{h}.$$

Applying Δ_i^h on both sides of (15), we get

$$\Delta_i^h \partial_x^m u(x,t) \approx (\Delta_i^h \partial_x^m \Gamma(t))*u_0 + \sum_{k=0}^{m}\binom{m}{k}\left[B(\Delta_i^h(\partial_x^k u), \partial_x^{m-k} u) + B(\partial_x^k u, \Delta_i^h(\partial_x^{m-k} u))\right].$$

Hence,

$$t^{(m+1)/2}|\Delta_i^h \partial_x^m u(x,t)| \le t^{(m+1)/2}|(\Delta_i^h \partial_x^m \Gamma(t))*u_0| + \sum_{k=0}^{m}\binom{m}{k}\Big(\big|B(t^{\frac{k+1}{2}}\Delta_i^h(\partial_x^k u), t^{\frac{m-k}{2}}\partial_x^{m-k} u)\big|$$
$$+ \big|B(t^{\frac{k}{2}}\partial_x^k u, t^{\frac{m-k}{2}}\Delta_i^h(\partial_x^{m-k} u))\big|\Big).$$

For $t \in (0,T_1)$,

$$t^{(m+1)/2}\|\Delta_i^h(\partial_x^m u(t))\|_{L_x^2} \overset{(7)}{\le} t^{(m+1)/2}\|\Delta_i^h \partial_x^m \Gamma(t)\|_{L_x^1}\|u_0\|_{L^2}$$
$$+ \sum_{k=0}^{m}\binom{m}{k} C\varepsilon^{-3/2}\sqrt{T_1}\,\|t^{\frac{k+1}{2}}\Delta_i^h(\partial_x^k u)\|_{\mathcal{X}_{0,T_1}}\|t^{\frac{m-k}{2}}\partial_x^{m-k} u\|_{\mathcal{X}_{0,T_1}}$$

$$\le t^{(m+1)/2}\|(\partial_x^{m+1}\Gamma)(t)\|_{L^1}\|u_0\|_{L^2} + \underbrace{\sum_{k=0}^{m-1}\binom{m}{k} C\varepsilon^{-3/2}\sqrt{T_1}\,\|t^{\frac{k+1}{2}}\Delta_i^h(\partial_x^k u)\|_{\mathcal{X}_{0,T_1}}\|t^{\frac{m-k}{2}}\partial_x^{m-k} u\|_{\mathcal{X}_{0,T_1}}}_{=M_1<\infty \text{ by the induction hypothesis}}$$
$$+ C\varepsilon^{-3/2}\sqrt{T_1}\,\|t^{(m+1)/2}\Delta_i^h(\partial_x^m u)\|_{\mathcal{X}_{0,T_1}}\|u\|_{\mathcal{X}_{0,T_1}}$$

$$\le t^{(m+1)/2}\|(\partial_x^{m+1}\Gamma)(t)\|_{L_x^1}\|u_0\|_{L^2} + M_1 + \frac{1}{2}\|t^{(m+1)/2}\Delta_i^h(\partial_x^m u)\|_{\mathcal{X}_{0,T_1}}. \qquad (16)$$

As explained in Part 1, page 16,

$$\| \partial_x^{m+1} \Gamma(t) \|_{L^1_x} \le \frac{C(m)}{t^{(m+1)/2}}.$$

Thus, (16) implies

$$t^{(m+1)/2} \| \Delta_i^h (\partial_x^m u(t)) \|_{L^2_x} \le C(m) \|u_0\|_{L^2} + M_1 + \frac{1}{2} \| t^{(m+1)/2} \Delta_i^h (\partial_x^m u) \|_{\mathcal{X}_{0,T_1}}.$$

Hence, $\| t^{(m+1)/2} \Delta_i^h (\partial_x^m u) \|_{\mathcal{X}_{0,T_1}} \le 2 \left(C(m) \|u_0\|_{L^2} + M_1 \right)$ for all $h \in (-1,1) \setminus \{0\}$.

Thus, $t^{(m+1)/2} \partial_x^{m+1} u$ exists and belongs to $\mathcal{X}_{0,T_1} = L^\infty_t L^2_x (\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)$. We have proved (14).

Next, we show by induction in $\ell \ge 0$ that

$$t^{\ell + \frac{m}{2}} \partial_t^\ell \partial_x^m u \in L^\infty_t L^2_x (\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3) \quad \forall m \ge 0. \qquad (17)$$

(17) is true for $\ell = 0$. Suppose that it is true for some $\ell \ge 0$. We show that it is true for $\ell + 1$. We'll work with the case $\ell = 0$ only. The case $\ell > 0$ can be done in the same way although the expressions look cumbersome.

Similarly to Eq. (33) in Part 1,

$$\partial_x^m u(x,t) = (\partial_x^m \Gamma)(t) * u_0 + \int_0^{t/2} \partial_x^m K'(t-s) * \left((u(s) + \eta_\varepsilon) \otimes u(s) \right) ds$$

$$+ \sum_{k=0}^{m} \binom{m}{k} \int_0^{t/2} K'(s) * \left((\partial_x^k u(t-s) + \eta_\varepsilon) \otimes \partial_x^{m-k} u(t-s) \right) ds. \qquad (18)$$

For $h \in (-1,1) \setminus \{0\}$ and function $v : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, we denote

$$\delta^h v(x,t) = \frac{v(x,t+h) - v(x,t)}{h}.$$

Applying δ^h to both sides of (18), we get

$$
\begin{aligned}
\delta^h(\partial_x^m u)(x,t) \simeq\ & \underbrace{((\partial_t\partial_x^m\Gamma)(t))*u_0}_{\{1\}} + \underbrace{\partial_x^m K'\left(\tfrac{t}{2}\right)*\left(\left(u\left(\tfrac{t}{2}\right)*\eta_\varepsilon\right)\otimes u\left(\tfrac{t}{2}\right)\right)}_{\{2\}} \\
& + \underbrace{\int_0^{t/2} \partial_t\partial_x^m K'(t-s)*((u(s)*\eta_\varepsilon)\otimes u(s))\,ds}_{\{3\}} + \underbrace{\sum_{k=0}^m \binom{m}{k} K'\left(\tfrac{t}{2}\right)*\left(\left(\partial_x^k u\left(\tfrac{t}{2}\right)*\eta_\varepsilon\right)\otimes\partial_x^{m-k}u\left(\tfrac{t}{2}\right)\right)}_{\{4\}} \\
& + \sum_{k=0}^m \binom{m}{k} \underbrace{\int_0^{t/2} K'(s)*((\delta^h\partial_x^k u(t-s)*\eta_\varepsilon)\otimes\partial_x^{m-k}u(t-s))\,ds}_{\{5\}} \\
& + \sum_{k=0}^m \binom{m}{k} \underbrace{\int_0^{t/2} K'(s)*((\partial_x^k u(t-s)*\eta_\varepsilon)\otimes\delta^h\partial_x^{m-k}u(t-s))\,ds}_{\{6\}}. \qquad (19)
\end{aligned}
$$

By Holder's inequality,

$$\|u(s)*\eta_\varepsilon\|_{L^\infty_x} \le \|u(s)\|_{L^2_x}\|\eta_\varepsilon\|_{L^2_x} = C\varepsilon^{-3/2}\|u(s)\|_{L^2_x}. \qquad (20)$$

With (20), we can estimate terms $\{1\},\dots,\{6\}$ by the same method as in Part 1, pages 18-19. Namely, for $t\in(0,T_1)$,

$$\|\{1\}\|_{L^2_x} \le C\|u_0\|_{L^2},$$

$$\|\{2\}\|_{L^2_x} \le \frac{C(m)\,\varepsilon^{-3/2}\|u(\frac{t}{2})\|^2_{L^2_x}}{t^{1/2+m/2}} \le \frac{C(m)\,\varepsilon^{-3/2}\|u\|^2_{\mathcal{X}_{0,T_1}}}{t^{1/2+m/2}},$$

$$\|\{3\}\|_{L^2_x} \le \frac{C(m)\,\varepsilon^{-3/2}\|u\|^2_{\mathcal{X}_{0,T_1}}}{t^{1/2+m/2}},$$

$$\|\{4\}\|_{L^2_x} = \|\{2\}\|_{L^2_x} \le \frac{C(m)\,\varepsilon^{-3/2}\|u\|^2_{\mathcal{X}_{0,T_1}}}{t^{1/2+m/2}}.$$

For $0 \leq k \leq m-1$,

$$\|\{5\}\|_{L^2_x} \leq C\varepsilon^{-3/2} \underbrace{\|\partial_t \partial_x^k u(t-s)\|_{L^2_x}}_{\leq \frac{C(k)}{(t-s)^{1+k/2}}} \underbrace{\|\partial_x^{m-k} u(t-s)\|_{L^2_x}}_{\leq \frac{C(k)}{(t-s)^{\frac{m-k}{2}}}}.$$

For $k=m$,

$$\|\{5\}\|_{L^2_x} \leq C\varepsilon^{-3/2}\sqrt{T_1}\,\|u\|_{\mathcal{X}_{0,T_1}} \|\delta^h \partial_x^m u\|_{\mathcal{X}_{0,T_1}} \leq \frac{1}{2}\|\delta^h \partial_x^m u\|_{\mathcal{X}_{0,T_1}}.$$

Note that we have used the same convention as in Part 1: the notation $C(m)$ is used to denote various quantities which depend only on m. We adopt such notations as $C(m)+1 = C(m)$, $C(m+1) = C(m)$, $2C(m) = C(m), \ldots$

Then (19) gives us an estimate

$$t^{1+\frac{m}{2}}\|\delta^h(\partial_x^m u)(t)\|_{L^2_x} \leq C\|u_0\|_{L^2} + C(m)\varepsilon^{-3/2}\sqrt{T_1}\,\|u\|^2_{\mathcal{X}_{0,T_1}} + C(m)\varepsilon^{-3/2}\sqrt{T_1}$$
$$+ \underbrace{C\varepsilon^{-3/2}\sqrt{T_1}\,\|u\|_{\mathcal{X}_{0,T_1}}}_{<\frac{1}{2}} \|t^{1+\frac{m}{2}}\delta^h \partial_x^m u\|_{\mathcal{X}_{0,T_1}}$$

Thus, $\|t^{1+\frac{m}{2}}\delta^h(\partial_x^m u)(t)\|_{L^2_x} \leq 2\left(C\|u_0\|_{L^2} + C(m)\varepsilon^{-3/2}\sqrt{T_1}\,\|u\|^2_{\mathcal{X}_{0,T_1}} + C(m)\varepsilon^{-3/2}\sqrt{T_1}\right)$ $\forall h$.

Thus, $\partial_t \partial_x^m u$ exists for $0<t<T_1$. Moreover, $t^{1+\frac{m}{2}}\partial_t\partial_x^m u \in \mathcal{X}_{0,T_1}$.

* <u>Energy identity</u>

By (10), $u(t) \in W^{m,2}(\mathbb{R}^3,\mathbb{R}^3)$ for all $m \geq 0$ and $t \in (0,T^*)$. By Sobolev's imbedding theorems (see Theorem 5.4, Eq. (3), Adams "Sobolev spaces", 1975)

$$u(t) \in W^{m,6}(\mathbb{R}^3,\mathbb{R}^3) \quad \forall m \geq 0 \ \forall t \in (0,T^*). \qquad (21)$$

By (5), $$p(t) \sim C \int_{\mathbb{R}^3} \frac{(\partial_x u(t) * \eta_\varepsilon)(y) \otimes u(y,t)}{|x-y|^2}\, dy \qquad (22)$$

$$\nabla p(t) \sim C \int_{\mathbb{R}^3} \frac{(\partial_x^2 u(t) * \eta_\varepsilon)(y) \otimes u(y,t)}{|x-y|^2}\, dy + C \int_{\mathbb{R}^3} \frac{(\partial_x u(t) * \eta_\varepsilon)(y) \otimes \partial_x u(y,t)}{|x-y|^2}\, dy$$

We have $$(23)$$

$$\underbrace{(\partial_x u(t) * \eta_\varepsilon)(y)}_{\in L^3_y} \otimes \underbrace{u(y,t)}_{\in L^2_y} \in L^{6/5}_y,$$

$$\underbrace{(\partial_x^2 u(t) * \eta_\varepsilon)(y)}_{\in L^3_y} \otimes \underbrace{u(y,t)}_{\in L^2_y} \in L^{6/5}_y,$$

$$\underbrace{(\partial_x u(t) * \eta_\varepsilon)(y)}_{\in L^3_y} \otimes \underbrace{\partial_x u(y,t)}_{\in L^2_y} \in L^{6/5}_y.$$

Recall the fractional interpolation (Theorem 4.18, p. 229, Bennett–Sharpley "Interpolation of Operators")

[For $f \in L^p(\mathbb{R}^n)$ and $I_\kappa f(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^{n-\kappa}}\, dy$, we have $\|I_\kappa f\|_q \le C_p \|f\|_p$

where $p > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\kappa}{n} > 0$.

Applying this result for $\kappa = 1$, $n = 3$, $p = 6/5$, $q = 2$, we have $p(t), \nabla p(t) \in L^2_x$.

Thus, $p(t) \in H^1_x$. Moreover, by (10) we have $\sqrt{t}\, p$, $t \nabla p \in L^\infty_t L^2_x(\mathbb{R}^3 \times (0, T_1))$.

Because of the regularity of u and p, they satisfy the differential equation

$$\partial_t u - \Delta u + ((u * \eta_\varepsilon) \cdot \nabla) u + \nabla p = 0. \qquad (24)$$

Thus, $t\partial_t u \in L^\infty_t L^2_x(\mathbb{R}^3 \times (0,T_1))$. Multiplying both sides of (24) by u and taking the integral over $x \in \mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} (\partial_t u) u\,dx - \underbrace{\int_{\mathbb{R}^3} u \Delta u\,dx}_{\{1\}} + \underbrace{\int_{\mathbb{R}^3} [((u * \eta_\varepsilon)\cdot\nabla)u] u\,dx}_{\{2\}} + \underbrace{\int_{\mathbb{R}^3} u \nabla p\,dx}_{\{3\}} = 0 \quad (25)$$

Because $u(t) \in W^{2,2}(\mathbb{R}^3,\mathbb{R}^3)$,

$$\{1\} = -\int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

Because $u(t) \in W^{1,2}(\mathbb{R}^3,\mathbb{R}^3)$ and $p(t) \in W^{1,2}(\mathbb{R}^3)$,

$$\{3\} = \int_{\mathbb{R}^3} p \underbrace{(\nabla\cdot u)}_{=0} dx = 0.$$

We have

$$\{2\} = \int_{\mathbb{R}^3} (u_j * \eta_\varepsilon)(x)\, u_{i,j}(x)\, u_i(x)\,dx$$

$$= \int_{\mathbb{R}^3} (u_j * \eta_\varepsilon)(x) \left(\frac{|u|^2}{2}\right)_{,j} dx$$

$\underline{\underline{u(t) \in W^{1,4}(\mathbb{R}^3,\mathbb{R}^3)}}$

$$- \int_{\mathbb{R}^3} (\underbrace{u_{j,j}}_{=0} * \eta_\varepsilon)(x) \frac{|u|^2}{2} dx$$

$$= 0.$$

Thus, (25) becomes $\displaystyle\int_{\mathbb{R}^3} (\partial_t u) u\,dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx = 0.$

Taking the integral both sides over $t \in [t_1, t_2] \subset (0, T_1)$, we get

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3} (\partial_t u) u\,dx\,dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\nabla u|^2 dx\,dt = 0.$$

By Fubini's theorem, the first term is equal to

$$\int_{\mathbb{R}^3}\int_{t_1}^{t_2}(\partial_t u)u\,dt\,dx = \frac{1}{2}\int_{\mathbb{R}^3}\frac{|u|^2}{2}\Big|_{t=t_1}^{t=t_2}dx = \frac{1}{2}\int_{\mathbb{R}^3}|u(t_2)|^2dx - \frac{1}{2}\int_{\mathbb{R}^3}|u(t_1)|^2dx.$$

Thus, $\frac{1}{2}\int_{\mathbb{R}^3}|u(t_2)|^2dx + \int_{t_1}^{t_2}\int_{\mathbb{R}^3}|\nabla u(t)|^2dx\,dt = \frac{1}{2}\int_{\mathbb{R}^3}|u(t_1)|^2dx$.

Letting $t_1\to 0^+$ and using (11), we get the <u>energy identity</u>

$$\frac{1}{2}\int_{\mathbb{R}^3}|u(t)|^2dx + \int_0^t\int_{\mathbb{R}^3}|\nabla u(s)|^2dx\,ds = \frac{1}{2}\int_{\mathbb{R}^3}|u_0|^2dx. \qquad (26)$$

* <u>Global-in-time existence</u>

Let $(0,T^*)$ be the maximal time-interval of existence of a mild solution to $(NSE)_\varepsilon$. If $T^*<\infty$ then by (9), $\lim_{t\to(T^*)^-}\|u(t)\|_{L^2_x}=\infty$. However, the energy identity (26) doesn't allow this to happen. Therefore, $T^*=\infty$, i.e. $(NSE)_\varepsilon$ has a global-in-time mild solution.

• <u>Making detail Step 2</u>:

Denote by $u_\varepsilon=(u_{\varepsilon 1},u_{\varepsilon 2},u_{\varepsilon 3})$ the global-in-time mild solution to $(NSE)_\varepsilon$ and the corresponding pressure p_ε. Then

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + ((u_\varepsilon * \eta_\varepsilon)\cdot\nabla)u_\varepsilon + \nabla p_\varepsilon = 0. \qquad (27)$$

Recall the notation $\mathcal{N}=\{\varphi\in\mathcal{D}(\mathbb{R}^3\times\mathbb{R},\mathbb{R}^3): \operatorname{div}\varphi(t)=0\ \forall t\geq 0\}$.

Multiplying both sides of (27) by $\varphi\in\mathcal{N}$ and taking the integral over $x\in\mathbb{R}^3$, we get:

$$\frac{d}{dt}\int_{\mathbb{R}^3} u_\varepsilon(t)\varphi(t)\,dt - \int_{\mathbb{R}^3} u_\varepsilon(t)\partial_t\varphi(t)\,dx + \int_{\mathbb{R}^3}\nabla u_\varepsilon(t)\cdot\nabla\varphi(t)\,dx +$$

$$+ \underbrace{\int_{\mathbb{R}^3}\left[((u_\varepsilon * \eta_\varepsilon)\cdot\nabla)u_\varepsilon\right]\varphi(t)\,dx}_{\{1\}} - \int_{\mathbb{R}^3} p_\varepsilon(t)\underbrace{\nabla\cdot\varphi(t)}_{=0}\,dx = 0 \quad (28)$$

We have
$$\{1\} = \int_{\mathbb{R}^3} (u_{\varepsilon j}(t) * \eta_\varepsilon)(x)\, u_{\varepsilon i,j}(x,t)\, \varphi_i(x,t)\,dx\,.$$

Taking the integral both sides of (28) over $t \in [t_1, t_2] \subset (0,\infty)$, we get

$$\int_{\mathbb{R}^3} u_\varepsilon(t_2)\varphi(t_2)\,dx - \int_{\mathbb{R}^3} u_\varepsilon(t_1)\varphi(t_1)\,dx + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla u_\varepsilon(t)\cdot\nabla\varphi(t)\,dx\,dt - \int_{t_1}^{t_2}\int_{\mathbb{R}^3} u_\varepsilon(t)\partial_t\varphi(t)\,dx\,dt$$

$$+ \int_{t_1}^{t_2}\int_{\mathbb{R}^3} (u_{\varepsilon j}(t) * \eta_\varepsilon)\, u_{\varepsilon i,j}(t)\, \varphi_i(t)\,dx\,dt = 0. \quad (29)$$

By (11), $\lim_{t\to 0^+} \|u_\varepsilon(t) - u_0\|_{L^2_x} = 0$. The energy identity (26) reads

$$\frac{1}{2}\int_{\mathbb{R}^3} |u_\varepsilon(t)|^2 dx + \int_0^t\int_{\mathbb{R}^3} |\nabla u_\varepsilon(s)|^2 dx\,ds = \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx \quad \forall t>0. \quad (30)$$

Thus,
$$\lim_{t\to 0^+}\int_0^t\int_{\mathbb{R}^3} |\nabla u_\varepsilon(s)|^2 dx\,ds = 0.$$

Letting $t_1 \to 0^+$ in (29), we get

$$\int_{\mathbb{R}^3} u_\varepsilon(t)\varphi(t)\,dx - \int_0^t\int_{\mathbb{R}^3} u_\varepsilon(s)\partial_t\varphi(s)\,dx\,ds + \int_0^t\int_{\mathbb{R}^3} \nabla u_\varepsilon(s)\cdot\nabla\varphi(s)\,dx\,ds +$$

$$+ \int_0^t\int_{\mathbb{R}^3} (u_{\varepsilon j}(s) * \eta_\varepsilon)\, u_{\varepsilon i,j}(s)\,\varphi_i(s)\,dx\,ds = 0 \quad \forall t>0, \forall \varphi \in \mathcal{N}. \quad (31)$$

By (30), $(\nabla u_\varepsilon)_{\varepsilon>0}$ is a bounded family in $L^2(\mathbb{R}^3\times(0,\infty), \mathbb{R}^{3\times 3})$. Thus, there is a ~~sub~~sequence $(\varepsilon_n)\downarrow 0$ such that $(\nabla u_{\varepsilon_n})$ converges in the weak topology of $L^2(\mathbb{R}^3\times(0,\infty), \mathbb{R}^{3\times 3})$. Hereafter, instead of writing u_{ε_n} and η_{ε_n}, we write $u^{(n)}$ and $\eta^{(n)}$. The components of $u^{(n)}$ are denoted by $u_1^{(n)}, u_2^{(n)}, u_3^{(n)}$. Write

$$\nabla u^{(n)} = (\nabla u_1^{(n)}, \nabla u_2^{(n)}, \nabla u_3^{(n)}) \longrightarrow (v_1, v_2, v_3) = v \in L^2(\mathbb{R}^3\times(0,\infty), \mathbb{R}^{3\times 3}). \quad (32)$$

In order to pass (31) to the limit as $\varepsilon\to 0$, we need to find a subsequence $(u^{(n')})$ of $(u^{(n)})$ such that $(u^{(n')}(t))$ converges in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ for almost every $t\in(0,\infty)$. Suppose that this is proved. For a.e. $t\in(0,\infty)$, we denote by $u(t)$ the limit of $(u^{(n')}(t))$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. By the energy identity (30),

$$\|u(t)\|_{L^2_x}, \ \|u^{(n')}(t)\|_{L^2_x} \leqslant \|u_0\|_{L^2} \qquad \text{a.e. } t\in(0,\infty). \quad (33)$$

Thus, $\|u^{(n')}(t) - u(t)\|_{L^2_x} \leqslant 2\|u_0\|_{L^2}$ for a.e. $t\in(0,\infty)$. For any $T\in(0,\infty)$, by Lebesgue's Dominated Convergence theorem,

$$\int_0^T \|u^{(n')}(t) - u(t)\|^2_{L^2_x}\,dt \longrightarrow 0 \qquad \text{as } n'\to\infty.$$

Hence, $u^{(n')}\to u$ in $L^2(\mathbb{R}^3\times(0,T), \mathbb{R}^3)$. Because $\nabla u^{(n')}\to v$ in $L^2(\mathbb{R}^3\times(0,T), \mathbb{R}^3)$, u has weak derivatives (with respect to x) in $L^2(\mathbb{R}^3\times(0,T), \mathbb{R}^3)$ and $\nabla u = v$. Thus,

$$\nabla u^{(n')} \longrightarrow \nabla u \quad \text{in } L^2(\mathbb{R}^3\times(0,\infty), \mathbb{R}^{3\times 3}). \quad (34)$$

Consequently, $$\int_0^\infty\int_{\mathbb{R}^3} |\nabla u(s)|^2\,dx\,ds \leqslant \liminf_{n'\to\infty} \int_0^\infty\int_{\mathbb{R}^3} |\nabla u^{(n')}(s)|^2\,dx\,ds \overset{(30)}{\leqslant} \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2\,dx. \quad (35)$$

By (33) and (35), $u \in \left(L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1\right)(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^3)$. (36)

By (31) we have

$$\underbrace{\int_{\mathbb{R}^3} u^{(n')}(t)\varphi(t)\,dx}_{\{1\}} - \underbrace{\int_0^t\int_{\mathbb{R}^3} u^{(n')}(s)\,\partial_t\varphi(s)\,dx\,ds}_{\{2\}} + \underbrace{\int_0^t\int_{\mathbb{R}^3} \nabla u^{(n')}(s)\cdot\nabla\varphi(s)\,dx\,ds}_{\{3\}}$$

$$+ \underbrace{\int_0^t\int_{\mathbb{R}^3} \left(u_j^{(n')}(s) * \eta^{(n')}\right) u_{i,j}^{(n')}(s)\,\varphi_i(s)\,dx\,ds}_{\{4\}} = 0 \quad \forall t\in(0,\infty)\ \forall \varphi \in \mathcal{N}. \tag{37}$$

Fix $\varphi \in \mathcal{N}$. We have

$\{1\} \longrightarrow \int_{\mathbb{R}^3} u(t)\varphi(t)\,dx$ a.e. $t\in(0,\infty)$ because $u^{(n')}(t) \xrightarrow{L_x^2} u(t)$ a.e. $t\in(0,\infty)$;

$\{2\} \longrightarrow \int_0^t\int_{\mathbb{R}^3} u(s)\,\partial_t\varphi(s)\,dx\,ds$ $\forall t\in(0,\infty)$ because $u^{(n')} \to u$ in $L^2(\mathbb{R}^3\times(0,T),\mathbb{R}^3)$;

$\{3\} \longrightarrow \int_0^t\int_{\mathbb{R}^3} \nabla u(s)\cdot\nabla\varphi(s)\,dx\,ds$ $\forall t\in(0,\infty)$ because of (34).

We have

$$\|u_j^{(n')}(s)*\eta^{(n')} - u_j(s)\|_{L_x^2} \le \|(u_j^{(n')}(s)-u(s))*\eta^{(n')}\|_{L_x^2} + \|u(s)*\eta^{(n')} - u(s)\|_{L_x^2}$$

$$\le \underbrace{\|u_j^{(n')}(s)-u(s)\|_{L_x^2}}_{\to 0}\ \underbrace{\|\eta^{(n')}\|_{L_x^1}}_{=1} + \underbrace{\|u(s)*\eta^{(n')} - u(s)\|_{L_x^2}}_{\to 0 \text{ for a.e. } s\in(0,\infty)}$$

Thus, $\lim_{n'\to\infty} \|u_j^{(n')}(s)*\eta^{(n')} - u_j(s)\|_{L_x^2} = 0$ for a.e. $s\in(0,\infty)$.

Moreover, $\|u_j^{(n')}(s)*\eta^{(n')} - u_j(s)\|_{L_x^2} \le \|u_j^{(n')}(s)*\eta^{(n')}\|_{L_x^2} + \|u_j(s)\|_{L_x^2}$

$$\le \|u_j^{(n')}(s)\|_{L_x^2}\ \underbrace{\|\eta^{(n')}\|_{L^1}}_{=1} + \|u_j(s)\|_{L_x^2}$$

$\le 2\|u_0\|_{L^2}$ (by (33)) $\quad \forall s\in(0,\infty)$.

By Lebesgue's Dominated Convergence theorem, $\|u_j^{(n')}(s)*\eta^{(n')} - u_j(s)\|_{L^2_x} \to 0$ in $L^2((0,T))$. Thus,

$$u_j^{(n')}(s) * \eta^{(n')} \longrightarrow u_j(s) \quad \text{in} \quad L^2(\mathbb{R}^3 \times (0,T), \mathbb{R}^3). \qquad (38)$$

Because of (34), $u_{i,j}^{(n')} \rightharpoonup u_{i,j}$ in $L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^3)$. $\qquad (39)$

Recall the following lemma: (Brezis "Functional Analysis, Sobolev Spaces and PDE", 2011, p.63)

[Let X be a Hilbert space and $(x_n), (y_n)$ be two sequences in X. Suppose that $x_n \to x$ and $y_n \rightharpoonup y$. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Applying this lemma for $X = L^2(\mathbb{R}^3 \times (0,T), \mathbb{R}^3)$, we conclude from (38) and (39) that

$$\{4\} \longrightarrow \int_0^t \int_{\mathbb{R}^3} u_j(s)\, u_{i,j}(s)\, \varphi_i(s)\, dx\, ds.$$

Therefore, as $n' \to \infty$, (37) yields

$$\int_{\mathbb{R}^3} u(t)\varphi(t)\,dx - \int_0^t \int_{\mathbb{R}^3} u(s)\,\partial_t \varphi(s)\,dx\,ds + \int_0^t \int_{\mathbb{R}^3} \nabla u(s) \cdot \nabla \varphi(s)\,dx\,ds$$
$$+ \int_0^t \int_{\mathbb{R}^3} u_j(s)\, u_{i,j}(s)\, \varphi_i(s)\,dx\,ds = 0 \qquad \text{a.e. } t \in (0,\infty). \qquad (40)$$

Now that we have (36) and (40), to say u is a Leray's weak solution to (NSE), we need to show that $\lim_{t \to 0^+} \|u(t) - u_0\|_{L^2_x} = 0$. First, we show that for each $\Psi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, the sequence $\left(\int_{\mathbb{R}^3} u^{(n')}(t)\,\Psi\,dx \right)_{n'}$ is equicontinuous in ~~$t \in (0,T)$~~. $t \in (0,\infty)$.

Let $w(x) = \int_{\mathbb{R}^3} -\dfrac{\Psi(x-y)}{4\pi |y|}\,dy$ be the Newtonian potential of Ψ.

Then $w \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$, $\Delta w = \Psi$ and for every multi-index α,

$$D^\alpha w(x) = \int_{\mathbb{R}^3} -\frac{D^\alpha \Psi(y)}{4\pi|x-y|}\,dy = \int_{B_R(0)} -\frac{D^\alpha \Psi(y)}{|x-y|}\,dy,$$

where $R>0$ is a number such that $\operatorname{supp}\Psi \subset B_R(0)$. Thus, $D^\alpha w \in L^p(\mathbb{R}^3)$ for all $p \in (1,\infty)$ (Gilbarg-Trudinger 1998, Theorem 9.9, p.230). On the other hand,

$$D^\alpha w(x) = \int_{\mathbb{R}^3} -\frac{D^\alpha \Psi(x-y)}{|y|}\,dy \longrightarrow 0 \quad \text{as } x \to \infty.$$

Thus, $D^\alpha w \in L^\infty(\mathbb{R}^3)$. Hence, $D^\alpha w \in L^p(\mathbb{R}^3)$ $\forall\, 1<p\leq\infty$. (41)

We have the identity

$$\Psi = \underbrace{\nabla(\operatorname{div} w)}_{} - \underbrace{\operatorname{curl}(\operatorname{curl} w)}_{}$$
$$= \Psi_1 + \Psi_2 .$$

By (41), $\Psi_1, \Psi_2 \in W^{m,p}(\mathbb{R}^3,\mathbb{R}^3)$ for all $m\geq 0$, $1<p\leq\infty$.

$$\int_{\mathbb{R}^3} u^{(n')}(t)\Psi\,dx = \underbrace{\int_{\mathbb{R}^3} u^{(n')}(t)\,\nabla(\operatorname{div} w)\,dx}_{} + \int_{\mathbb{R}^3} u^{(n')}(t)\,\Psi_2\,dx .$$
$$= -\int_{\mathbb{R}^3} \underbrace{(\operatorname{div} u^{(n')}(t))}_{=0} \operatorname{div} w\,dx$$

By replacing Ψ with Ψ_2, we can assume that $\Psi \in (C^\infty \cap W^{m,p})(\mathbb{R}^3;\mathbb{R}^3)$ for all $m\geq 0$, $1<p\leq\infty$, and $\operatorname{div}\Psi = 0$. By (37) we have

$$\int_{\mathbb{R}^3} u^{(n')}(t)\Psi\,dx + \int_0^t\!\!\int_{\mathbb{R}^3} \nabla u^{(n')}(s)\cdot\nabla\Psi\,dx\,ds + \int_0^t\!\!\int_{\mathbb{R}^3} \left(u_j^{(n')}(s) * \eta^{(n')}\right) u_{i,j}^{(n')}(s)\,\Psi_i\,dx\,ds = 0$$
$$\forall\, t\in[0,\infty). \quad (42)$$

For $0<t_1<t_2<\infty$,

$$\left|\int_{\mathbb{R}^3}\left(u^{(n')}(t_2)-u^{(n')}(t_1)\right)\psi\,dx\right| \le \underbrace{\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\nabla u^{(n')}(s)\cdot\nabla\psi\,dx\,ds\right|}_{\{1\}} + \underbrace{\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\left(u_j^{(n')}(s)*\eta^{(n')}\right)u_{i,j}^{(n')}(s)\psi_i\,dx\,ds\right|}_{\{2\}} \qquad (43)$$

We estimate $\{1\}$ and $\{2\}$ as follows.

$$\{1\} \overset{\text{Schwarz}}{\le} \int_{t_1}^{t_2}\|\nabla u^{(n')}(s)\|_{L^2_x}\|\nabla\psi\|_{L^2}\,ds \overset{\text{Schwarz}}{\le} \|\nabla\psi\|_{L^2}\sqrt{t_2-t_1}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|\nabla u^{(n')}(s)|\,dx\,ds\right)^{1/2}$$

$$\overset{(30)}{\le} \|\nabla\psi\|_{L^2}\left(\frac{1}{2}\|u_0\|_{L^2}^2\right)^{1/2}\sqrt{t_2-t_1}. \qquad (44)$$

$$\{2\} \overset{\text{Schwarz}}{\le} \int_{t_1}^{t_2}\underbrace{\|u_j^{(n')}(s)*\eta^{(n')}\|_{L^2_x}}_{\le \|u_j^{(n')}(s)\|_{L^2_x}\le\|u_0\|_{L^2}}\|u_{i,j}^{(n')}(s)\|_{L^2_x}\|\psi\|_{L^\infty}\,dx\,ds$$

$$\le \|\psi\|_{L^\infty}\|u_0\|_{L^2}\int_{t_1}^{t_2}\|u_{i,j}^{(n')}(s)\|_{L^2_x}\,ds$$

$$\overset{\text{Schwarz}}{\le} \|\psi\|_{L^\infty}\|u_0\|_{L^2}\sqrt{t_2-t_1}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^3}|\nabla u(s)|^2\,dx\,ds\right)^{1/2}$$

$$\overset{(30)}{\le} \|\psi\|_{L^\infty}\|u_0\|_{L^2}\left(\frac{1}{2}\|u_0\|_{L^2}^2\right)^{1/2}\sqrt{t_2-t_1}. \qquad (45)$$

Replacing (44) and (45) into (43), we get

$$\left|\int_{\mathbb{R}^3}\left(u^{(n')}(t_2)-u^{(n')}(t_1)\right)\psi\,dx\right| \le \left(C\|\nabla\psi\|_{L^2}\|u_0\|_{L^2}+C\|\psi\|_{L^\infty}\|u_0\|_{L^2}^2\right)\sqrt{t_2-t_1}. \qquad (46)$$

Thus, the sequence $\left(\int_{\mathbb{R}^3}u^{(n')}(t)\psi\,dx\right)_{n'}$ is equicontinuous in $t\in(0,\infty)$.

Next, we show that $u(t)\to u_0$ in $L^2(\mathbb{R}^3,\mathbb{R}^3)$ as $t\to 0^+$. For each $\psi\in\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}(u(t)-u_0)\psi\,dx = \underbrace{\int_{\mathbb{R}^3}\left(u^{(n')}(t)-u_0\right)\psi\,dx}_{\{3\}} - \int_{\mathbb{R}^3}\left(u^{(n')}(t)-u(t)\right)\psi\,dx. \qquad (47)$$

For each n', $\|u^{(n')}(t_1) - u_0\|_{L^2_x} \to 0$ as $t_1 \to 0^+$ (by (11)). In (46), letting $t_1 \to 0^+$ we get

$$|\{3\}| = \left|\int_{\mathbb{R}^3} (u^{(n')}(t) - u_0)\,\Psi\, dx\right| \leq \left(C\|\nabla\Psi\|_{L^2}\|u_0\|_{L^2} + C\|\Psi\|_{L^\infty}\|u_0\|^2_{L^2}\right)\sqrt{t}.$$

Thus, for each $\varepsilon > 0$, there exists $\delta > 0$ depending only on Ψ, u_0, ε such that $|\{3\}| < \varepsilon$ for all n' and $t \in (0,\delta)$. From (47),

$$\left|\int_{\mathbb{R}^3} (u(t) - u_0)\,\psi\, dx\right| < \varepsilon + \left|\int_{\mathbb{R}^3} (u^{(n')}(t) - u(t))\,\psi\right| \qquad \forall n',\ \forall t \in (0,\delta)$$

Letting $n' \to \infty$ and using the fact that $u^{(n')}(t) \to u(t)$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ for a.e. $t \in (0,\infty)$, we get

$$\left|\int_{\mathbb{R}^3} (u(t) - u_0)\,\psi\, dx\right| < \varepsilon \qquad \text{a.e. } t \in (0,\delta).$$

Thus, $u(t) \rightharpoonup u_0$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. On the other hand, $\|u(t)\|_{L^2_x} \leq \|u_0\|_{L^2}$ because of (33). Thus, $u(t) \to u_0$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ as $t \to 0^+$. This means u is a Leray's weak solution.

Therefore, all we need to show is the existence of a subsequence $(u^{(n')})$ of $(u^{(n)})$ such that $(u^{(n')}(t))$ converges in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ for a.e. $t \in (0,\infty)$. We have

$$\int_0^\infty \liminf_{n\to\infty} \|\nabla u^{(n)}(s)\|^2_{L^2_x}\, ds \overset{\text{Fatou}}{\leq} \liminf_{n\to\infty} \int_0^\infty \|\nabla u^{(n)}(s)\|^2_{L^2_x}\, ds \overset{(30)}{\leq} \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx$$

Put $A = \{s \in (0,\infty) : \liminf_{n\to\infty} \|\nabla u^{(n)}(s)\|_{L^2_x} < \infty\}$. Then $(0,\infty)\setminus A$ is of measure zero. We are going to show the following:

(i) For each $t \in A$, there exists a subsequence $(u^{(n')}(t))$ which converges in $L^2(\mathbb{R}^3, \mathbb{R}^3)$.

(ii) For each $\psi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, the sequence $\left(\int_{\mathbb{R}^3} u^{(n)}(t)\,\psi\,dx\right)_n$ is equicontinuous in $t \in (0, \infty)$.

(iii) There is a subsequence $(u^{(n')})$ of $(u^{(n)})$ such that for each $t \in A$, $(u^{(n')}(t))$ converges in $L^2(\mathbb{R}^3, \mathbb{R}^3)$.

Proof of (i)

Fix $t \in A$. Because $\liminf_{n \to \infty} \|\nabla u^{(n)}(t)\|_{L^2_x} < \infty$, there exists a subsequence $(\nabla u^{(n')}(t))$ that is bounded in L^2_x. We want to show that the set $\{u^{(n')}(t)\}_n$ is precompact in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. Theorem 2.22, page 33, Adams "Sobolev Spaces", 1975 gives a criterion for precompactness in L^p, $1 \le p < \infty$. Accordingly, if we have two following properties

(a) For each bounded open subset B of $\mathbb{R}^3$, the set $\{u^{(n')}(t)\}_{n'}$ is precompact in $L^2(B)$,

(b) For each $\varepsilon > 0$, there exists a number $R > 0$ such that

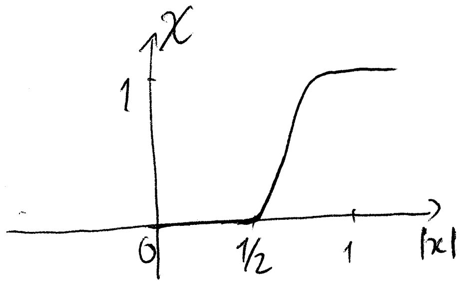
$$\int_{|x|>R} |u^{(n')}(t)|^2\,dx < \varepsilon \qquad \forall n',$$

then $\{u^{(n')}(t)\}_n$ is a precompact subset of $L^2(\mathbb{R}^3, \mathbb{R}^3)$. Fix a bounded open subset B of $\mathbb{R}^3$. By Rellich-Kondrachov's theorem, the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(B)$ is compact. Because both $(u^{(n')}(t))_{n'}$ and $(\nabla u^{(n')}(t))_{n'}$ are bounded sequences in $L^2(\mathbb{R}^3)$, $(u^{(n')}(t))_{n'}$ is bounded in $H^1(\mathbb{R}^3)$. Thus,

$\{u^{(n')}(t)\}_{n'}$ is precompact in $L^2(B)$. Thus, (a) is proved.

Now we prove (b). Let $p^{(n')}$ be the corresponding pressure. From Step 1,

$$\partial_t u^{(n')} - \Delta u^{(n')} + ((u^{(n')} + \eta^{(n')}) \cdot \nabla) u^{(n')} + \nabla p^{(n')} = 0. \qquad (48)$$



Define a map $\chi : \mathbb{R}^3 \to \mathbb{R}$,

$$\chi(x) = \begin{cases} 0 & \text{if } |x| \le \frac{1}{2}, \\ 1 & \text{if } |x| \ge 1, \\ -16|x|^3 + 36|x|^2 - 24|x| + 5 & \text{if } \frac{1}{2} \le |x| \le 1. \end{cases}$$

Then $\chi \in C^1(\mathbb{R}^3)$ and $0 \le \chi(x) \le 1$. For each $R > 0$, define

$$\chi_R(x) = \chi\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^3.$$

Then $\chi_R \in C^1(\mathbb{R}^3)$, $0 \le \chi_R(x) \le 1$ and $\chi_R(x) = \begin{cases} 0 & \text{if } |x| \le \frac{R}{2}, \\ 1 & \text{if } |x| \ge R. \end{cases}$

Multiplying both sides of (48) by $\chi_R u^{(n')}$ and taking the integral over $\mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} \chi_R u^{(n')} \partial_t u^{(n')} dx + \int_{\mathbb{R}^3} \nabla u^{(n')} \cdot \nabla(\chi_R u^{(n')}) dx + \int_{\mathbb{R}^3} [((u^{(n')} + \eta^{(n')}) \cdot \nabla) u^{(n')}] \chi_R u^{(n')} dx$$
$$- \int_{\mathbb{R}^3} p^{(n')} \nabla \cdot (\chi_R u^{(n')}) dx = 0.$$

To simplify the notations, we denote $u^{(n')}$ by v, $\eta^{(n')}$ by ζ, and $p^{(n')}$ by q.

Then
$$\int_{\mathbb{R}^3} \chi_R v \partial_t v \, dx + \underbrace{\int_{\mathbb{R}^3} \nabla v \cdot \nabla(\chi_R v) dx}_{\{4\}} + \underbrace{\int_{\mathbb{R}^3} [((v+\zeta)\cdot\nabla) v] \chi_R v}_{\{5\}} - \underbrace{\int_{\mathbb{R}^3} q \nabla \cdot (\chi_R v) dx}_{\{6\}} = 0. \qquad (49)$$

We have
$$\{4\} = \int_{\mathbb{R}^3} |\nabla v|^2 \chi_R \, dx + \int_{\mathbb{R}^3} (\nabla v) v (\nabla \chi_R) dx. \qquad (50)$$

$$\{5\} = \int_{\mathbb{R}^3} (v_j + \zeta) v_{i,j} \chi_R v_i \, dx = \frac{1}{2} \int_{\mathbb{R}^3} (v_j + \zeta) \chi_R (|v|^2)_{,j} \, dx$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \big[(\underbrace{v_{j,j}}_{=0} + \zeta) \chi_R + (v_j + \zeta) \chi_{R,j} \big] dx$$

$$= -\frac{1}{2} \int_{\mathbb{R}^3} |v|^2 (v + \zeta) \nabla \chi_R \, dx. \qquad (51)$$

$$\{6\} = \int_{\mathbb{R}^3} q \underbrace{(\nabla \cdot v)}_{=0} \chi_R \, dx + \int_{\mathbb{R}^3} q v \nabla \chi_R \, dx = \int_{\mathbb{R}^3} q v \nabla \chi_R \, dx. \qquad (52)$$

Substituting (50), (51), (52) into (49), we get

$$\int_{\mathbb{R}^3} \chi_R v \, \partial_t v \, dx = -\int_{\mathbb{R}^3} |\nabla v|^2 \chi_R \, dx - \int_{\mathbb{R}^3} (\nabla v) v (\nabla \chi_R) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 (v + \zeta) \nabla \chi_R \, dx$$

$$+ \int_{\mathbb{R}^3} q v \nabla \chi_R \, dx$$

$$\leq \underbrace{-\int_{\mathbb{R}^3} (\nabla v) v (\nabla \chi_R) \, dx}_{\{7\}} + \underbrace{\frac{1}{2} \int_{\mathbb{R}^3} |v|^2 (v + \zeta) \nabla \chi_R \, dx}_{\{8\}} + \underbrace{\int_{\mathbb{R}^3} q v \nabla \chi_R \, dx}_{\{9\}} \qquad (53)$$

By the definition of χ_R, $\nabla \chi_R = \frac{1}{R} \nabla \chi\left(\frac{x}{R}\right)$. Thus,

$$|\nabla \chi_R| \leq \frac{1}{R} \max_{\mathbb{R}^3} |\nabla \chi| = \frac{C}{R}. \qquad (54)$$

$$\{7\} \leq \int_{\mathbb{R}^3} |\nabla v| |v| |\nabla \chi_R| \, dx \overset{(54)}{\leq} \frac{C}{R} \int_{\mathbb{R}^3} |\nabla v| |v| \, dx$$

$$\overset{\text{Schwarz}}{\leq} \frac{C}{R} \|\nabla v\|_{L^2_x} \|v\|_{L^2_x}$$

$$\overset{(33)}{\leq} \frac{C}{R} \|\nabla v\|_{L^2_x} \|u_0\|_{L^2}. \qquad (55)$$

$$\{8\} \overset{(54)}{\leq} \frac{C}{R} \int_{\mathbb{R}^3} |v|^2 |v * \zeta| \, dx \overset{\text{Schwarz}}{\leq} \frac{C}{R} \|v\|_{L^4_x}^2 \|v * \zeta\|_{L^2_x} .$$

By Holder's inequality, $\|v\|_{L^4_x} \leq \|v\|_{L^6_x}^{3/4} \|v\|_{L^2_x}^{1/4}$. By Young's inequality, $\|v * \zeta\|_{L^2_x} \leq \|v\|_{L^2_x} \|\zeta\|_{L^1_x} = \|v\|_{L^2_x}$. Then we get

$$\{8\} \leq \frac{C}{R} \|v\|_{L^6_x}^{3/2} \|v\|_{L^2_x}^{3/2} \overset{(33)}{\leq} \frac{C}{R} \|v\|_{L^6_x}^{3/2} \|u_0\|_{L^2}^{3/2} .$$

By (10), $v(t) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$. By Sobolev's imbedding theorem, $\|v(t)\|_{L^6_x} \leq C \|\nabla v(t)\|_{L^2_x}$.

Thus, $$\{8\} \leq \frac{C}{R} \|\nabla v\|_{L^2_x}^{3/2} \|u_0\|_{L^2}^{3/2} . \qquad (56)$$

$$\{9\} \overset{(54)}{\leq} \frac{C}{R} \int_{\mathbb{R}^3} |q| |v| \, dx \overset{\text{Schwarz}}{\leq} \frac{C}{R} \|q\|_{L^2_x} \|v\|_{L^2_x} \overset{(33)}{\leq} \frac{C}{R} \|u_0\|_{L^2} \|q\|_{L^2_x} . \qquad (57)$$

By (22), page 15 and the fractional interpolation mentioned on the same page, we have

$$\begin{aligned} \|q\|_{L^2_x} \leq C \|(\nabla v) v\|_{L^{6/5}_x} &\overset{\text{Holder}}{\leq} C \|\nabla v\|_{L^2_x} \|v\|_{L^3_x} \\ &\overset{\text{Holder}}{\leq} C \|\nabla v\|_{L^2_x} \|v\|_{L^6_x}^{1/2} \|v\|_{L^2_x}^{1/2} \\ &\overset{(33)}{\leq} C \|u_0\|_{L^2}^{1/2} \|\nabla v\|_{L^2_x} \|v\|_{L^6_x}^{1/2} \\ &\underset{\text{imbedding}}{\overset{\text{Sobolev}}{\leq}} C \|u_0\|_{L^2}^{1/2} \|\nabla v\|_{L^2_x}^{3/2} . \end{aligned}$$

Then (57) implies $$\{9\} \leq \frac{C}{R} \|u_0\|_{L^2}^{3/2} \|\nabla v\|_{L^2_x}^{3/2} \qquad (58)$$

Substituting (55), (56), (58) into (53), we get

$$\int_{\mathbb{R}^3} \chi_R v \partial_t v \, dx \leq \frac{C}{R} \|u_0\|_{L^2} \|\nabla v\|_{L^2_x} + \frac{C}{R} \|u_0\|_{L^2}^{3/2} \|\nabla v\|_{L^2_x}^{3/2}$$

Integrating both sides over $[0, t]$, we get

$$\int_{\mathbb{R}^3} \chi_R \int_0^t v\,\partial_t v\,dx \leq \frac{C}{R}\|u_0\|_{L^2}\int_0^t \|\nabla v(s)\|_{L^2_x}^2\,ds + \frac{C}{R}\|u_0\|_{L^2}^{3/2}\int_0^t \|\nabla v(s)\|_{L^2_x}^{3/2}\,ds$$

$$\overset{\text{Holder}}{\leq} \frac{C}{R}\|u_0\|_{L^2} t^{1/2}\left(\int_0^t \|\nabla v(s)\|_{L^2_x}^2\,dx\right)^{1/2}$$

$$+ \frac{C}{R}\|u_0\|_{L^2}^{3/2} t^{1/4}\left(\int_0^t \|\nabla v(s)\|_{L^2_x}^2\,ds\right)^{3/4}$$

$$\overset{(30)}{\leq} \frac{C}{R}\|u_0\|_{L^2}^2 t^{1/2} + \frac{C}{R}\|u_0\|_{L^2}^3 t^{1/4}.$$

Thus, $\frac{1}{2}\int_{\mathbb{R}^3}(|v(t)|^2 - |u_0|^2)\chi_R\,dx \leq \frac{C}{R}\|u_0\|_{L^2}^2 t^{1/2} + \frac{C}{R}\|u_0\|_{L^2}^3 t^{1/4}$.

Thus,

$$\int_{|x|>R} |v(t)|^2 dx \leq \int_{\mathbb{R}^3} |v(t)|^2 \chi_R\,dx$$

$$\leq \int_{\mathbb{R}^3} |u_0|^2 \chi_R\,dx + 2\left(\frac{C}{R}\|u_0\|_{L^2}^2 t^{1/2} + \frac{C}{R}\|u_0\|_{L^2}^3 t^{1/4}\right)$$

$$\leq \int_{|x|>\frac{R}{2}} |u_0|^2 dx + \frac{C}{R}\|u_0\|_{L^2}^2 t^{1/2} + \frac{C}{R}\|u_0\|_{L^2}^3 t^{1/4}. \qquad (59)$$

For each $\varepsilon > 0$, there exists $R > 0$ depending only on u_0 and t such that RHS(59) $< \varepsilon$. Thus,

$$\int_{|x|>R} |u^{(n')}(t)|^2 dx < \varepsilon \qquad \forall n'.$$

Proof of (ii) Take $\Psi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$. The way we show that the sequence $\left(\int_{\mathbb{R}^3} u^{(n)}(t)\Psi\,dx\right)_n$ is equicontinuous in $t \in (0,\infty)$ is exactly the way we derived (46). We rewrite the result:

$$\left|\int_{\mathbb{R}^3}(u^{(n)}(t_2) - u^{(n)}(t_1))\Psi\,dx\right| \leq \left(C\|\nabla\Psi\|_{L^2}\|u_0\|_{L^2} + C\|\Psi\|_{L^\infty}\|u_0\|_{L^2}^2\right)\sqrt{t_2 - t_1}, \qquad (60)$$

$$\forall n \in \mathbb{N}.$$

Proof of (iii)

Let A' be a countable dense subset of A. By Part (i) and the Cantor's diagonal method, there exists a subsequence $(u^{(n')})$ of $(u^{(n)})$ such that for every $t \in A'$, $(u^{(n')}(t))$ converges in $L^2(\mathbb{R}^3,\mathbb{R}^3)$. Take any $t_0 \in A$ and $\psi \in \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$. We show that the sequence $\left(\int_{\mathbb{R}^3} u^{(n')}(t_0)\psi\,dx\right)_{n'}$ converges. There exists a sequence (t_m) in A' such that $t_m \to t_0$. We have

$$\left|\int_{\mathbb{R}^3}\left(u^{(n')}(t_0)-u^{(k')}(t_0)\right)\psi\,dx\right| \leq \underbrace{\left|\int_{\mathbb{R}^3}\left(u^{(n')}(t_0)-u^{(n')}(t_m)\right)\psi\,dx\right|}_{\{1\}}$$
$$+\underbrace{\left|\int_{\mathbb{R}^3}\left(u^{(n')}(t_m)-u^{(k')}(t_m)\right)\psi\,dx\right|}_{\{2\}}$$
$$+\underbrace{\left|\int_{\mathbb{R}^3}\left(u^{(k')}(t_m)-u^{(k')}(t_0)\right)\psi\,dx\right|}_{\{3\}} \qquad (61)$$

Terms $\{1\}$ and $\{3\}$ can be estimated using (60). Thus,

$$|\{1\}|+|\{3\}| \leq \left(C\|\nabla\psi\|_{L^2}\|u_0\|_{L^2}+C\|\psi\|_{L^\infty}\|u_0\|_{L^2}^2\right)|t_m-t_0| \quad \forall m \in \mathbb{N}. \qquad (62)$$

For each $\varepsilon > 0$, we choose $m \in \mathbb{N}$ such that $RHS(62) < \frac{\varepsilon}{2}$. Because the sequence $(u^{(n')}(t_m))_{n'}$ converges in $L^2(\mathbb{R}^3,\mathbb{R}^3)$, the sequence $\left(\int_{\mathbb{R}^3} u^{(n')}(t_m)\psi\,dx\right)_{n'}$ converges, and thus is a Cauchy sequence. Thus, there exists $N \in \mathbb{N}$ such that

$$|\{2\}| < \frac{\varepsilon}{2} \quad \forall n', k' > N. \qquad (63)$$

Replacing (62) and (63) into (61), we get

$$\left| \int_{\mathbb{R}^3} \left(u^{(n')}(t_0) - u^{(k')}(t_0) \right) \Psi \, dx \right| < \varepsilon \quad \forall n', k' > N.$$

Thus, the sequence $\left(\int_{\mathbb{R}^3} u^{(n')}(t_0) \Psi \, dx \right)_{n'}$ converges.

Next, we show that $u^{(n')}(t_0) \rightharpoonup u(t_0)$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. Define a functional $T : \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}$, $T(\Psi) = \lim_{n' \to \infty} \int_{\mathbb{R}^3} u^{(n')}(t_0) \Psi \, dx$. Then T is well-defined and linear. Moreover,

$$|T(\Psi)| \leq \limsup_{n' \to \infty} \left| \int_{\mathbb{R}^3} u^{(n')}(t_0) \Psi \, dx \right| \leq \limsup_{n' \to \infty} \| u^{(n')}(t_0) \|_{L^2_x} \| \Psi \|_{L^2}$$

$$\overset{(33)}{\leq} \| u_0 \|_{L^2} \| \Psi \|_{L^2} \quad \forall \Psi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3).$$

This means T can extend to a linear continuous functional on $L^2(\mathbb{R}^3, \mathbb{R}^3)$. By Riesz's Representation theorem, there exists a function $u(t_0) \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ such that $T(\Psi) = \int_{\mathbb{R}^3} u(t_0) \Psi \, dx$ for all $\Psi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$. Thus, $u^{(n')}(t_0) \rightharpoonup u(t_0)$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$.

Next, we show that $u^{(n')}(t_0) \to u(t_0)$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. It suffices to show that $\limsup_{n' \to \infty} \| u^{(n')}(t_0) \|_{L^2_x} \leq \| u(t_0) \|_{L^2_x}$. Because the sequence $(u^{(n')}(t_0))$ is bounded in $L^2(\mathbb{R}^3, \mathbb{R}^3)$, there exists a subsequence $(u^{(n'')}(t_0))$ such that

$$\lim_{n'' \to \infty} \| u^{(n'')}(t_0) \|_{L^2_x} = \limsup_{n' \to \infty} \| u^{(n')}(t_0) \|_{L^2_x}.$$

By Part (i), $(u^{(n'')}(t_0))$ has a subsequence $(u^{(n''')}(t_0))$ that converges in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. Because $u^{(n'')}(t_0) \rightharpoonup u(t_0)$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$, $(u^{(n''')}(t_0))$ must converge to $u(t_0)$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. Thus,

$$\|u(t_0)\|_{L^2_x} = \lim_{n'''\to\infty} \|u^{(n''')}(t_0)\|_{L^2_x} = \lim_{n''\to\infty} \|u^{(n'')}(t_0)\|_{L^2_x} = \limsup_{n'\to\infty} \|u^{(n')}(t_0)\|_{L^2_x}.$$

This completes the proof. As a consequence of (33) and (34),

$$\int_{\mathbb{R}^3} |u(t)|^2 dx \leq \liminf_{n'\to\infty} \int_{\mathbb{R}^3} |u^{(n')}(t)|^2 dx \qquad \text{a.e. } t\in(0,\infty),$$

$$\int_0^t \int_{\mathbb{R}^3} |\nabla u(s)|^2 dx ds \leq \liminf_{n'\to\infty} \int_0^t \int_{\mathbb{R}^3} |\nabla u^{(n')}(s)|^2 dx ds \qquad \forall t\in(0,\infty).$$

Hence, u satisfies the energy inequality:

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(s)|^2 dx ds \leq \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx \qquad \text{a.e. } t\in(0,\infty). \quad (64)$$

3 Weak-strong uniqueness

Let $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, $\operatorname{div} u_0 = 0$ in sense of distribution, a be a mild solution on the maximal time interval $(0, T^*)$, and u be a Leray's weak solution obtained from the construction in the previous section (note that we didn't prove the uniqueness of u). We show that $u = a$ almost everywhere in $\mathbb{R}^3 \times (0,T^*)$ in two following cases:

(i) $u_0 \in (L^2 \cap L^\infty)(\mathbb{R}^3, \mathbb{R}^3)$ and a is the mild solution in the subcritical setting (see Part 1, Sections 1, 2, 3, 4).

(ii) $u_0 \in (L^2 \cap L^3)(\mathbb{R}^3, \mathbb{R}^3)$ and a is the mild solution in the critical setting (see Part 1, Sections 1, 2, 3).

Proof for case (i)

Replacing $u = v + a$ into (64), we get

$$\frac{1}{2}\int_{\mathbb{R}^3}\left(|v(t)|^2+2v(t)\cdot a(t)\right)dx+\int_0^t\int_{\mathbb{R}^3}\left(|\nabla v(s)|^2+2\nabla v(s)\cdot\nabla a(s)\right)dx\,ds$$

$$+\frac{1}{2}\int_{\mathbb{R}^3}|a(t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla a(s)|^2dx\,ds\leq\frac{1}{2}\int_{\mathbb{R}^3}|u_0|^2dx\quad \text{a.e. } t\in(0,\infty). \tag{65}$$

Because a is a mild solution on the interval $(0,T^*)$, it ~~suffices~~ satisfies the energy identity (see Eq. (49) Part 1):

$$\frac{1}{2}\int_{\mathbb{R}^3}|a(t)|^2dx+\int_0^t\int_{\mathbb{R}^3}|\nabla a(s)|^2dx\,ds=\frac{1}{2}\int_{\mathbb{R}^3}|u_0|^2dx\qquad\forall t\in(0,T^*).$$

Thus, (65) implies

$$\int_{\mathbb{R}^3}\left(\frac{1}{2}|v(t)|^2+v(t)\cdot a(t)\right)dx+\int_0^t\int_{\mathbb{R}^3}\left(|\nabla v(s)|^2+2\nabla v(s)\cdot\nabla a(s)\right)dx\,ds\leq 0\quad \text{a.e. } t\in(0,T^*). \tag{66}$$

Recall that by the definition of Leray's weak solutions on Page 4, we have

$$\int_{\mathbb{R}^3}u(t)\varphi(t)dx-\int_0^t\int_{\mathbb{R}^3}u(s)\partial_t\varphi(s)dx\,ds+\int_0^t\int_{\mathbb{R}^3}\nabla u(s)\cdot\nabla\varphi(s)\,dx\,ds$$

$$+\int_0^t\int_{\mathbb{R}^3}u_j(s)u_{i,j}(s)\varphi_i(s)dx\,ds=\int_{\mathbb{R}^3}u_0\varphi(0)dx\qquad \text{a.e. } t\in(0,\infty),$$

where $\varphi\in\mathcal{D}(\mathbb{R}^3\times\mathbb{R},\mathbb{R}^3)$, $\operatorname{div}\varphi(t)=0$ for all $t\geq 0$. We want to be able to take $\varphi=a$. To do so, we need to modify the derivation of the above equation in pages 18–21. We rewrite (27):

$$\partial_t u_\varepsilon-\Delta u_\varepsilon+((u_\varepsilon*\eta_\varepsilon)\cdot\nabla)u_\varepsilon+\nabla p_\varepsilon=0. \tag{67}$$

We know the following properties of u_ε, p_ε and a:

$$t^{m+\frac{\ell}{2}}\partial_t^\ell\partial_x^m u_\varepsilon\in L_t^\infty L_x^2(\mathbb{R}^3\times(0,T),\mathbb{R}^3)\qquad\forall T\in(0,\infty)\ \forall m,\ell\geq 0$$

(see (10)).

$p_\varepsilon(t) \in H^1(\mathbb{R}^3) \quad \forall t \in (0,\infty)$ (see page 15).

$t^m \partial_x^m a \in (L^\infty_{t,x} \cap L^\infty_t L^2_x)(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3) \quad \forall T_1 \in (0,T^*) \ \forall m, l \geq 0.$

(See (24) and (56) of Part 1)

Multiplying both sides of (67) by $a(t)$ and taking the integral over $\mathbb{R}^3$, we get

$$\int_{\mathbb{R}^3} (\partial_t u_\varepsilon) a(t) dx + \int_{\mathbb{R}^3} (\nabla u_\varepsilon \cdot \nabla a) dx + \int_{\mathbb{R}^3} [((u_\varepsilon * \eta_\varepsilon)\cdot\nabla)u_\varepsilon] a dx - \int_{\mathbb{R}^3} p_\varepsilon \underbrace{\nabla\cdot a(t)}_{=0} dx = 0$$

Now take the integral over $t \in [t_1, t_2] \subset (0,\infty)$:

$$\int_{\mathbb{R}^3}\int_{t_1}^{t_2} (\partial_t u_\varepsilon) a(t) dt dx + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla u_\varepsilon \cdot \nabla a \, dx dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} [((u_\varepsilon * \eta_\varepsilon)\cdot\nabla)u_\varepsilon] a \, dx dt = 0.$$

Thus,

$$\int_{\mathbb{R}^3} u_\varepsilon(t_2) a(t_2) dx - \underbrace{\int_{\mathbb{R}^3} u_\varepsilon(t_1) a(t_1) dx}_{\{1\}} - \underbrace{\int_{\mathbb{R}^3}\int_{t_1}^{t_2} u_\varepsilon(t) \partial_t a \, dt dx}_{\{2\}} + \underbrace{\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla u_\varepsilon(t)\cdot\nabla a(t) dx dt}_{\{3\}}$$

$$+ \underbrace{\int_{t_1}^{t_2}\int_{\mathbb{R}^3} [((u_\varepsilon * \eta_\varepsilon)\cdot\nabla) u_\varepsilon] a \, dx dt}_{\{4\}} = 0. \qquad (68)$$

We consider the limit of LHS(68) as $t_1 \to 0^+$. Because

$$\lim_{t_1\to 0^+} \| u_\varepsilon(t_1) - u_0 \|_{L^2_x} = 0 \qquad (\text{by } (11))$$

and $$\lim_{t_1\to 0^+} \| a(t_1) - u_0 \|_{L^2_x} = 0 \qquad (\text{by the argument on Page 20, Part 1}),$$

we have $\{1\} \longrightarrow \int_{\mathbb{R}^3} |u_0|^2 dx$ as $t_1 \to 0^+$. $\qquad$ (69)

Because a is a mild solution, it satisfies the differential equation

$$\partial_t a - \Delta a + (a\cdot\nabla)a + \nabla q = 0.$$

Thus, $$\{2\} = \int_{\mathbb{R}^3}\int_{t_1}^{t_2} u_\varepsilon(t)\big(\Delta a - (a\cdot\nabla)a - \nabla q\big)\,dt\,dx$$

$$= \int_{\mathbb{R}^3}\int_{t_1}^{t_2} u_\varepsilon(t)\Delta a(t)\,dt\,dx - \int_{\mathbb{R}^3}\int_{t_1}^{t_2} u_\varepsilon(t)\,(a\cdot\nabla)a\,dt\,dx + \int_{t_1}^{t_2}\int_{\mathbb{R}^3} q\,\underbrace{\nabla\cdot u_\varepsilon(t)}_{=0}\,dx\,dt$$

$$= -\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \nabla u_\varepsilon(t)\cdot\nabla a(t)\,dx\,dt - \int_{t_1}^{t_2}\int_{\mathbb{R}^3} u_\varepsilon(t)\,(a\cdot\nabla)a\,dx\,dt. \qquad (70)$$

Because u_ε and a satisfy the energy identity,

$$\int_0^\infty\int_{\mathbb{R}^3} |\nabla u_\varepsilon(t)||\nabla a(t)|\,dx\,dt \le \left(\int_0^\infty\int_{\mathbb{R}^3} |\nabla u_\varepsilon(t)|^2\,dx\,dt\right)^{1/2}\left(\int_0^\infty\int_{\mathbb{R}^3} |\nabla a(t)|^2\,dx\,dt\right)^{1/2} < \infty. \quad (71)$$

Also, $$\int_0^{T_1}\int_{\mathbb{R}^3} |u_\varepsilon(t)||\nabla a||a|\,dx\,dt \le \|a\|_{L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} \int_0^{T_1} \|u_\varepsilon(t)\|_{L^2_x}\|\nabla a(t)\|_{L^2_x}\,dt$$

$$\le \|u_0\|_{L^2}\|a\|_{L^\infty_{t,x}} \int_0^{T_1} \|\nabla a(t)\|_{L^2_x}\,dt$$

$$\le \|u_0\|_{L^2}\|a\|_{L^\infty_{t,x}} \sqrt{T_1}\left(\int_0^{T_1} \|\nabla a(t)\|_{L^2_x}^2\,dt\right)^{1/2}$$

$$\le C\|u_0\|_{L^2}^2\|a\|_{L^\infty_{t,x}}\sqrt{T_1}$$

$$< \infty.$$

Thus, as $t_1 \to 0^+$, (70) gives

$$\{2\} \to -\int_0^{t_2}\int_{\mathbb{R}^3} \nabla u_\varepsilon(t)\cdot\nabla a(t)\,dx\,dt - \int_0^{t_2}\int_{\mathbb{R}^3} u_\varepsilon(t)\,(a\cdot\nabla)a\,dx\,dt. \qquad (72)$$

Because of (71), $$\{3\} \to \int_0^{t_2}\int_{\mathbb{R}^3} \nabla u_\varepsilon(t)\cdot\nabla a(t)\,dx\,dt \quad \text{as } t_1 \to 0^+. \qquad (73)$$

We have
$$\int_0^{T_1}\int_{\mathbb{R}^3} |u_\varepsilon * \eta_\varepsilon| |\nabla u_\varepsilon| |a| \, dx dt \leq \int_0^{T_1} \|u_\varepsilon * \eta_\varepsilon\|_{L^\infty_x} \int_{\mathbb{R}^3} |\nabla u_\varepsilon| |a| \, dx dt$$
$$\leq \int_0^{T_1} \underbrace{\|u_\varepsilon(t)\|_{L^2_x}}_{\leq \|u_0\|_{L^2}} \underbrace{\|\eta_\varepsilon\|_{L^2_\varepsilon}}_{= C\varepsilon^{-3/2}} \left(\|\nabla u_\varepsilon(t)\|_{L^2_x} \underbrace{\|a(t)\|_{L^2_x}}_{\leq \|u_0\|_{L^2}} \right) dt$$
$$\leq C\varepsilon^{-3/2} \|u_0\|_{L^2}^2 \int_0^{T_1} \|\nabla u_\varepsilon(t)\|_{L^2_x} dt$$
$$\leq C\sqrt{T_1}\, \varepsilon^{-3/2} \|u_0\|_{L^2}^2 \left(\int_0^{T_1}\int_{\mathbb{R}^3} |\nabla u_\varepsilon(t)|^2 dx dt \right)^{1/2}$$
$$\leq C\sqrt{T_1}\, \varepsilon^{-3/2} \|u_0\|_{L^2}^3$$
$$< \infty.$$

Thus, as $t_1 \to 0^+$, $\{4\} \to \int_0^{t_2}\int_{\mathbb{R}^3} [((u_\varepsilon * \eta_\varepsilon)\cdot\nabla) u_\varepsilon] a \, dx dt.$ (74)

Thanks to (69), (72), (73), (74), we get the limit of (68) as $t_1 \to 0^+$:
$$\int_{\mathbb{R}^3} u_\varepsilon(t) a(t) dx - \int_{\mathbb{R}^3} |u_0|^2 dx + 2\int_0^t\int_{\mathbb{R}^3} \nabla u_\varepsilon(s)\cdot\nabla a(s) dx ds + \int_0^t\int_{\mathbb{R}^3} u_\varepsilon(s)[(a(s)\cdot\nabla)a(s)] dx ds$$
$$+ \int_0^t\int_{\mathbb{R}^3} [((u_\varepsilon * \eta_\varepsilon)\cdot\nabla) u_\varepsilon] a \, dx ds = 0 \quad \forall t \in (0, T^*). \quad (75)$$

Here we have replaced t_2 by t. ~~From~~ By Section $\boxed{1}$, we showed that there exists a sequence $(\varepsilon_n) \downarrow 0$ such that
$$u^{(n)}(t) \to u(t) \quad \text{in } L^2(\mathbb{R}^3, \mathbb{R}^3) \quad \text{a.e. } t \in (0, \infty),$$
$$\nabla u^{(n)} \rightharpoonup \nabla u \quad \text{in } L^2(\mathbb{R}^3 \times (0,\infty), \mathbb{R}^3).$$

From (75) we have
$$\underbrace{\int_{\mathbb{R}^3} u^{(n)}(t) a(t) dx}_{\{5\}} + \underbrace{2\int_0^t\int_{\mathbb{R}^3} \nabla u^{(n)}(s)\cdot\nabla a(s) dx ds}_{\{6\}} - \int_{\mathbb{R}^3} |u_0|^2 dx =$$

$$= -\underbrace{\int_0^t\int_{\mathbb{R}^3} u^{(n)}(s)\left[(a(s)\cdot\nabla)a(s)\right]dxds}_{\{7\}} - \underbrace{\int_0^t\int_{\mathbb{R}^3}\left[((u^{(n)}*\eta^{(n)})\cdot\nabla)u^{(n)}\right]a\,dxds}_{\{8\}} \quad \forall t\in(0,T^*) \tag{76}$$

As $n\to\infty$,

$$\{5\} \longrightarrow \int_{\mathbb{R}^3} u(t)a(t)dx \qquad \text{a.e. } t\in(0,T^*), \tag{77}$$

$$\{6\} \longrightarrow \int_0^t\int_{\mathbb{R}^3} \nabla u(s)\cdot\nabla a(s)dxds \qquad \forall t\in(0,T^*), \tag{78}$$

We have

$$\int_0^{T_1}\int_{\mathbb{R}^3} |\nabla a(t)||a(t)|\,dxds \leq \int_0^{T_1}\|\nabla a(t)\|_{L^2_x}\|a(t)\|_{L^2_x}dx$$

$$\leq \|u_0\|_{L^2}\sqrt{T_1}\left(\int_0^{T_1}\int_{\mathbb{R}^3}|\nabla a(t)|^2dxdt\right)^{1/2}$$

$$\leq C\|u_0\|^2_{L^2}\sqrt{T_1}.$$

Thus, $(a(s)\cdot\nabla)a(s)\in L^2(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$. By Lebesgue's Dominated Convergence theorem,

$$\{7\} \longrightarrow \int_0^t\int_{\mathbb{R}^3} u(s)\left[(a(s)\cdot\nabla)a(s)\right]dxds. \tag{79}$$

By (38), $u^{(n)}*\eta^{(n)} \longrightarrow u^{(n)}$ in $L^2(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$.

Because $a\in L^\infty_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$, $(\nabla u^{(n)})a \to (\nabla u)a$ in $L^2(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)$.

Thus, by the lemma stated on page 21, we have

$$\{8\} \longrightarrow \int_0^t\int_{\mathbb{R}^3}\left[(u\cdot\nabla)u\right]a\,dxds \qquad \forall t\in(0,T^*). \tag{80}$$

Thanks to (77)-(80), we can take the limit both sides of (76) as $n\to\infty$:

$$\int_{\mathbb{R}^3} u(t)a(t)dx + 2\int_0^t\int_{\mathbb{R}^3}\nabla u(s)\cdot\nabla a(s)dxds - \int_{\mathbb{R}^3}|u_0|^2dx$$

$$= -\int_0^t\int_{\mathbb{R}^3} u(s)\left[(a(s)\cdot\nabla)a(s)\right]dxds - \int_0^t\int_{\mathbb{R}^3}\left[(u\cdot\nabla)u\right]a\,dxds \quad \text{a.e. } t\in(0,T^*) \tag{81}$$

$$-RHS(81) = \int_0^t \int_{\mathbb{R}^3} u_i(s) a_j(s) a_{i,j}(s)\,dx\,ds + \int_0^t \int_{\mathbb{R}^3} u_i(s) u_{j,i}(s) a_j(s)\,dx\,ds$$

$$= \int_0^t \int_{\mathbb{R}^3} \big(v_i(s)+a_i(s)\big) a_j(s)\, a_{i,j}(s)\,dx\,ds + \int_0^t \int_{\mathbb{R}^3} \big(v_i(s)+a_i(s)\big)\big(v_{j,i}(s)+a_{j,i}(s)\big) a_j(s)\,dx\,ds$$

$$= \underbrace{\int_0^t \int_{\mathbb{R}^3} v_i a_j a_{i,j}\,dx\,ds}_{\{9\}} + \underbrace{\int_0^t \int_{\mathbb{R}^3} a_i a_j a_{i,j}\,dx\,ds}_{\{10\}} + \int_0^t \int_{\mathbb{R}^3} v_i v_{j,i} a_j\,dx\,ds$$

$$+ \underbrace{\int_0^t \int_{\mathbb{R}^3} v_i a_{j,i} a_j\,dx\,ds}_{\{11\}} + \underbrace{\int_0^t \int_{\mathbb{R}^3} v_{j,i} a_j a_i\,dx\,ds}_{\{12\}} + \underbrace{\int_0^t a_i a_j a_{j,i}\,dx\,ds}_{\{13\}}. \qquad (82)$$

We have $$\{10\} = \{13\} = \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} a_i \left(\frac{|a|^2}{2}\right)_{,i} dx\,ds = -\frac{1}{2}\int_0^t \int_{\mathbb{R}^3} \frac{|a|^2}{2} \underbrace{a_{i,i}}_{=0}\,dx\,ds = 0. \qquad (83)$$

$$\{9\} = -\int_0^t \int_{\mathbb{R}^3} (v_i a_j)_{,j}\, a_i\,dx\,ds = -\int_0^t \int_{\mathbb{R}^3} \big(v_{i,j} a_j + v_i \underbrace{a_{j,j}}_{=0}\big) a_i\,dx\,ds$$

$$= -\int_0^t \int_{\mathbb{R}^3} v_{i,j} a_j a_i\,dx\,ds = -\{12\}. \qquad (84)$$

Because $u^{(n)}(t) \to u(t)$ a.e. $t \in (0,\infty)$ and $\operatorname{div} u^{(n)}(t) \equiv 0$, we have $\operatorname{div} u(t) = 0$ in sense of distribution. Because $u(t) \in L^2_x$ for a.e. $t \in (0,\infty)$,

$$\int_{\mathbb{R}^3} u_i(t)\, \varphi_{,i}\,dx = 0 \quad \forall \varphi \in H^1(\mathbb{R}^3);$$

Because $a \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ and $\operatorname{div} a(t) \equiv 0$, we get

$$\int_{\mathbb{R}^3} v_i(t)\, \varphi_{,i}\,dx = 0 \qquad \forall \varphi \in H^1(\mathbb{R}^3).$$

By (61) in Part 1, $a(t) \in W^{1,4}(\mathbb{R}^3, \mathbb{R}^3)$. Hence $|a|^2 \in W^{1,2}(\mathbb{R}^3) = H^1(\mathbb{R}^3)$.

We can apply the above identity for $\varphi = \frac{|a(t)|^2}{2}$. Then

$$\int_{\mathbb{R}^3} v_i(t) \frac{|a(t)|^2}{2} dx = 0.$$

Thus, $\{11\} = 0$ for a.e. $t \in (0, T^*)$. (85)

Replacing (83), (84), (85) into (82), we get

$$-RHS(81) = \int_0^t \int_{\mathbb{R}^3} v_i v_{j,i} a_j \, dx \, ds.$$

Thus, (81) becomes

$$\int_{\mathbb{R}^3} u(t) a(t) dx + 2\int_0^t \int_{\mathbb{R}^3} \nabla u(s).\nabla a(s) \, dx\, ds - \int_{\mathbb{R}^3} |u_0|^2 dx = -\int_0^t \int_{\mathbb{R}^3} v_i v_{j,i} a_j \, dx\, ds.$$

Because $u = v + a$ and a satisfies the energy identity, we get

$$\int_{\mathbb{R}^3} v(t) a(t) dx + 2\int_0^t \int_{\mathbb{R}^3} \nabla v(s).\nabla a(s) \, dx\, ds = -\int_0^t \int_{\mathbb{R}^3} v_{j,i} v_i a_j \, dx\, ds.$$

Replacing this identity into (66), we get

$$\int_{\mathbb{R}^3} \frac{|v(t)|^2}{2} dx + \int_0^t \int_{\mathbb{R}^3} |\nabla v(s)|^2 dx\, ds \le \int_0^t \int_{\mathbb{R}^3} v_i v_{j,i} a_j \, dx\, ds$$

Replacing this identity into (66), we get

$$\int_{\mathbb{R}^3} \frac{|v(t)|^2}{2} dx + \int_0^t \int_{\mathbb{R}^3} |\nabla v(s)|^2 dx\, ds \le \int_0^t \int_{\mathbb{R}^3} v_i v_{j,i} a_j \, dx\, ds \quad \text{a.e. } t \in (0, T^*) \tag{86}$$

$$RHS(86) \le \int_0^t \int_{\mathbb{R}^3} |v(s)| |\nabla v(s)| |a(s)| \, dx\, ds$$

$$\overset{\text{Cauchy}}{\le} \int_0^t \int_{\mathbb{R}^3} \frac{\frac{1}{2}|a(s)|^2 |v(s)|^2 + 2|\nabla v(s)|^2}{2} dx\, ds$$

$$\le \frac{1}{2} \|a\|^2_{L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)} \int_0^t \int_{\mathbb{R}^3} \frac{|v(s)|^2}{2} dx\, ds + \int_0^t \int_{\mathbb{R}^3} |\nabla v(s)|^2 dx\, ds$$

where $T_1 \in (0,T^*)$ and $t \in (0,T_1)$. Thus, (86) implies

$$\int_{\mathbb{R}^3} \frac{|v(t)|^2}{2}\,dx \leq \frac{1}{2}\,\|a\|_{L^\infty_{t,x}} \int_0^t \int_{\mathbb{R}^3} \frac{|v(s)|^2}{2}\,dx\,ds \qquad \text{a.e. } t \in (0,T_1).$$

Put $A = \frac{1}{2}\|a\|_{L^\infty_{t,x}(\mathbb{R}^3 \times (0,T_1),\mathbb{R}^3)}$ and $f(t) = \int_{\mathbb{R}^3} \frac{|v(t)|^2}{2}\,dx$, we have

$$0 \leq f(t) \leq A \int_0^t f(s)\,ds \qquad \text{a.e. } t \in (0,T_1).$$

Also, $f \in L^\infty((0,T_1))$ because $u, a \in L^\infty_t L^2_x(\mathbb{R}^3 \times (0,T_1),\mathbb{R}^3)$. Because $u(t), a(t)$ converges to u_0 in $L^2(\mathbb{R}^3,\mathbb{R}^3)$ as $t \to 0^+$, we have $f(t) \to 0$ as $t \to 0^+$. More precisely, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $f(t) < \varepsilon$ for a.e. $t \in (0,\delta)$. Define

$$g(t) = e^{-At} \int_0^t f(s)\,ds.$$

Then $g \in C([0,T_1])$. Moreover,

$$\frac{dg}{dt} = \left(f(t) - A\int_0^t f(s)\,ds\right) g(t) \leq 0 \qquad \text{a.e. } t \in (0,T_1).$$

Thus, $g(t) \leq g(0) = 0 \quad \forall t \in [0,T_1]$. Thus $f(s) = 0$ a.e. $t \in (0,T_1)$. Hence, $v(x,t) = 0$ a.e. $(x,t) \in \mathbb{R}^3 \times (0,T_1)$. Because T_1 is any value in $(0,T^*)$, $v(x,t) = 0$ a.e. $(x,t) \in \mathbb{R}^3 \times (0,T^*)$.

Proof for case (ii)

By Page 20 of Part 1,

$$t^{l+\frac{m}{2}} \partial_t^l \partial_x^m a \in L^5_{t,x}(\mathbb{R}^3 \times (0,T_1),\mathbb{R}^3) \quad \forall T_1 \in (0,T^*),\ \forall m,l \geq 0. \qquad (87)$$

We now show that a satisfies

$$t^{l+\frac{m}{2}} \partial_t^l \partial_x^m a \in L^\infty_t L^2_x(\mathbb{R}^3 \times (0,T_1),\mathbb{R}^3) \quad \forall T_1 \in (0,T^*),\ \forall m,l \geq 0. \qquad (88)$$

As mentioned in the bottom of page 21, Part 1, this property will be proved if we can show $a \in L_t^\infty L_x^2(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)$ for all $T_1 \in (0,T^*)$. Recall that $a(t) = \Gamma(t) * u_0 + B(a,a)$. By Eq. (62), Homework # 1, Topics in PDE, Spring 2014, we have

$$\|B(a,a)\|_{L_t^\infty L_x^3(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} \leq C \|a\|_{L^5_{t,x}} \|\sqrt{t}\,\partial_x a\|_{L^5_{t,x}}.$$

Thus, $$\|a(t)\|_{L^3_x} \leq \underbrace{\|\Gamma(t)\|_{L^1_x}}_{=1} \|u_0\|_{L^3} + \|B(a,a)\|_{L^3_x} \leq \|u_0\|_{L^3} + C\|a\|_{L^5_{t,x}} \|\sqrt{t}\,\partial_x a\|_{L^5_{t,x}}$$
$\forall\, t \in (0,T_1)$.

Thus, $$\|a\|_{L_t^\infty L_x^3(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} \leq \|u_0\|_{L^3} + C\|a\|_{L^5_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)} \|\sqrt{t}\,\partial_x a\|_{L^5_{t,x}}. \tag{89}$$

We have

$$\begin{aligned}
\|B(a,a)\|_{L^2_x} &= \left\| \int_0^t K'(t-s) * (a(s)\otimes a(s))\,ds \right\|_{L^2_x} \\
&\leq \int_0^t \|K'(t-s)\|_{L^1_x} \|a(s)\otimes a(s)\|_{L^2_x}\,ds \\
&\leq \int_0^t \frac{C}{\sqrt{t-s}} \|a(s)\|^2_{L^4_x}\,ds \\
&\overset{\text{Holder}}{\leq} \int_0^t \frac{C}{\sqrt{t-s}} \|a(s)\|^{3/4}_{L^3_x} \|a(s)\|^{5/4}_{L^5_x}\,ds \\
&\overset{(89)}{\leq} C\|a\|_{L_t^\infty L_x^3} \int_0^t \frac{1}{\sqrt{t-s}} \|a(s)\|^{5/4}_{L^5_x}\,ds \\
&\overset{\text{Holder}}{\leq} C\|a\|_{L_t^\infty L_x^3} \left(\int_0^t \frac{1}{(t-s)^{2/3}}\,ds\right)^{3/4} \left(\int_0^t \|a(s)\|^5_{L^5_x}\,ds\right)^{1/4} \\
&\leq C\|a\|_{L_t^\infty L_x^3}\, T_1^{1/4}\, \|a\|^{5/4}_{L^5_{t,x}(\mathbb{R}^3\times(0,T_1),\mathbb{R}^3)}.
\end{aligned} \tag{90}$$

Thus, $\|a(t)\|_{L^2_x} \leq \|\Gamma(t)*u_0\|_{L^2_x} + \|B(a,a)\|_{L^2_x}$

$$\leq \|u_0\|_{L^2_x} + C\|a\|_{L^\infty_t L^3_x} T_1^{1/4} \|a\|^{5/4}_{L^5_{t,x}} \qquad \forall t \in (0,T_1).$$

Hence, $a \in L^\infty_t L^2_x(\mathbb{R}^3 \times (0,T_1), \mathbb{R}^3)$ for all $T_1 \in (0,T^*)$.

By (88), a satisfies the energy identity, the justification of which is the same as Section [4], Part 1.

$$\frac{1}{2}\int_{\mathbb{R}^3} |a(t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla a(s)|^2 dx\, ds = \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx .$$

Now we repeat the arguments in Case (i). There are a few points that need verifying differently, namely

$$\lim_{t\to 0^+} \|a(t) - u_0\|_{L^2_x} = 0, \qquad (91)$$

$$\int_0^{T_1} \int_{\mathbb{R}^3} |u_\varepsilon(t)||\nabla a||a|\, dx\, dt < \infty, \qquad (92)$$

$$\int_{\mathbb{R}^3} a_i \left(\frac{|a|^2}{2}\right)_{,i} dx = -\int_{\mathbb{R}^3} a_{i,i} \frac{|a|^2}{2} dx \qquad \text{a.e. } t \in (0,T_1), \qquad (93)$$

and how to achieve $v = 0$ a.e. $(x,t) \in \mathbb{R}^3 \times (0,T^*)$ from (86). We have

$$\|a(t) - u_0\|_{L^2_x} \leq \underbrace{\|\Gamma(t)*u_0 - u_0\|_{L^2_x}}_{\to 0 \text{ as } t\to 0^+} + \underbrace{\|B(a,a)\|_{L^2_x}}_{\substack{\to 0 \text{ as } t\to 0^+ \\ \text{because of (90)}}}$$

Thus, (91) is verified. Because $\displaystyle\int_0^\infty \int_{\mathbb{R}^3} |\nabla u_\varepsilon(t)|^2 dx\, dt < \infty$,

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon(t)|^2 dx < \infty \qquad \text{a.e. } t \in (0,\infty).$$

By the Sobolev's imbedding theorem, $\|u_\varepsilon(t)\|_{L^6_x} \leq C\|\nabla u_\varepsilon(t)\|_{L^2_x}$ a.e $t\in(0,\infty)$.

Thus,

$$\int_0^{T_1}\int_{\mathbb{R}^3} |u_\varepsilon(t)|\,|\nabla a|\,|a|\,dx\,dt \overset{\text{Holder}}{\leq} \int_0^{T_1} \|u_\varepsilon(t)\|_{L^6_x} \|\nabla a(t)\|_{L^2_x} \|a(t)\|_{L^3_x}\,dt$$

$$\leq C\|a\|_{L^\infty_t L^3_x} \int_0^{T_1} \|\nabla u_\varepsilon(t)\|_{L^2_x} \|\nabla a(t)\|_{L^2_x}\,dt$$

$$\overset{\text{Schwarz}}{\leq} C\|a\|_{L^\infty_t L^3_x} \left(\int_0^{T_1}\int_{\mathbb{R}^3} |\nabla u_\varepsilon(t)|^2 dx\,dt\right)^{1/2} \left(\int_0^{T_1}\int_{\mathbb{R}^3} |\nabla a(t)|^2 dx\,dt\right)^{1/2}$$

$$\leq C\|a\|_{L^\infty_t L^3_x} \|u_0\|^2_{L^2}.$$

Hence, (92) is verified. Because of (88) and the Sobolev's imbedding theorem, we have $\partial_x^m a(t) \in L^6_x$ for a.e. $t\in(0,T^*)$. Thus,

$$\partial_x^m a(t) \in L^2_x \cap L^6_x \quad \text{a.e. } t\in(0,T^*).$$

This implies $a(t), |a(t)|^2 \in H^1_x$ for a.e $t\in(0,T^*)$. Thus, (93) is verified.

Now suppose that we have (86):

$$\int_{\mathbb{R}^3} \frac{|v(t)|^2}{2} dx + \int_0^t\int_{\mathbb{R}^3} |\nabla v(s)|^2 dx\,ds \leq \int_0^t\int_{\mathbb{R}^3} v_i\, v_{j,i}\, a_j\, dx\,ds \quad \text{a.e. } t\in(0,T^*).$$

We need to show that $v=0$ a.e. $(x,t)\in\mathbb{R}^3\times(0,T^*)$. By the Sobolev's imbedding theorem, $\|v(t)\|_{L^6_x} \leq C\|\nabla v(t)\|_{L^2_x} \leq C\|\nabla u(t)\|_{L^2_x} + C\|\nabla a(t)\|_{L^2_x} < \infty$ for a.e. $t\in(0,T^*)$.

$$\text{RHS}(86) \leq \int_0^t\int_{\mathbb{R}^3} |v(s)|\,|\nabla v(s)|\,|a(s)|\,dx\,ds$$

$$= \int_0^t\int_{\mathbb{R}^3} \underbrace{|a(s)|}_{\in L^5_x}\, \underbrace{|v(s)|^{\frac{2}{5}}}_{\in L^5_x}\, \underbrace{|v(s)|^{\frac{3}{5}}}_{\in L^{10}_x}\, \underbrace{|\nabla v(s)|}_{\in L^2_x}\,dx\,ds$$

$$\overset{\text{Hölder}}{\leq} \int_0^t \|a(s)\|_{L^5_x} \|v(s)\|_{L^2_x}^{2/5} \|v(s)\|_{L^6_x}^{3/5} \|\nabla v(s)\|_{L^2_x} \, ds$$

$$\leq C \int_0^t \underbrace{\|a(s)\|_{L^5_x} \|v(s)\|_{L^2_x}^{2/5}}_{A} \underbrace{\|\nabla v(s)\|_{L^2_x}^{8/5}}_{B} \, ds$$

Applying Young's inequality $AB \leq CA^5 + B^{5/4}$, we get

$$RHS(86) \leq C \int_0^t \|a(s)\|_{L^5_x}^5 \|v(s)\|_{L^2_x}^2 \, ds + \int_0^t \|\nabla v(s)\|_{L^2_x}^2 \, ds.$$

Thus, (86) implies

$$\int_{\mathbb{R}^3} \frac{|v(t)|^2}{2} dx \leq C \int_0^t \|a(s)\|_{L^5_x}^5 \|v(s)\|_{L^2_x}^2 \, ds.$$

Put $b(s) = \|a(s)\|_{L^5_x}^5 \in L^1((0,T_1))$ and $f(s) = \|v(s)\|_{L^2_x}^2 \in L^\infty((0,T_1))$. We get

$$0 \leq f(t) \leq C \int_0^t b(s) f(s) \, ds.$$

By Gronwall's inequality, $f = 0$ a.e. in $(0,T_1)$. Thus, $v = 0$ a.e $(x,t) \in \mathbb{R}^3_x \times (0,T_1)$.

Because T_1 is arbitrary in $(0,T^*)$, we have $v = 0$ a.e. $(x,t) \in \mathbb{R}^3 \times (0,T^*)$.

Comments: We have showed in this section that if u is a Leray's weak solution obtained from the construction in Section $\boxed{2}$ then it coincides the mild solution (in either subcritical or critical setting) on the maximal time-interval where the mild solution exists. The key of the proof is to show that one can substitute $\varphi = a$ into part (ii) of Section $\boxed{1}$ (definition of Leray's weak solution). Other than this, there is no need to restrict u to be obtained from the construction in Section $\boxed{2}$. In other words, if we modify the definition of Leray's weak solution by enlarging the space $\mathcal{N}$ in Part (ii) of Section $\boxed{1}$ so that it includes the mild solution a then every Leray's weak solution, not necessarily obtained from the construction in Section $\boxed{2}$, coincides with the mild solution on the maximal time-interval where the mild solution exists.

$\boxed{4}$ The set of singular times

Let $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$, $\operatorname{div} u_0 = 0$ in sense of distribution, and u be a Leray's weak solution obtained from the construction in Section $\boxed{2}$. A time-interval (t_1, t_2) is called an interval of regularity if it is a maximal open interval on which u coincides a regular solution. Let S be the union of all intervals of regularity. We show that $(0,\infty)\setminus S$ is ~~countable~~ at most countable and is bounded.

Recall from Section $\boxed{2}$ that $A = \{t \in (0,\infty) : \liminf_{n\to\infty} \|\nabla u^{(n)}(t)\|_{L^2_x} < \infty\}$. The set $(0,\infty)\setminus A$ is of measure zero. Because u satisfies the energy inequality, the set $S' = \{t \in A : \|\nabla u(t)\|_{L^2_x} < \infty\}$ satisfies $(0,\infty)\setminus S'$ is of measure zero. Take $t_0 \in S'$ arbitrarily. Then $\tilde{u}_0 = u(t_0) \in H^1(\mathbb{R}^3, \mathbb{R}^3)$. Moreover, $\operatorname{div} \tilde{u}_0 = 0$ in sense of distribution because $\tilde{u}_0 = \lim_{n\to\infty} u^{(n)}$ in $L^2(\mathbb{R}^3, \mathbb{R}^3)$ and $\operatorname{div} u^{(n)} = 0$. We show that the function $\tilde{u}(x,t) = u(x, t+t_0)$ is a Leray's solution to the Navier-Stokes equation

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + (\tilde{u}.\nabla)\tilde{u} + \nabla \tilde{p} = 0, \\ \operatorname{div} \tilde{u} = 0, \\ \tilde{u}(0) = \tilde{u}_0, \end{cases} \qquad (I)$$

where $\tilde{p}(x,t) = p(x, t+t_0)$. The conditions (i) and (ii) in the definition of

Leray's weak solutions are automatically satisfied. We only need to check (iii), which say $\lim_{t\to t_0^+} \|u(t)-u(t_0)\|_{L^2_x} = 0$. First, we show $u(t) \rightharpoonup u(t_0)$ in $L^2(\mathbb{R}^3,\mathbb{R}^3)$ as $t\to t_0^+$. For each $\psi\in\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$, by (46) we have

$$\int_{\mathbb{R}^3} (u^{(n)}(t_2)-u^{(n)}(t_1))\psi\,dx \leqslant \left(C\|\nabla(\mathbb{P}\psi)\|_{L^2}\|u_0\|_{L^2} + C\|\mathbb{P}\psi\|_{L^\infty}\|u_0\|^2_{L^2}\right)\sqrt{t_2-t_1}$$
$$\forall n\in\mathbb{N},\ \forall\, 0<t_1<t_2<\infty,$$

where $\mathbb{P}\psi$ is the divergence-free component of ψ. In case $t_1,t_2\in A$, we have $u^{(n)}(t_2)\rightharpoonup u(t_2)$ and $u^{(n)}(t_1)\rightharpoonup u(t_1)$. Thus,

$$\int_{\mathbb{R}^3} (u(t)-u(t_0))\psi\,dx \leq \left(C\|\nabla(\mathbb{P}\psi)\|_{L^2}\|u_0\|_{L^2} + C\|\mathbb{P}\psi\|_{L^\infty}\|u_0\|^2_{L^2}\right)\sqrt{t-t_0}$$
$$\forall t\in A,\ t>t_0.$$

Hence, $u(t)\rightharpoonup u(t_0)$ as $t\to t_0^+$. Because each $u^{(n)}$ satisfies the energy identity, we have $\|u^{(n)}(t)\|_{L^2_x}\leq \|u^{(n)}(t_0)\|_{L^2_x}$ if $t>t_0$. Thus,

$$\|u(t_0)\|_{L^2_x} = \lim_{n\to\infty}\|u^{(n)}(t_0)\|_{L^2_x} \geqslant \lim_{n\to\infty}\|u^{(n)}(t)\|_{L^2_x} = \|u(t)\|_{L^2_x} \quad \forall t\geqslant t_0.$$

Thus, $\|u(t_0)\|_{L^2_x} \geqslant \limsup_{t\to t_0^+}\|u(t)\|_{L^2_x}$. Hence $u(t)\to u(t_0)$ in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ as $t\to t_0^+$.

We have showed that $\tilde{u}$ is a Leray's weak solution to Problem (I). Because $\tilde{u}_0\in H^1(\mathbb{R}^3,\mathbb{R}^3)$, $\tilde{u}_0\in (L^2\cap L^6)(\mathbb{R}^3,\mathbb{R}^3)$. By Section $\boxed{1}$, Part 1 - Mild solutions, Problem (I) has a mild solution in the critical setting on a short time-interval $(0,\tau_{t_0})$. By Section $\boxed{3}$, $\tilde{u}$ coincides this mild solution on the interval $(0,\tau_{t_0})$. Thus, the time-interval $(t_0, t_0+\tau_{t_0})$ is contained in an interval of regularity. Thus,

$$\bigcup_{t\in S'} \underbrace{(t, t+\tau_t)}_{I_t} \subset S.$$

For any open interval (a,b) and number $\lambda > 0$, we denote $\lambda(a,b)$ to be an open interval centered at $(a+b)/2$ with length $\lambda(b-a)$. We have

$$S' = \bigcup_{t\in S'} t \subseteq \bigcap_{\lambda>1} \Big(\bigcup_{t\in S'} \lambda I_t\Big).$$

Thus,

$$\underbrace{(0,\infty)\setminus S'}_{\text{measure zero}} \supset \bigcup_{\lambda>1} \Big[(0,\infty)\setminus \Big(\bigcup_{t\in S'} \lambda I_t\Big)\Big].$$

Hence, $(0,\infty)\setminus\Big(\bigcup_{t\in S'}\lambda I_t\Big)$ is of measure zero for all $\lambda > 1$. We have

$$|(0,\infty)\setminus S| \leq \Big|(0,\infty)\setminus\Big(\bigcup_{t\in S'} I_t\Big)\Big| = \lim_{\lambda\to 1^+} \Big|(0,\infty)\setminus\Big(\bigcup_{t\in S'}\lambda I_t\Big)\Big| = 0.$$

Thus, $(0,\infty)\setminus S$ is of measure zero.

By definition, two intervals of regularity are either the same or disjoint. Thus, S is a union of at most countably many intervals of regularity $(t_1, t_1'), (t_2, t_2'), (t_3, t_3'), \ldots$ Since $(0,\infty)\setminus S$ is of measure zero, we must have $t_1' = t_2$, $t_2' = t_3, \ldots$ Thus, S is the union of the intervals $(T_0, T_1), (T_1, T_2), (T_2, T_3), \ldots$ where $T_0 = 0$. This implies that $(0,\infty)\setminus S$ is at most countable.

If $u(t_0) \in (L^2\cap L^3)(\mathbb{R}^3, \mathbb{R}^3)$ then by Theorem 1, Kato's paper (1984), the mild solution in critical setting with initial condition $u(t_0)$, which coincides u, satisfies $\sqrt{t-t_0}\, u(t) \in L^\infty_{t,x}(\mathbb{R}^3\times(t_0, t_0+\tau_{t_0}), \mathbb{R}^3)$.

Thus, $u(t) \in L^\infty_x$ for $t \in (t_0, t_0 + \tau_{t_0})$. Thus, $u(t) \in L^\infty_x$ for a.e. $t \in (0,\infty)$.

For each finite interval (T_i, T_{i+1}), there exists $T_i' \in (T_i, \frac{T_i + T_{i+1}}{2})$ such that $u(T_i') \in H^1_x \cap L^\infty_x$. Because the mild solution exists on the finite interval (T_i', T_{i+1}) and blows up near T_{i+1}, by Section 6 of Part 1 - Mild Solutions we have

$$\|\nabla u(t)\|^2_{L^2_x} \geq \frac{C}{(T_{i+1} - t)^{1/2}} \qquad \forall t \in (T_i', T_{i+1}).$$

Thus, $$\int_{T_i'}^{T_{i+1}} \|\nabla u(t)\|^2_{L^2_x}\, dt \geq \int_{T_i'}^{T_{i+1}} \frac{C}{(T_{i+1}-t)^{1/2}}\, dt = C\sqrt{T_{i+1} - T_i'} \geq C\sqrt{T_{i+1} - T_i}.$$

Thus, $$\sum_i \sqrt{T_{i+1} - T_i} \leq C \sum_i \int_{T_i'}^{T_{i+1}} \|\nabla u(t)\|^2_{L^2_x}\, dt \leq \int_0^\infty \|\nabla u(t)\|^2_{L^2_x}\, dt \leq C\|u_0\|^2_{L^2}.$$

Thus, $$\sup\{T_i \mid T_{i+1} < \infty\} = \sum_i (T_{i+1} - T_i) \leq \Big(\sum_i \sqrt{T_{i+1} - T_i}\Big)^2 \leq C\|u_0\|^4_{L^2}.$$

Therefore, all singular times are before $C\|u_0\|^4_{L^2}$. This means $(0,\infty) \setminus S$ is bounded in $(0,\infty)$.

$\boxed{5}$ Asymptotic behavior as time goes to infinity

Let $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ in sense of distribution, and u be a Leray's weak solution obtained from the construction in Section $\boxed{2}$. By Section $\boxed{4}$, the set of singular times is bounded. Thus, there exists $t_0 \in (0,\infty)$ such that $u(t_0) \in (H^1 \cap L^\infty)(\mathbb{R}^3, \mathbb{R}^3)$ and that u coincides the mild solution on the time interval (t_0, ∞). Thus, the asymptotic behavior of $u(t)$ as $t\to\infty$ is determined by the asymptotic behavior of the mild solution as $t\to\infty$. Thus, the problem reduces to the following:

Let $u_0 \in (H^1\cap L^\infty)(\mathbb{R}^3, \mathbb{R}^3)$, $\operatorname{div} u_0 = 0$ in sense of distribution, and u be a global-in-time mild solution to the problem (NSE)

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0, \\ \operatorname{div} u = 0 \\ u(0) = u_0. \end{cases}$$

What are the behaviors of $u(t)$ as $t\to\infty$?

Put $V(t) = \|u(t)\|_{L^\infty_x}$, $W(t) = \|u(t)\|_{L^2_x}$ and $J(t) = \|\nabla u(t)\|_{L^2_x}$ for $t\geq 0$.
We show that

$$\text{(i)}\quad J(t) \leq \frac{C\|u_0\|_{L^2}}{t^{1/2}} \quad \text{for all } t \geq C\|u_0\|_{L^2}^4.$$

$$\text{(ii)}\quad V(t) \leq \frac{C\|u_0\|_{L^2}}{t^{3/4}} \quad \text{for all } t > C\|u_0\|_{L^2}^4.$$

Proof of (i)

By Eq. (79), page 36 in Part 1 - Mild Solutions, we have

$$V(t) \leq \frac{CJ(0)}{t^{1/4}} \qquad \forall t\in(0,\tau] \tag{94}$$

where $\tau = CJ(0)^{-4}$. We have

$$u(t) = \Gamma(t) * u_0 + \int_0^t K'(t-s) * (u(s)\otimes u(s))\, ds.$$

Thus, $$\nabla u(t) \sim \Gamma(t) * \nabla u_0 + \int_0^t K'(t-s) * (u(s)\otimes \nabla u(s))\, ds$$

Thus, $$\|\nabla u(t)\|_{L^2_x} \leq \|\Gamma(t)\|_{L^1_x}\|\nabla u_0\|_{L^2} + \int_0^t \|K'(t-s)\|_{L^1_x}\|u(s)\otimes\nabla u(s)\|_{L^2_x}\, ds$$

$$\leq \|\nabla u_0\|_{L^2} + \int_0^t \frac{C}{\sqrt{t-s}} \|u(s)\|_{L^\infty_x} \|\nabla u(s)\|_{L^2_x} ds.$$

Hence, $$J(t) \leq J(0) + \int_0^t \frac{C}{\sqrt{t-s}} V(s) J(s)\, ds$$

$$\overset{(94)}{\leq} J(0) + \int_0^t \frac{C J(0)}{(t-s)^{1/2} s^{1/4}} J(s)\, ds \qquad \forall t \in (0,\tau]. \qquad (95)$$

Suppose that there exists a continuous function $\Psi : [0,\infty) \to \mathbb{R}$ such that

$$\Psi(t) \geq J(0) + \int_0^t \frac{C J(0)}{(t-s)^{1/2} s^{1/4}} \Psi(s)\, ds \qquad \forall t \in (0,\tau] \qquad (96)$$

and $\Psi(0) > J(0)$. Then $J(t) \leq \Psi(t)$ for all $t \in (0,\tau]$.

Choose $\Psi(t) \equiv (1+A) J(0)$ where $A > 0$ is a number to be determined.

Then (96) becomes $$A \geq \int_0^t \frac{C(1+A) J(0)}{(t-s)^{1/2} s^{1/4}} ds,$$

which is equivalent to $$\int_0^t \frac{ds}{(t-s)^{1/2} s^{1/4}} \leq \frac{CA}{1+A} \frac{1}{J(0)}. \qquad (97)$$

$$LHS(97) \leq \int_0^{t/2} \frac{C\, ds}{t^{1/2} s^{1/4}} + \int_{t/2}^t \frac{C\, ds}{(t-s)^{1/2} t^{1/4}} = C t^{1/4}.$$

Take any $A > 0$, for example $A = 1$. Then (97) is satisfied if $C t^{1/4} \leq \frac{1}{J(0)}$, which is equivalent to $t \leq C J(0)^{-4} = \tau$. Thus,

$$J(t) \leq \Psi(t) = C J(0) \qquad \forall t \in (0,\tau].$$

In other words, if $t \leq \tau$ then $J(t) \leq C J(0)$. In other words, if $J(0) \leq t^{-1/4}$ then $J(0) \geq C J(t)$. Thus,

$$J(0) \geq \min\left\{ \frac{1}{t^{1/4}}, C J(t) \right\} \qquad \forall t > 0.$$

This property can be generalized as $J(s) \geq \min\left\{\frac{1}{t^{1/4}}, CJ(t)\right\}$ $\forall t > s$.

Integrating the square of both sides over $s \in (0,t)$, we get

$$\int_0^t J(s)^2 ds \geq \int_0^t \min\left\{\frac{1}{t^{1/2}}, CJ(t)^2\right\} ds.$$

Because u satisfies the energy inequality,

$$\int_0^t \min\left\{\cancel{\frac{1}{(1+s)^{1/2}}}\ \frac{1}{t^{1/2}}, CJ(t)^2\right\} ds \leq C\|u_0\|_{L^2}^2. \qquad (98)$$

We have $\frac{1}{t^{1/2}} \geq CJ(t)^2 \Leftrightarrow J(t) \leq \frac{C}{t^{1/4}}$.

If $J(t) \leq \frac{C}{t^{1/4}}$ then (98) becomes $\int_0^t CJ(t)^2 ds \leq C\|u_0\|_{L^2}^2$, which implies $J(t) \leq \frac{C\|u_0\|_{L^2}}{t^{1/2}}$.

If $J(t) \geq \frac{C}{t^{1/4}}$ then (98) becomes $\int_0^t \frac{ds}{t^{1/2}} \leq C\|u_0\|_{L^2}^2$, which is equivalent to $t \leq C\|u_0\|_{L^2}^4$. Therefore, if $t > C\|u_0\|_{L^2}^4$ then $J(t) \leq \frac{C\|u_0\|_{L^2}}{t^{1/2}}$.

Proof of (ii)

Eq. (94) can be generalized as $V(t) \leq \frac{CJ(s)}{(t-s)^{1/4}}$ $\forall\, 0 < t-s < \tau = CJ(s)^{-4}$.

By Part (i), if $s > C\|u_0\|_{L^2}^4$ then $J(s) \leq \frac{C\|u_0\|_{L^2}}{s^{1/2}}$, which implies

$$J(s)^{-4} \geq C\|u_0\|_{L^2}^{-4} s^2.$$

Thus, the condition $0 \leq t-s < CJ(s)^{-4}$ is satisfied if $0 < t-s < C\|u_0\|_{L^2}^{-4} s^2$.

The latter is satisfied if $C\|u_0\|_{L^2}^2\sqrt{t} < s < t$. Thus,

$$V(t) \le \frac{C\,J(s)}{(t-s)^{1/4}} \quad \forall\, C\|u_0\|_{L^2}^2\sqrt{t} < s < t.$$

Integrating both sides over $s \in (C\|u_0\|_{L^2}^2\sqrt{t}, t)$, we get

$$(t - C\|u_0\|_{L^2}^2\sqrt{t})\,V(t) \le \int_{C\|u_0\|_{L^2}^2\sqrt{t}}^{t} \frac{C J(s)}{(t-s)^{1/4}}\,ds \le \int_0^t \frac{C J(s)}{(t-s)^{1/4}}\,ds$$

$$\overset{\text{Schwarz}}{\le} C\left(\int_0^t J(s)^2\,ds\right)^{1/2}\left(\int_0^t \frac{ds}{(t-s)^{1/2}}\right)^{1/2}$$

$$\le C\|u_0\|_{L^2}\, t^{1/4}.$$

Thus, $$V(t) \le \frac{C\|u_0\|_{L^2}}{t^{1/4}\left(\sqrt{t} - C\|u_0\|_{L^2}^2\right)}.$$

If $t > C\|u_0\|_{L^2}^4$ then $$V(t) \le \frac{C\|u_0\|_{L^2}}{t^{1/4}\left(\sqrt{t} - \frac{\sqrt{t}}{2}\right)} = \frac{C\|u_0\|_{L^2}}{t^{3/4}}.$$