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Math 8651: Theory of Probability
Problems for Final Exams

(1) Let ν be an outer measure on a set Ω and Σ be the collection of ν-sets. Let $A, B \in \Sigma$ with $AB = \phi$. We show that $\nu(X(A\cup B)) = \nu(XA) + \nu(XB)$ for all $X \subset \Omega$.

Take $X \subset \Omega$. Because $A \in \Sigma$,

$$\nu(X(A\cup B)) = \nu(X(A\cup B)A) + \nu(X(A\cup B)A^c) = \nu(XA) + \nu(XBA^c) \quad (1)$$

Because $AB = \phi$, $B \subset A^c$. Then $BA^c = B$. Then (1) becomes $\nu(X(A\cup B)) = \nu(XA) + \nu(XB)$. As a consequence, for $X = \Omega$ we get the identity $\nu(A\cup B) = \nu(A) + \nu(B)$.

(2) Let $F: \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function. For $-\infty \le a \le b \le \infty$, we denote $(a,b] = \{x \in \mathbb{R} : a < x \le b\}$. Let $\mathcal{E}$ be the collection of finite unions of disjoint intervals of type $(a,b]$. Define a map $R: \mathcal{E} \to \mathbb{R}$ as follows.

$$R((a,b]) = F(b) - F(a) \text{ with the convention } F(\infty) = \lim_{x\to\infty} F(x),$$
$$F(-\infty) = \lim_{x\to-\infty} F(x),$$

$$R\left(\bigcup_{i=1}^{n}(a_i,b_i]\right) = \sum_{i=1}^{n} R((a_i,b_i]) \text{ where } (a_1,b_1], \dots, (a_n,b_n] \text{ are pairwise disjoint.}$$

First, we show that the map R is well-defined. That is to show that for each $A \in \mathcal{E}$, the definition of $R(A)$ does not depend on the way A is represented as a finite union of disjoint intervals of type $(a, b]$. Suppose

$$A = \bigcup_{i=1}^{n} (a_i, b_i] = \bigcup_{j=1}^{m} (c_j, d_j], \qquad (1)$$

where $(a_1, b_1], \ldots, (a_n, b_n]$ are pairwise disjoint, and $(c_1, d_1], \ldots, (c_m, d_m]$ are pairwise disjoint. We show that $\sum_{i=1}^{n} R((a_i, b_i]) = \sum_{j=1}^{m} R((c_j, d_j])$.

This is equivalent to showing

$$\sum_{i=1}^{n} (F(b_i) - F(a_i)) = \sum_{j=1}^{m} (F(d_j) - F(c_j)).$$

We can assume n is the smallest number for which a presentation $A = \bigcup_{i=1}^{n} (a_i, b_i]$ is possible. Because $(a_i, b_i] \cap (a_{i+1}, b_{i+1}] = \phi$, $b_i \leq a_{i+1}$.

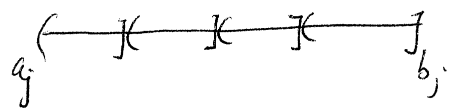
Suppose $b_i = a_{i+1}$ for some $1 \leq i \leq n-1$. Then $(a_i, b_i] \cup (a_{i+1}, b_{i+1}] = (a_i, b_{i+1}]$. Thus, the number of disjoint intervals of form $(a, b]$ with union equal to A is reduced by 1. This contradicts the minimality of n. Hence, $b_i < a_{i+1}$ for all $1 \leq i \leq n-1$.

Then $(a_1, b_1], \ldots, (a_n, b_n]$ are connected components of A. Because each $(c_i, d_i]$ is connected, it is contained in one of the intervals $(a_1, b_1], \ldots, (a_n, b_n]$. Suppose

$$(c_i, d_i] \subset (a_1, b_1] \quad \forall m_0 = 0 < i \leq m_1,$$
$$(c_i, d_i] \subset (a_2, b_2] \quad \forall m_1 < i \leq m_2,$$
$$\vdots$$
$$(c_i, d_i) \subset (a_n, b_n] \quad \forall m_{n-1} < i \leq m_n = m.$$

Then $$A = \bigcup_{i=1}^{m} (c_i, d_i] = \bigcup_{j=1}^{n} \underbrace{\bigcup_{m_{j-1} < i \le m_j} (c_i, d_i]}_{\subset (a_j, b_j]} \subset \bigcup_{j=1}^{n} (a_j, b_j] = A.$$

Thus, $$\bigcup_{m_{j-1} < i \le m_j} (c_i, d_i] = (a_j, b_j] \quad \forall 1 \le j \le n. \qquad (2)$$



Taking the infimum and supremum of both sides of (2), we get $c_{m_{j-1}+1} = a_j$ and $d_{m_j} = b_j$. Because LHS(2) is a connected set, it is necessary that $d_i = c_{i+1}$ for all $m_{j-1} < i < m_j$. Then
$$\sum_{m_{j-1} < i \le m_j} (F(d_i) - F(c_i)) = F(d_{m_j}) - F(c_{m_{j-1}+1}) = F(b_j) - F(a_j).$$

Summing both sides over $1 \le j \le n$, we get
$$\sum_{j=1}^{n} (F(b_j) - F(a_j)) = \sum_{j=1}^{n} \sum_{m_{j-1} < i \le m_j} (F(d_i) - F(c_i)) = \sum_{i=1}^{m} (F(d_i) - F(c_i)).$$

Next, we show that $\mathcal{E}$ is closed under finite unions. That is to show $A \cup B \in \mathcal{E}$ for all $A, B \in \mathcal{E}$. Write $A = \bigcup_{i=1}^{n} (a_i, b_i]$ and $B = \bigcup_{i=1}^{m} (c_i, d_i]$. We relabel the intervals $(a_1, b_1], \dots, (a_n, b_n], (c_1, d_1], \dots, (c_m, d_m]$ as $(\alpha_1, \beta_1], \dots, (\alpha_{m+n}, \beta_{m+n}]$. Then $$A \cup B = \bigcup_{i=1}^{m+n} (\alpha_i, \beta_i].$$

Let I be a connected component of $A \cup B$. Put $\alpha = \inf I$ and $\beta = \sup I$. There exists a sequence (y_k) in I that converges to β. Because each y_k belongs to one of the intervals $(\alpha_1, \beta_1], \dots, (\alpha_{m+n}, \beta_{m+n}]$, there is at least one interval, called

$(\alpha_e, \beta_e]$, that contains infinitely many terms of (y_k). Thus, (y_k) has a subsequence contained in $(\alpha_e, \beta_e]$. Its limit is, therefore, less than or equal to β_e. This means $\beta \le \beta_e$. On the other hand, because $(\alpha_e, \beta_e]$ is connected and $(\alpha_e, \beta_e] \cap I \neq \phi$, $(\alpha_e, \beta_e] \subset I$. Thus, $\beta_e \le \sup I = \beta$. We get $\beta = \beta_e \in I$. Suppose by contradiction that $\alpha \in I$. Then α belongs to some interval $(\alpha_s, \beta_s]$. Because $I \cap (\alpha_s, \beta_s] = \phi$ and $(\alpha_s, \beta_s]$ is connected, $(\alpha_s, \beta_s] \subset I$. Then $\frac{\alpha + \alpha_s}{2} \in I$. However, $\frac{\alpha + \alpha_s}{2} < \frac{\alpha + \alpha}{2} = \alpha = \inf I$. This is a contradiction. Thus, $\alpha \notin I$. We get $I = (\alpha, \beta]$. Because $\beta \in \{\beta_1, \dots, \beta_{m+n}\}$ and the connected components of $A \cup B$ are pairwise disjoint, $A \cup B$ has only finitely many connected components. Thus, $A \cup B$ is a finite union of disjoint intervals of type $(\alpha, \beta]$. In otherwords, $A \cup B \subset \mathcal{E}$.

Next, we show that R is additive. Let $A, B \in \mathcal{E}$, $AB = \phi$. Write

$$A = \bigcup_{i=1}^{n} (a_i, b_i] \quad \text{and} \quad B = \bigcup_{j=1}^{m} (c_j, d_j],$$

where $(a_i, b_i]$'s are pairwise disjoint and $(c_j, d_j]$'s are pairwise disjoint. Because $A \cap B = \phi$, $(a_i, b_i] \cap (c_j, d_j] = \phi$. Thus,

$$A \cup B = \left(\bigcup_{i=1}^{n} (a_i, b_i] \right) \cup \left(\bigcup_{j=1}^{m} (c_j, d_j] \right)$$

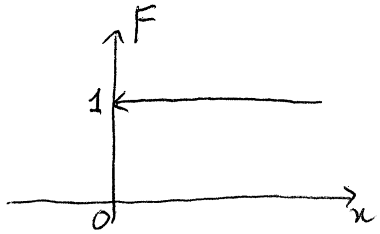
is a union of disjoint intervals of type $(a, b]$. By the definition of R,

$$R(A \cup B) = \sum_{i=1}^{n} (F(b_i) - F(a_i)) + \sum_{j=1}^{m} (F(d_j) - F(c_j)) = R(A) + R(B).$$

We have showed that R is additive.

Next, we show that R may not be σ-additive in case F is not continuous.

Take $F(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$



This function is nondecreasing and left continuous. Also, $F(\infty) = \lim_{x\to\infty} F(x) = 1$. We have

$$(0,\infty] = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right] \cup (1,\infty].$$

By the definition of R, $R((0,\infty]) = F(\infty) - F(0) = 1 - 0 = 1$.

If R is σ-additive then

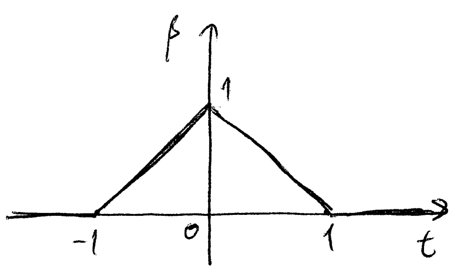
$$R((0,\infty]) = \sum_{n=1}^{\infty} R\left(\left(\frac{1}{n+1}, \frac{1}{n}\right]\right) + R((1,\infty])$$

$$= \sum_{n=1}^{\infty} \left(F\left(\frac{1}{n}\right) - F\left(\frac{1}{n+1}\right)\right) + (F(\infty) - F(1))$$

$$= \sum_{n=1}^{\infty} (1-1) + (1-1) = 0.$$

This is a contradiction. Thus, R is not σ-additive.

(3) Consider a function $\beta : \mathbb{R} \to \mathbb{C}$, $\beta(t) = \max\{1-|t|, 0\}$. First we show that β is a characteristic function of a distribution on $\mathbb{R}$.



We have $\beta(0) = 1$, $\int_{-\infty}^{\infty} |\beta(t)|\,dt = \int_{-1}^{1} (1-|t|)\,dt < \infty$,

and β is continuous.

According to the proof of Theorem 14, Fristedt-Gray, page 231, if we could show that

$$f(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \beta(t)\,dt \geq 0 \quad \forall x \in \mathbb{R}$$

then β is the characteristic function of a distribution on $\mathbb{R}$ whose density function is f.

$$Re(2\pi f(x)) = \int_{-\infty}^{\infty} \cos(-tx)\,\beta(t)\,dt = \int_{-1}^{1} \underbrace{\cos(tx)\,(1-|t|)}_{\text{even function}}\,dt$$

$$= 2\int_0^1 \cos(tx)\,(1-t)\,dt$$

$$= 2\left(\frac{\sin(tx)}{x}(1-t)\Big|_{t=0}^{t=1} - \int_0^1 -\frac{\sin(tx)}{x}\,dt \right)$$

$$= 2\int_0^1 \frac{\sin(tx)}{x}\,dt$$

$$= 2\,\frac{1-\cos x}{x^2} \geq 0.$$

$$Im(2\pi f(x)) = \int_{-\infty}^{\infty} \sin(-tx)\,\beta(t)\,dt = \int_{-1}^{1} \underbrace{\sin(-tx)(1-|t|)}_{\text{odd function}}\,dt = 0.$$

We have showed that β is the characteristic function of a distribution on $\mathbb{R}$ whose density function ~~is~~ is f.

Next, for some $n \in \mathbb{N}$, $c_1, c_2, \dots, c_n > 0$, $a_1, \dots, a_n \geq 0$, $\sum_{k=1}^{n} a_k = 1$, we show that the function $\tilde{\beta}: \mathbb{R} \to \mathbb{C}$, $\tilde{\beta}(t) = \sum_{k=1}^{n} a_k \beta(c_k t)$ is the characteristic function of a distribution on $\mathbb{R}$ whose density function is

$$\tilde{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\tilde{\beta}(t)\,dt.$$

We have $\tilde{\beta}(0) = \sum_{k=1}^{n} a_k \beta(0) = \beta(0) = 1$,

$$\int_{-\infty}^{\infty} |\tilde{\beta}(t)|\,dt = \int_{-\infty}^{\infty} \sum_{k=1}^{n} a_k \beta(c_k t)\,dt = \sum_{k=1}^{n} a_k \int_{-\infty}^{\infty} \beta(c_k t)\,dt = \sum_{k=1}^{n} \frac{a_k}{c_k}\int_{-\infty}^{\infty} \beta(t)\,dt < \infty.$$

Moreover, $\tilde{\beta}$ is continuous because β is continuous. To conclude that $\tilde{\beta}$ is the characteristic function of a distribution on $\mathbb{R}$ whose density function is $\tilde{f}$, we need to show that $\tilde{f}(x) \geq 0$ for all $x \in \mathbb{R}$.

$$\tilde{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} \sum_{k=1}^{n} a_k \beta(c_k t)\,dt = \sum_{k=1}^{n} a_k \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\beta(c_k t)\,dt$$

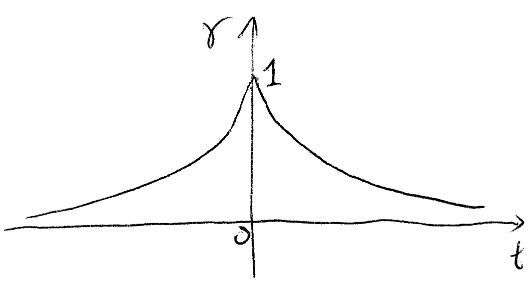
$$= \sum_{k=1}^{n} \frac{a_k}{c_k}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ic_k t x}\beta(t)\,dt = \sum_{k=1}^{n} \frac{a_k}{c_k} f(c_k x).$$

Because $f(y) \geq 0$ for all $y \in \mathbb{R}$, $\tilde{f}(x) \geq 0$ for all $x \in \mathbb{R}$.

Next, we prove a result by Polya. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\begin{cases} \gamma(0) = 1, \\ \gamma(t) \geq 0 \quad \forall t \in \mathbb{R} \\ \gamma \text{ is even, i.e. } \gamma(t) = \gamma(-t) \quad \forall t \in \mathbb{R}, \\ \gamma \text{ is continuous} \\ \gamma|_{[0,\infty)} \text{ is convex and nonincreasing.} \end{cases}$$

We show that γ is the characteristic function of a distribution on $\mathbb{R}$.



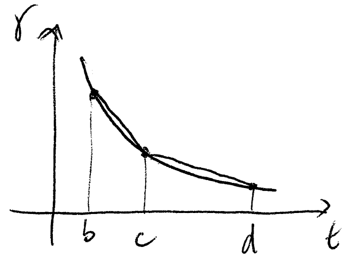
We call functions of the form $\sum_{k=1}^{n} a_k \beta(c_k t)$ for some $n \in \mathbb{N}$, $a_1, \dots, a_n \geq 0$, $\sum_{k=1}^{n} a_k = 1$ and $c_1, \dots, c_n > 0$ to be of Type (A).

Suppose that γ is a pointwise limit of a sequence of functions of Type (A). Write $\gamma(t) = \lim_{n\to\infty} \beta_n(t)$ for all $t \in \mathbb{R}$. As proved earlier, β_n is the characteristic function of a distribution on $\mathbb{R}$. Because (β_n) converges pointwise to γ and γ

is continuous at 0, by Theorem 15, Fristedt-Gray, page 260, γ is the characteristic function of a distribution on $\mathbb{R}$. Therefore, it suffices to approximate γ by a sequence of functions of Type (A).

We recall an important property of convex functions. For $0 \le b < c < d$,

$$\frac{\gamma(c)-\gamma(b)}{c-b} \le \frac{\gamma(d)-\gamma(c)}{d-c}. \qquad (1)$$



Indeed,

$$(1) \Leftrightarrow \frac{d-b}{c-b}\gamma(c) \le \frac{d-c}{c-b}\gamma(b) + \gamma(d)$$

$$\Leftrightarrow \gamma(c) \le \frac{d-c}{d-b}\gamma(b) + \frac{c-b}{d-b}\gamma(d)$$

$$\Leftrightarrow \gamma(\alpha b + (1-\alpha)d) \le \alpha\gamma(b) + (1-\alpha)\gamma(d),$$

where $\alpha = \dfrac{d-c}{d-b} \in (0,1)$. This is true because γ is convex.

Return to the problem. First, we consider the case where γ is strictly decreasing. For each $n \in \mathbb{N}$, $n \ge 2$, we put

$$d_k = \frac{k}{n} \quad \forall k = 0, 1, 2, \dots, n^2-1,$$

$$e_k = \begin{cases} \dfrac{\gamma(d_k)-\gamma(d_{k-1})}{d_k - d_{k-1}} & \text{if } 1 \le k \le n^2-1, \\ 0 & \text{if } k = n^2+1, \end{cases}$$

$$d_{n^2} = d_{n^2-1} - \frac{\gamma(d_{n^2-1})}{e_{n^2-1}},$$

$$e_{n^2} = \frac{0 - \gamma(d_{n^2-1})}{d_{n^2} - d_{n^2-1}},$$

$$a_k = d_k(e_{k+1} - e_k) \quad \forall k = 1, 2, \dots, n^2.$$

Note that $e_k < 0$ for $1 \le k \le n^2$ because γ is strictly decreasing.

Define a function $\gamma_n : \mathbb{R} \to \mathbb{R}$, $\quad \gamma_n(t) = \sum_{k=1}^{n^2} a_k \, \beta\left(\frac{t}{d_k}\right).$

We show that $\gamma_n(d_r) = \gamma(d_r)$ for all $r = 0, 1, \dots, n^2 - 1$.

$$\gamma_n(d_r) = \sum_{k=1}^{n^2} a_k \, \beta\left(\frac{d_r}{d_k}\right) = \sum_{k=r}^{n^2} a_k \left(1 - \frac{d_r}{d_k}\right) = \sum_{k=r}^{n^2} (e_{k+1} - e_k)(d_k - d_r)$$

$$= \sum_{k=r}^{n^2} e_{k+1} d_k - \sum_{k=r}^{n^2} e_k d_k - \sum_{k=r}^{n^2} (e_{k+1} - e_k) d_r$$

$$= \sum_{k=r+1}^{n^2+1} e_k d_{k-1} - \sum_{k=r}^{n^2} e_k d_k - (\underbrace{e_{n^2+1}}_{=0} - e_r) d_r$$

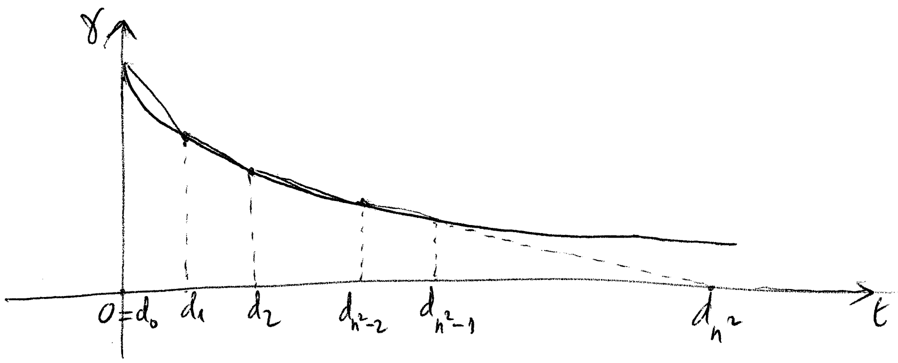
$$= \sum_{k=r+1}^{n^2} e_k d_{k-1} - \sum_{k=r}^{n^2} e_k d_k + e_r d_r$$

$$= \sum_{k=r+1}^{n^2} e_k (d_{k-1} - d_k)$$

$$= \sum_{k=r+1}^{n^2-1} e_k (d_{k-1} - d_k) + e_{n^2} (d_{n^2-1} - d_{n^2})$$

$$= \sum_{k=r+1}^{n^2-1} \big(\gamma(d_{k-1}) - \gamma(d_k)\big) + \gamma(d_{n^2-1})$$

$$= \gamma(d_r).$$



As a consequence, $\sum_{k=1}^{n^2} a_k = \gamma_n(d_0) = \gamma(d_0) = \gamma(0) = 1$.

Because of the convexity of γ, $e_{k+1} \geq e_k$ for all $1 \leq k \leq n^2 - 2$.

$$e_{n^2} = \frac{\gamma(d_{n^2-1})}{d_{n^2-1} - d_{n^2}} = \frac{\gamma(d_{n^2-1})}{\frac{\gamma(d_{n^2-1})}{e_{n^2-1}}} = e_{n^2-1} < 0,$$

$$e_{n^2+1} = 0 > e_{n^2}.$$

Thus, $a_k \geq 0$ for all $1 \leq k \leq n^2$. This implies γ_n is a function of Type (A). It remains to show that (γ_n) converges pointwise to γ in $\mathbb{R}$. Since γ_n and γ are even, it suffices to show that (γ_n) converges to γ pointwise on $[0, \infty)$.

We see that

$$\gamma_n(t) = \sum_{k=1}^{n^2} a_k \beta\left(\frac{t}{d_k}\right) = \sum_{k=1}^{n^2} a_k \left(1 - \frac{t}{d_k}\right) I_{t > d_k} \qquad \forall t \in [0, \infty).$$

Thus, γ_n is an affine function on each interval $[d_r, d_{r+1}]$, $0 \leq r \leq n^2 - 1$. Because $\gamma_n(d_r) = \gamma(d_r)$ for all $0 \leq r \leq n^2 - 1$, γ_n is the linear interpolation of γ on $[d_r, d_{r+1}]$ for $0 \leq r \leq n^2 - 2$. Because γ is decreasing on $[0, \infty)$, so is γ_n. Because γ is convex in $[0, \infty)$,

$$\gamma(t) \leq \gamma_n(t) \quad \forall t \in [d_r, d_{r+1}], \quad 0 \leq r \leq n^2 - 2$$

Because γ_n is decreasing, $\gamma_n(t) \leq \gamma_n(d_r) = \gamma(d_r)$. Thus,

$$0 \leq \gamma_n(t) - \gamma(t) \leq \gamma(d_r) - \gamma(t) \quad \forall t \in [d_r, d_{r+1}], \; 0 \leq r \leq n^2 - 2.$$

Thus,
$$0 \leq \gamma_n(t) - \gamma(t) \leq \sup_{|x-y| \leq \frac{1}{n}} |\gamma(x) - \gamma(y)| \qquad \forall t \in [0, d_{n^2-1}].$$

For $t_0 \in [0, \infty)$ and $n \in \mathbb{N}$, $n > t_0 + 1$, we have $d_{n^2-1} = \frac{n^2-1}{n} = n - \frac{1}{n} \geq n - 1 > t_0$.

Thus, $t_0 \in [0, d_{n^2-1}]$ and $0 \leq \gamma_n(t_0) - \gamma(t_0) \leq \sup_{|x-y| \leq \frac{1}{n}} |\gamma(x) - \gamma(y)|$.

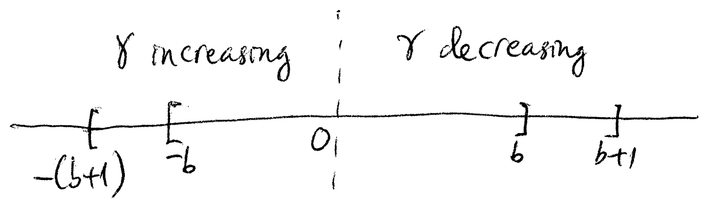
To conclude that $\gamma_n(t_0) \to \gamma(t_0)$ as $n\to\infty$, it suffices to show that γ is uniformly continuous.

Let $\varepsilon > 0$. Put $\alpha = \lim_{t\to\infty} \gamma(t)$. There exists a number $b > 1$ such that

$$\gamma(t) - \alpha < \varepsilon \quad \forall t \in \mathbb{R},\ |t| > b.$$

Because γ is uniformly continuous on $[-(b+1), b+1]$, there exists $\delta \in (0,1)$ such that

$$|\gamma(x) - \gamma(y)| < \varepsilon \quad \forall x, y \in [-(b+1), b+1],\ |x-y| < \delta.$$



For any $z, t \in \mathbb{R}$, $0 \le t - z < \delta$, we have three following cases.

- $z \le -(b+1)$.

  Then $t < z + \delta < -(b+1) + 1 = -b$. Then $|\gamma(z) - \gamma(t)| = \gamma(t) - \gamma(z) \le \gamma(t) - \alpha < \varepsilon$.

- $-(b+1) < z \le b$.

  Then $t < z + \delta < b+1$. Then $z, t \in [-(b+1), b+1]$. We get $|\gamma(z) - \gamma(t)| < \varepsilon$.

- $b < z$.

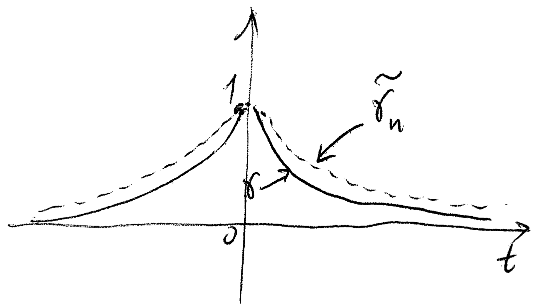
  Then $|z|, |t| > b$. Thus, $|\gamma(z) - \gamma(t)| = \gamma(z) - \gamma(t) < (\alpha + \varepsilon) - \alpha = \varepsilon$.

Therefore, γ is uniformly continuous on $\mathbb{R}$. We have finished the case where γ is strictly decreasing.

Finally, we consider the general case for γ. For each $n \in \mathbb{N}$, we define a map $\tilde{\gamma}_n : \mathbb{R} \to \mathbb{R}$,

$$\tilde{\gamma}_n(t) = \frac{\gamma(t) + \frac{1}{n} e^{-|t|}}{1 + \frac{1}{n}}.$$

Then



$$\begin{cases} \tilde{\gamma}_n(0) = 1, \\ \tilde{\gamma}_n(t) \geq 0 \ \forall t \in \mathbb{R}, \\ \tilde{\gamma}_n \text{ is an even function,} \\ \tilde{\gamma}_n \text{ is continuous} \\ \tilde{\gamma}_n|_{[0,\infty)} \text{ is convex because it is the (rescaled)} \\ \qquad \text{sum of two convex functions on } [0,\infty). \end{cases}$$

Moreover, for $0 \leq x < y$,

$$\tilde{\gamma}_n(y) - \tilde{\gamma}_n(x) = \underbrace{\frac{\gamma(y) - \gamma(x)}{1 + \frac{1}{n}}}_{\leq 0} + \underbrace{\frac{\frac{1}{n}(e^{-y} - e^{-x})}{1 + \frac{1}{n}}}_{< 0} < 0.$$

Thus, $\tilde{\gamma}_n$ is strictly decreasing. As proved in the first case, $\tilde{\gamma}_n$ is the characteristic function of a distribution on $\mathbb{R}$. Because $(\tilde{\gamma}_n)$ converges pointwise to γ on $\mathbb{R}$, and γ is continuous at 0, γ is also the characteristic function of a distribution on $\mathbb{R}$.

4 Let a_k^n, a_k, b_k, for $k, n \in \mathbb{N}$, be numbers such that

$$\begin{cases} \lim_{n \to \infty} a_k^n = a_k, \\ |a_k^n| \leq b_k, \\ \sum_{k=1}^{\infty} b_k < \infty. \end{cases}$$

We show that $\lim_{n \to \infty} \sum_{k=1}^{\infty} a_k^n = \sum_{k=1}^{\infty} a_k$.

We want to model the problem so that the Dominated Convergence Theorem can be applied. In solving ~~the Dominated Convergence~~ Problem 4 of Homework #4, we introduced the counting measure on the set $\mathbb{N}$.

$$\mu(A) = \# A \quad \forall A \subset \mathbb{N}.$$

We pointed out that the measurable functions are the sequences in $\mathbb{R}$, and that the integrals over $\mathbb{N}$ are the infinite sums. Define the functions $f, f_n, g : \mathbb{N} \to \mathbb{R}$,

$$f(k) = a_k, \quad f_n(k) = a_k^n, \quad g(k) = b_k.$$

They are measurable functions. Also,

$$\begin{cases} \lim_{n\to\infty} f_n(k) = f(k), \\ |f_n(k)| \le g(k) \quad \forall n, k \in \mathbb{N}, \\ \int_{\mathbb{N}} g(k)\, \mu(dk) < \infty . \end{cases}$$

By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{N}} f_n(k)\, \mu(dk) = \int_{\mathbb{N}} f(k)\, \mu(dk).$$

In other words, $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k^n = \sum_{k=1}^{\infty} a_k$.

(5) Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and $f : \Omega \to \mathbb{R}$ be a nonnegative measurable function. Suppose $\int_{\Omega} f(x)\, \mu(dx) < \infty$. Let (A_n) be a sequence in $\mathcal{F}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$. We show that $\lim_{n\to\infty} \int_{\Omega} f(x) I_{A_n}(x)\, \mu(dx) = 0$.

For each $m \in \mathbb{N}$, we put $B_m = \{x \in \Omega : f(x) \le m\} = f^{-1}([0, m])$. Then $B_m \in \mathcal{F}$ because f is measurable. Moreover, $B_1 \subset B_2 \subset B_3 \subset \dots$ and $\bigcup_{m=1}^{\infty} B_m = \Omega$. Put $f_m(x) = f(x) I_{B_m}(x)$. Then

$$0 \le f_1(x) \le f_2(x) \le f_3(x) \le \dots$$

$$\lim_{m\to\infty} f_m(x) = f(x).$$

By the Monotone Convergence Theorem, $\lim\limits_{m\to\infty} \int_\Omega f_m(x)\mu(dx) = \int_\Omega f(x)\mu(dx) < \infty$.

Put $B_m^c = \Omega \setminus B_m$. Then

$$\int_\Omega f_m(x)\mu(dx) = \int_\Omega f(x)\left(1 - I_{B_m^c}(x)\right)\mu(dx) = \int_\Omega f(x)\mu(dx) - \int_\Omega f(x) I_{B_m^c}(x)\mu(dx).$$

Thus,
$$\lim_{m\to\infty} \int_\Omega f(x) I_{B_m^c}(x)\mu(dx) = 0.$$

For each $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $\int_\Omega f(x) I_{B_M^c}(x)\mu(dx) < \frac{\varepsilon}{2}$.

Because $\lim\limits_{n\to\infty} \mu(A_n) = 0$, there exists $N \in \mathbb{N}$ such that $M\mu(A_n) < \frac{\varepsilon}{2}$ for all $n > N$. Then

$$\int_\Omega f(x) I_{A_n}(x)\mu(dx) = \underbrace{\int_\Omega f(x) I_{A_n}(x) I_{B_M}(x)\mu(dx)}_{\{1\}} + \underbrace{\int_\Omega f(x) I_{A_n}(x) I_{B_M^c}(x)\mu(dx)}_{\{2\}} \qquad (1)$$

We have

$$\{1\} = \int_{B_M} f(x) I_{A_n}(x)\mu(dx) \le \int_{B_M} M I_{A_n}(x)\mu(dx) \le \int_\Omega M I_{A_n}(x)\mu(dx) = M\mu(A_n) < \frac{\varepsilon}{2} \quad \forall n > N.$$

$$\{2\} \le \int_\Omega f(x) I_{B_M^c}(x)\,dx < \frac{\varepsilon}{2}.$$

Then by (1) we get

$$\int_\Omega f(x) I_{A_n}(x)\mu(dx) \le \{1\} + \{2\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall n > N.$$

Therefore, $\lim\limits_{n\to\infty} \int_\Omega f(x) I_{A_n}(x)\mu(dx) = 0$.

(6) List all rational numbers in $(0,1)$ as a sequence $r_1, r_2, r_3, \dots$ Let X be a random variable such that $\mathbb{P}(X=r_n) = cn^{-2}$, where $c = \left(\sum_{k=1}^{\infty} k^{-2}\right)^{-1}$. Let $F: \mathbb{R} \to [0,1]$, $F(x) = \mathbb{P}(X \le x)$ be the distribution function of X.

First, we show that F is discontinuous at every point in $\mathbb{Q} \cap (0,1)$. Take $x \in \mathbb{Q} \cap (0,1)$. We know that $\mathbb{P}(X=x) > 0$ and

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}(X < x) + \mathbb{P}(X = x) = F(x-) + \mathbb{P}(X=x).$$

Then $F(x) > F(x-) = \lim_{y \to x^-} F(y)$. This implies F is not continuous at x.

Next, we show that F is continuous at every point $x \in \mathbb{R} \setminus (\mathbb{Q} \cap (0,1))$. We know that F is right-continuous at x. It remains to show that F is left-continuous at x, i.e. $F(x) = F(x-)$. Since $x \notin \mathbb{Q} \cap (0,1)$,

$$\mathbb{P}(X=x) \le 1 - \mathbb{P}(X \in \mathbb{Q} \cap (0,1)) = 1 - \sum_{n=1}^{\infty} \mathbb{P}(X = r_n)$$
$$= 1 - c \sum_{n=1}^{\infty} n^{-2} = 0.$$

Thus, $\mathbb{P}(X=x) = 0$. Hence,

$$F(x) = F(x-) + \mathbb{P}(X=x) = F(x-).$$

(7) Let (X_n) be a sequence of pairwise independent random variables. Suppose

$$\mathbb{P}(X_n \in (a,b)) = (1-2^{-n})(b-a) \quad \forall\, 0 \le a \le b \le 1,$$
$$\mathbb{P}(X_n = 2^n) = 2^{-n}.$$

We show that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k = \frac{1}{2}$ almost surely.

First, we compute the distribution function of X_n. By the hypotheses,

$$P(X_n \in (0,1) \cup \{2^n\}) = P(X_n \in (0,1)) + P(X_n = 2^n) = (1-2^{-n})(1-0) + 2^{-n} = 1.$$

Consequently, $X_n \geq 0$ almost surely. We can redefine X_n on a set of measure zero, if necessary, to make it a nonnegative random variable. Let $F_n : \mathbb{R} \to [0,1]$ be the distribution function of X_n. Then

$$F_n(1) - F_n(0) = P(X_n \in (0,1]) \geq P(X_n \in (0,1)) = 1 - 2^{-n},$$

$$F_n(2^n) - F(2^n-) = P(X_n = 2^n) = 2^{-n}.$$

Thus, $F_n(2^n) - F_n(0) = \underbrace{(F_n(2^n) - F_n(2^n-))}_{=2^{-n}} + \underbrace{(F_n(2^n-) - F(0))}_{\geq F_n(1) - F_n(0)} \geq (1-2^{-n}) + 2^{-n} = 1.$

The equality must hold. Then

$$\begin{cases} F_n(1) - F_n(0) = 1 - 2^{-n} \\ F_n(2^n) = 1, \ F_n(0) = 0 \end{cases}$$

Then $F_n(1) = 1 - 2^{-n}$. For $x \in (0,1)$,

$$F_n(x) = F_n(0) + P(X \in (0,x]) \geq 0 + P(X \in (0,x)) = (1-2^{-n})x,$$

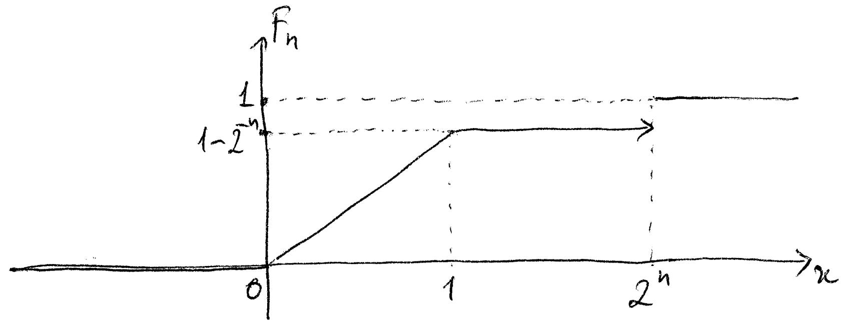
$$F_n(x) = F_n(1) - P(X \in (x,1]) \leq (1-2^{-n}) - P(X \in (x,1)) = (1-2^{-n}) - (1-2^{-n})(1-x)$$
$$= (1-2^{-n})x.$$

Thus, $F_n(x) = (1-2^{-n})x$.

We have $F_n(2^n-) = F_n(2^n) - P(X_n = 2^n) = 1 - 2^{-n} = F_n(1)$.

For $x \in (1, 2^n)$, $F_n(1) \leq F_n(x) \leq F_n(2^{+n}-)$. Thus $F_n(x) = 1 - 2^{-n}$. We get

$$F_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ (1-2^{-n})x & \text{if } 0 < x < 1, \\ 1 - 2^{-n} & \text{if } 1 \leq x < 2^n, \\ 1 & \text{if } x \geq 2^n. \end{cases}$$



Next, we compute EX_n. In Problem 4 of Homework #6, we verified the formula

$$EX_n = \int_0^\infty (1-F_n(x))\,dx.$$

This is also Corollary 20, Fristedt-Gray, page 58. Then we get

$$\begin{aligned} EX_n &= \int_0^1 [1-(1-2^{-n})x]\,dx + \int_1^{2^n} [1-(1-2^{-n})]\,dx + \int_{2^n}^\infty (1-1)\,dx \\ &= \left[x - (1-2^{-n})\frac{x^2}{2}\right]_{x=0}^{x=1} + 2^{-n}(2^n-1) \\ &= \frac{3-2^{-n}}{2}. \end{aligned}$$

Put $Y_n = X_n I_{X_n \le n} + E(X_n I_{X_n > n})$. We show that $\sum_{n=1}^\infty \frac{\operatorname{Var} Y_n}{n^2} < \infty$.

Put $Z_n = X_n I_{X_n \le n}$. First, we compute the distribution function G_n of Z_n.

For $x \in [0,1)$,

$$\{\omega \in \Omega : Z_n(\omega) \le x\} = \{\omega : X_n(\omega) I_{X_n(\omega) \le n} \le x\} = \{\omega : X_n(\omega) > n\} \cup \{\omega : X_n(\omega) \le x\}.$$

Thus,

$$G_n(x) = P(Z_n \le x) = P(X_n > n) + P(X_n \le x) = (1-F_n(n)) + F_n(x).$$

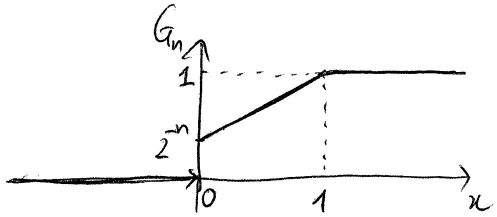
Because $1 \le n < 2^n$, $F_n(n) = 1-2^{-n}$. Then

$$G_n(x) = [1-(1-2^{-n})] + (1-2^{-n})x = 1-(1-2^{-n})(1-x).$$

Consequently, $P(Z_n \le 1) = G_n(1) = 1$. This implies $G_n(x) = 1$ for all $x \ge 1$.

Because $Z_n \ge 0$, $G_n(x) = 0$ for all $x < 0$. Therefore,

$$G_n(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1-(1-2^{-n})(1-x) & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$



Next, we compute EZ_n and EZ_n^2.

$$EZ_n = \int_0^\infty (1-G_n(x))\,dx = \int_0^1 (1-2^{-n})(1-x)\,dx = (1-2^{-n})\left(x-\frac{x^2}{2}\right)\Big|_0^1 = \frac{1-2^{-n}}{2}.$$

Let $\tilde{G}_n$ be the distribution function of Z_n^2. Then

$$\tilde{G}_n(x) = P(Z_n^2 \le x) = \begin{cases} 0 & \text{if } x < 0 \\ P(Z_n \le \sqrt{x}) & \text{if } x \geq 0 \end{cases} = G_n(\sqrt{x}).$$

Thus,

$$EZ_n^2 = \int_0^\infty (1-G_n(\sqrt{x}))\,dx = \int_0^1 (1-2^{-n})(1-\sqrt{x})\,dx = (1-2^{-n})\left(x-\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^1$$
$$= \frac{1-2^{-n}}{3}.$$

Therefore,

$$\begin{aligned} \operatorname{Var} Y_n = \operatorname{Var}(Z_n + E(X_n I_{X_n > n})) &= \operatorname{Var} Z_n \\ &= EZ_n^2 - (EZ_n)^2 \\ &\le EZ_n^2 < \frac{1}{3}. \end{aligned}$$

Consequently, $\displaystyle\sum_{n=1}^\infty \frac{\operatorname{Var} Y_n}{n^2} \le \frac{1}{3}\sum_{n=1}^\infty \frac{1}{n^2} < \infty.$

Next, we show that $Y_1, Y_2, Y_3, \dots$ are pairwise independent. For each $n \in \mathbb{N}$, put $h_n : \mathbb{R} \to \mathbb{R}$, $h_n(x) = x I_{x \le n} + E(X_n I_{X_n > n})$. This map is Borel-measurable. Because $X_1, X_2, X_3, \ldots$ are pairwise independent and $Y_n = h_n(X_n)$, the random variables $Y_1, Y_2, Y_3, \ldots$ are also pairwise independent.

We now compute $\mu_n = EY_n$.

$$EY_n = EZ_n + E(X_n I_{X_n > n}) = EZ_n + E(X_n - Z_n) = EX_n = \frac{3 - 2^{-n}}{2}.$$

Next, we compute the limit of $\frac{\mu_1 + \cdots + \mu_n}{n}$.

$$\frac{\mu_1 + \cdots + \mu_n}{n} = \frac{1}{n} \sum_{k=1}^{n} \frac{3 - 2^{-k}}{2} = \frac{1}{n}\left(\frac{3n}{2} - \frac{1}{2}\sum_{k=1}^{n} 2^{-k}\right) = \frac{3}{2} - \underbrace{\frac{1}{2n}\sum_{k=1}^{n} 2^{-k}}_{< 1/2n}$$

Then $\lim_{n\to\infty} \frac{\mu_1 + \cdots + \mu_n}{n} = \frac{3}{2}$.

The following lemma is from the lecture on 10/22/2014.

Let $Y_n \geq 0$, $n = 1, 2, \ldots$ be pairwise uncorrelated or negatively correlated random variables. Assume $\sum_{n=1}^{\infty} \frac{\operatorname{Var} Y_n}{n^2} < \infty$ and that for $\mu_n = EY_n$ the limit

$$\lim_{n\to\infty} \frac{\mu_1 + \cdots + \mu_n}{n} = \mu$$

exists. Then $\frac{1}{n}\sum_{k=1}^{n} Y_k \to \mu$ a.s.

Applying this lemma, we get $\frac{1}{n}\sum_{k=1}^{n} Y_k \to \frac{3}{2}$ almost surely.

$$E(X_n I_{X_n > n}) = E(X_n - Z_n) = EX_n - EZ_n = \frac{3 - 2^{-n}}{2} - \frac{1 - 2^{-n}}{2} = 1.$$

Hence, $Y_n = Z_n + 1$. Then

$$\frac{1}{n}\sum_{k=1}^{n} Z_k = \frac{1}{n}\sum_{k=1}^{n} Y_k - 1 \longrightarrow \frac{3}{2} - 1 = \frac{1}{2} \quad \text{a.s.}$$

For each $n \in \mathbb{N}$, we put $A_n = \{\omega : X_n(\omega) > n\}$. Then

$$P(A_n) = P(X_n > n) = 1 - F_n(n) = 2^{-n}.$$

Thus, $\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty$. By Borel's lemma (Lemma 3, Fristedt-Gray, page 78), $P(\overline{\lim} A_n) = 0$. Put

$$A = \{\omega : \frac{1}{n}\sum_{k=1}^{n} Z_k(\omega) \not\to \frac{1}{2}\} \cup \overline{\lim} A_n .$$

Then $P(A) = 0$. Take any $\omega \in \Omega \setminus A$. Because $\omega \notin \overline{\lim} A_n$, there exists $m \in \mathbb{N}$ such that $\omega \notin A_n$ for all $n > m$. This means $X_n(\omega) \le n$ for all $n > m$. Thus, $Z_n(\omega) = X_n(\omega)$ for all $n > m$. Then

$$\frac{1}{n}\sum_{k=1}^{n} X_k(\omega) = \frac{1}{n}\sum_{k=1}^{m} X_k(\omega) + \frac{1}{n}\sum_{k=m+1}^{n} Z_k(\omega)$$

$$= \underbrace{\frac{1}{n}\sum_{k=1}^{m}\left(X_k(\omega) - Z_k(\omega)\right)}_{\to 0} + \underbrace{\frac{1}{2}\sum_{k=1}^{n} Z_k(\omega)}_{\to \frac{1}{2}} \to \frac{1}{2} \text{ as } n \to \infty .$$

Therefore, $\frac{1}{n}\sum_{k=1}^{n} X_k \to \frac{1}{2}$ a.s.

(8) Let (X, ρ) be a metric space and $f : X \to \mathbb{R}$ be a function. Denote by Δ_f the set of all points in X at which f is not continuous. We show that Δ_f is Borel-measurable.

For $x \in X, r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r. For each $n \in \mathbb{N}$, we define $\varphi_n : X \to [0, \infty]$,

$$\varphi_n(x) = \sup\{|f(y) - f(z)| : y, z \in B(x, \tfrac{1}{n})\}.$$

Then $\varphi_1(x) \ge \varphi_2(x) \ge \varphi_3(x) \ge \cdots \ge 0$. We show that f is continuous at x

if and only if $\inf_{n\in\mathbb{N}} \varphi_n(x) = 0$.

Suppose f is continuous at x. For each $\varepsilon>0$, there exists $\delta>0$ such that $|f(y)-f(x)|<\frac{\varepsilon}{2}$ for all $y\in B(x,\delta)$. Then

$$|f(y)-f(z)| \le |f(y)-f(x)| + |f(z)-f(x)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \quad \forall y,z\in B(x,\delta).$$

For all $n\in\mathbb{N}$, $n>\delta^{-1}$, we have

$$|f(y)-f(z)|<\varepsilon \quad \forall y,z\in B(x,\tfrac{1}{n}).$$

Then $\varphi_n(x)\le\varepsilon$. This implies $\inf_n \varphi_n(x) = 0$.

Conversely, suppose $\inf_n \varphi_n(x)=0$. For each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $\varphi_{n_0}(x)\le\varepsilon$. Thus,

$$|f(y)-f(z)|\le\varepsilon \quad \forall y,z\in B(x,\tfrac{1}{n_0}).$$

In particular, $|f(y)-f(x)|\le\varepsilon$ for all $y\in B(x,\frac{1}{n_0})$. This implies f is continuous at x.

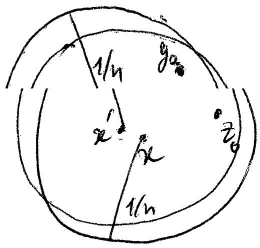
For each $m,n\in\mathbb{N}$, we put $A_{n,m}=\{x\in X : \varphi_n(x)>\frac{1}{m}\}$. Then

$$\begin{aligned}
f \text{ is discontinuous at } x &\iff \inf_n \varphi_n(x)>0\\
&\iff \exists m\in\mathbb{N} : \varphi_n(x)>\tfrac{1}{m} \quad \forall n\in\mathbb{N}\\
&\iff \exists m\in\mathbb{N} : x\in A_{n,m} \quad \forall n\in\mathbb{N}\\
&\iff \exists m\in\mathbb{N} : x\in \bigcap_{n=1}^{\infty} A_{n,m}\\
&\iff x\in \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} A_{n,m}.
\end{aligned}$$

Therefore, $\Delta_f = \bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty} A_{n,m}$.

To show that Δ_f is Borel-measurable, it suffices to show that each set $A_{n,m}$ is Borel-measurable. In fact, we show that $A_{n,m}$ is open in X. Take any $x \in A_{n,m}$. Because $\varphi_n(x) > \frac{1}{m}$, there exist $y_0, z_0 \in B(x, \frac{1}{n})$ such that

$$|f(y_0) - f(z_0)| > \frac{1}{m}.$$



Put $r := \frac{1}{n} - \max\{\rho(x, y_0), \rho(x, z_0)\} > 0$.

For each $x' \in B(x, r)$,

$$\rho(x', y_0) \leq \rho(x', x) + \rho(x, y_0) < r + \rho(x, y_0) \leq \frac{1}{n},$$

$$\rho(x', z_0) \leq \rho(x', x) + \rho(x, z_0) < r + \rho(x, z_0) \leq \frac{1}{n}.$$

Thus, $y_0, z_0 \in B(x', \frac{1}{n})$. Then

$$\varphi_n(x') \geq |f(y_0) - f(z_0)| > \frac{1}{n}.$$

Thus, $x' \in A_{n,m}$. This implies $B(x, r) \subset A_{n,m}$. We have showed that $\cancel{A_{n,m}}$ $A_{n,m}$ is open in X.

(9) Let $C([0,1])$ be the space of all continuous function from $[0,1]$ to $\mathbb{R}$. It is a Banach space with norm $\|x\| = \sup_{t \in [0,1]} |x(t)|$. Put $f(t) = \sin \frac{1}{t}$ and

$$\Gamma_n = \left\{ x \in C([0,1]) : \sup_{t \in [\frac{1}{n}, 1]} |x(t) - f(t)| \leq \frac{1}{2} \text{ and } \|x\| \leq 2 \right\}.$$

Because $\|x\| \leq 2$ for all $x \in \Gamma_n$, Γ_n is a bounded subset of $C([0,1])$. For each $x \in \Gamma_{n+1}$,
$$\frac{1}{2} \geq \sup_{t \in [\frac{1}{n+1}, 1]} |x(t) - f(t)| \geq \sup_{t \in [\frac{1}{n}, 1]} |x(t) - f(t)|,$$

which implies $x \in \Gamma_n$. Thus, $\Gamma_{n+1} \subset \Gamma_n$. In other words, (Γ_n) is a decreasing sequence. Next, we show that Γ_n is closed in $C([0,1])$. Let (x_m) be a sequence in Γ_n that converges to some $x \in C([0,1])$. Then $\|x\| = \lim_{m\to\infty} \|x_m\| \leq 2$.

For each $t \in [\frac{1}{n}, 1]$,

$$|x(t) - f(t)| = \lim_{m\to\infty} |x_m(t) - f(t)|.$$

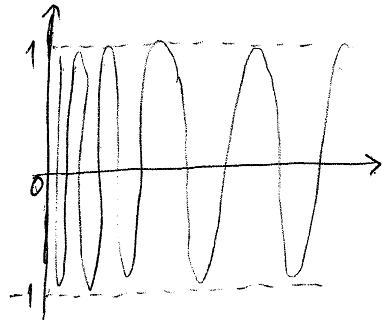
Since $|x_m(t) - f(t)| \leq \sup_{s\in[\frac{1}{n},1]} |x_m(s) - f(s)| \leq \frac{1}{2}$ for every $m \in \mathbb{N}$, $|x(t) - f(t)| \leq \frac{1}{2}$.

Because this is true for every $t \in [\frac{1}{n}, 1]$, $\sup_{t\in[\frac{1}{n},1]} |x(t) - f(t)| \leq \frac{1}{2}$. Thus $x \in \Gamma_n$.

We have showed that Γ_n is closed.

Next, we show that $\bigcap_{n=1}^{\infty} \Gamma_n = \phi$. Suppose by contradiction that there exists $x \in \bigcap_{n=1}^{\infty} \Gamma_n$. Then

$$\sup_{t\in[\frac{1}{n},1]} |x(t) - f(t)| \leq \frac{1}{2} \quad \forall n \in \mathbb{N}.$$



Thus, $|x(t) - f(t)| \leq \frac{1}{2} \quad \forall t \in [\frac{1}{n}, 1], \forall n \in \mathbb{N}$.

This implies

$$|x(t) - f(t)| \leq \frac{1}{2} \quad \forall t \in (0,1]. \qquad (1)$$

Put $a = x(0)$. Because x is continuous at 0, there exists $\delta \in (0,1)$ such that

$$|x(t) - a| < \frac{1}{2} \quad \forall 0 \leq t \leq \delta. \qquad (2)$$

Combining (1) and (2), we get

$$\begin{cases} |x(t) - a| < \frac{1}{2}, \\ |x(t) - f(t)| \leq \frac{1}{2} \end{cases} \quad \forall 0 < t \leq \delta.$$

Thus, $|f(t)-a| \le |x(t)-a| + |x(t)-f(a)| < \frac{1}{2}+\frac{1}{2} = 1 \quad \forall 0<t\le\delta$. (3)

Take $t_1 = \dfrac{1}{\frac{\pi}{2}+2k\pi}$ for some $k \in \mathbb{N}$, $k > (2\pi\delta)^{-1}$. Then $0<t_1\le\delta$ and

$$f(t_1) = \sin\left(\frac{\pi}{2}+2k\pi\right) = 1.$$

Substituting $t=t_1$ into (3), we get $|1-a|<1$.

Take $t_2 = \dfrac{1}{\frac{3\pi}{2}+2k\pi}$ for the same number k. Then $0<t_2\le\delta$ and

$$f(t_2) = \sin\left(\frac{3\pi}{2}+2k\pi\right) = -1.$$

Substituting $t=t_2$ into (3), we get $|-1-a|<1$. Then

$$2 > |1-a| + |-1-a| = |1-a|+|1+a| \ge |(1-a)+(1+a)| = 2.$$

This is a contradiction.

(10) Let Q and Q_n, $n=1,2,\dots$ be probability distributions on a Polish space X. Let μ be a σ-finite measure on X. Suppose

$$\begin{cases} Q \text{ has density function } f \text{ with respect to } \mu, \\ Q_n \text{ has density function } f_n \text{ with respect to } \mu, \\ f_n \to f \quad \mu\text{-a.e.} \end{cases}$$

We show that $Q_n \xrightarrow{w} Q$. Let $g: X \to \mathbb{R}$ be a continuous bounded function. We want to show

$$\lim_{n\to\infty} \int_X g(x)\, Q_n(dx) = \int_X g(x)\, Q(dx).$$

We know that f_n is the Randon-Nikodym derivative of Q_n with respect to μ. By the Change of Measure theorem (Theorem 19, Fristedt-Gray, page 116),

$$\int_X g(x)\, Q_n(dx) = \int_X g(x) f_n(x)\, \mu(dx).$$

Similarly, $\int_X g(x)\, Q(dx) = \int_X g(x) f(x)\, \mu(dx)$.

We need to show $\lim_{n\to\infty} \int_X g(x) f_n(x)\, \mu(dx) = \int_X g(x) f(x)\, \mu(dx)$.

Since g is bounded, there is a number $C>0$ such that $|g(x)| \leq C$ for all $x \in X$.

Then

$$\left| \int_X g(x) f_n(x) \mu(dx) - \int_X g(x) f(x) \mu(dx) \right| = \left| \int_X g(x) (f_n(x) - f(x)) \mu(dx) \right|$$
$$\leq \int_X |g(x)|\, |f_n(x) - f(x)|\, \mu(dx)$$
$$\leq C \int_X |f_n(x) - f(x)|\, \mu(dx).$$

Thus, it suffices to show $\lim_{n\to\infty} \int_X |f_n(x) - f(x)|\, \mu(dx) = 0$.

We know that f and f_n are μ-measurable, nonnegative and $f_n \to f$ a.e.

$$\int_X f_n(x) \mu(dx) = \int_X Q_n(dx) = Q_n(X) = 1 = Q(X) = \int_X f(x) \mu(dx) \quad \forall n \in \mathbb{N}.$$

By Scheffe's lemma, we conclude that $\lim_{n\to\infty} \int_X |f_n(x) - f(x)|\, \mu(dx) = 0$.

(11) Denote by $C([0,1])$ the space of all continuous functions from $[0,1]$ to $\mathbb{R}$. It is a Banach separable space with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. Thus, the metric space induced by this norm is a Polish space. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X_n, X : \Omega \to C([0,1])$ be random variables having probability distributions Q_n and Q respectively. Define $Y_n, Y : \Omega \to \mathbb{R}$,

$$Y_n(\omega) = \max_{t\in[0,1]} X_n(\omega)(t), \qquad Y(\omega) = \max_{t\in[0,1]} X(\omega)(t).$$

Suppose $X_n \xrightarrow{D} X$. We show that $Y_n \xrightarrow{D} Y$.

$$\begin{tikzcd} \Omega \arrow[r, "X_n"] \arrow[rr, bend right, "Y_n"'] & C([0,1]) \arrow[r, "h"] & \mathbb{R} \end{tikzcd}$$

Define a map $h : C([0,1]) \to \mathbb{R}$, $h(x) = \max_{t\in[0,1]} x(t)$. We first show that h is continuous. For $x, y \in C([0,1])$, put $x(t_1) = \max_{t\in[0,1]} x(t)$ and $y(t_2) = \max_{t\in[0,1]} y(t)$.

Then

$$h(x) - h(y) = x(t_1) - y(t_2) \leq x(t_1) - y(t_1) \leq \|x-y\|,$$
$$h(x) - h(y) = x(t_1) - y(t_2) \geq x(t_2) - y(t_2) \geq -\|x-y\|.$$

Thus, $|h(x) - h(y)| \leq \|x-y\|$. This implies h is continuous.

Since $Y_n = h\circ X_n$ and $Y = h\circ X$, Y_n and Y are random variables.

We have

$$\begin{cases} Q_n(B) = P(X_n^{-1}(B)), \\ Q(B) = P(X^{-1}(B)) \end{cases} \quad \forall B \in \mathcal{B}(C([0,1])).$$

Because $Q_n \xrightarrow{w} Q$, by Portmanteau's theorem (Theorem 6, Fristedt-Gray, page 353)

$$\underline{\lim}\, Q_n(O) \geq Q(O) \quad \forall O \text{ open in } C([0,1]).$$

Let R_n and R be the distribution of Y_n and Y respectively. That is

$$\begin{cases} R_n(B') = P(Y_n^{-1}(B')), \\ R(B') = P(Y^{-1}(B')) \end{cases} \quad \forall B' \in \mathcal{B}(\mathbb{R}).$$

To show $Y_n \xrightarrow{D} Y$ is to show $R_n \xrightarrow{w} R$. By Portmanteau's theorem, it is equivalent to showing

$$\underline{\lim}\, R_n(\mathcal{O}') \geq R(\mathcal{O}') \quad \forall\, \mathcal{O}' \text{ open in } \mathbb{R}.$$

Take an open subset $\mathcal{O}'$ of $\mathbb{R}$. We have

$$R_n(\mathcal{O}') = \mathbb{P}(Y_n^{-1}(\mathcal{O}')) = \mathbb{P}(X_n^{-1}(h^{-1}(\mathcal{O}'))) = \mathbb{P}(X_n^{-1}(\mathcal{O})) = Q_n(\mathcal{O})$$

with $\mathcal{O} = h^{-1}(\mathcal{O}')$. Since h is continuous, $\mathcal{O}$ is open in $C([0,1])$. Similarly, $R(\mathcal{O}') = Q(\mathcal{O})$. Because $\underline{\lim}\, Q_n(\mathcal{O}) \geq Q(\mathcal{O})$, we get $\underline{\lim}\, R_n(\mathcal{O}') \geq R(\mathcal{O}')$.

(12) Let (ε_n) be a sequence of independent and identically distributed random variables with $\mathbb{P}(\varepsilon_1 = 1) = \mathbb{P}(\varepsilon_1 = -1) = 1/2$. We show that

$$\xi := \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$$

is uniformly distributed on $[-1,1]$. For each $n \in \mathbb{N}$, we put $\xi_n = \sum_{k=1}^{n} \frac{\varepsilon_k}{2^k}$.

Then ξ_n is a random variable. Since $\xi = \lim \xi_n$, ξ is also a random variable.

The characteristic functions of ξ_n and ξ are $\varphi_{\xi_n}(t) = Ee^{it\xi_n}$ and $\varphi_\xi = Ee^{it\xi}$.

For each $t \in \mathbb{R}$, $\lim_{n\to\infty} e^{it\xi_n} = e^{it\xi}$ and $|e^{it\xi_n(\omega)}| = 1$ for all $\omega \in \Omega$.

By the Dominated Convergence Theorem, $\lim_{n\to\infty} Ee^{it\xi_n} = Ee^{it\xi}$. Thus,

$$\varphi_\xi(t) = \lim_{n\to\infty} \varphi_{\xi_n}(t) \quad \forall t \in \mathbb{R}. \qquad (1)$$

We have
$$\varphi_{\xi_n}(t) = Ee^{it\xi_n} = E\left(\prod_{k=1}^{n} e^{it\frac{\varepsilon_k}{2^k}}\right).$$

Fix $t \in \mathbb{R}$. Because $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are independent, $e^{it\frac{\varepsilon_1}{2}}, e^{it\frac{\varepsilon_2}{2^2}}, \dots, e^{it\frac{\varepsilon_n}{2^n}}$ are also independent. Thus,

$$\varphi_{\xi_n}(t) = E\left(\prod_{k=1}^{n} e^{it\frac{\varepsilon_k}{2^k}}\right) = \prod_{k=1}^{n} Ee^{it\frac{\varepsilon_k}{2^k}} = \prod_{k=1}^{n} \varphi_{\varepsilon_k}\left(\frac{t}{2^k}\right). \qquad (2)$$

Because $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ have the same distribution, $\beta_{\varepsilon_1} = \beta_{\varepsilon_2} = \dots = \beta_{\varepsilon_n}$.

$$\beta_{\varepsilon_1}(t) = E e^{it\varepsilon_1} = e^{it} P(\varepsilon_1 = 1) + e^{-it} P(\varepsilon_1 = -1)$$
$$= \frac{e^{it} + e^{-it}}{2} = \cos t.$$

Then (2) becomes
$$\beta_{\xi_n}(t) = \prod_{k=1}^{n} \beta_{\varepsilon_1}\left(\frac{t}{2^k}\right) = \prod_{k=1}^{n} \cos\left(\frac{t}{2^k}\right).$$

Multiplying both sides by $\sin\left(\frac{t}{2^n}\right)$ and using the identity $\sin x \cos x = \frac{1}{2}\sin 2x$ repeatedly, we get

$$\sin\left(\frac{t}{2^n}\right) \beta_{\xi_n}(t) = \cos\left(\frac{t}{2}\right)\cos\left(\frac{t}{2^2}\right)\cdots\cos\left(\frac{t}{2^{n-1}}\right)\cos\left(\frac{t}{2^n}\right)\sin\left(\frac{t}{2^n}\right)$$
$$= \frac{1}{2}\cos\left(\frac{t}{2}\right)\cdots\cos\left(\frac{t}{2^{n-2}}\right)\cos\left(\frac{t}{2^{n-1}}\right)\sin\left(\frac{t}{2^{n-1}}\right)$$
$$= \frac{1}{2^2}\cos\left(\frac{t}{2}\right)\cdots\cos\left(\frac{t}{2^{n-2}}\right)\sin\left(\frac{t}{2^{n-2}}\right)$$
$$= \cdots$$
$$= \frac{1}{2^{n-1}}\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)$$
$$= \frac{\sin t}{2^n}.$$

Thus, $\beta_{\xi_n}(t) = \dfrac{\sin t}{2^n \sin\left(\frac{t}{2^n}\right)}$ if $\sin\left(\frac{t}{2^n}\right) \neq 0$.

If $t \neq 0$, $\sin\left(\frac{t}{2^n}\right) \neq 0$ when n is sufficiently large. We compute $\lim_{n\to\infty} \beta_{\xi_n}(t)$ by using the identity $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$\beta_{\xi_n}(t) = \frac{\sin t}{t} \frac{t/2^n}{\sin(t/2^n)} \longrightarrow \frac{\sin t}{t} \quad \text{as } n\to\infty.$$

Then (1) gives

$$\varphi_\xi(t) = \frac{\sin t}{t} \quad \forall t \neq 0.$$

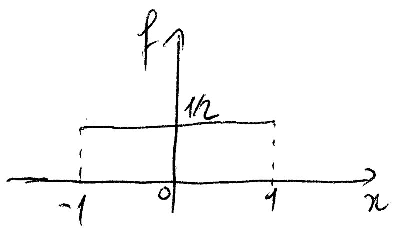
Since characteristic functions are always continuous,

$$\varphi_\xi(t) = \begin{cases} \frac{\sin t}{t} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases} \qquad (3)$$

Next, let X be a uniformly distributed random variable on $[-1,1]$.

The density function of X is

$$f(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [-1,1] \\ 0 & \text{otherwise.} \end{cases}$$



The characteristic function of X is

$$\varphi_X(t) = \int_{\mathbb{R}} e^{itx} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} e^{itx}\,dx$$

$$\underline{\underline{t \neq 0}} \quad \frac{e^{itx}}{2it}\Big|_{x=-1}^{x=1}$$

$$= \frac{e^{it} - e^{-it}}{2it} = \frac{\sin t}{t}.$$

Since characteristic functions are always continuous,

$$\varphi_X(t) = \begin{cases} \frac{\sin t}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases} \qquad (4)$$

By (3) and (4), $\varphi_\xi = \varphi_X$. This implies ξ and X induce the same probability distribution on $\mathbb{R}$. Therefore, ξ is uniformly distributed on $[-1,1]$.

(13) Let (u_n) be an equicontinuous sequence in $C([a,b])$. Suppose $\lim_{n\to\infty} u_n(t) = u(t)$ for every $t \in [a,b]$.

First, we show that u is continuous on $[a,b]$. For each $\varepsilon > 0$, there exists $\delta > 0$ such that

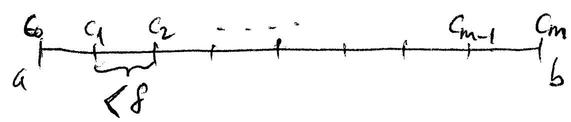
$$|u_n(t) - u_n(s)| < \frac{\varepsilon}{3} \quad \forall n \in \mathbb{N},\ \forall t, s \in [a,b],\ |t-s| < \delta.$$

For $t, s \in [a,b]$, $|t-s| < \delta$, we get

$$|u(t) - u(s)| = \lim_{n\to\infty} |u_n(t) - u_n(s)| \leq \frac{\varepsilon}{3} < \varepsilon.$$

This implies u is (uniformly) continuous on $[a,b]$.

Next, we show that (u_n) converges to u uniformly in $[a,b]$. Take an integer $m > \delta^{-1}(b-a)$. Partition the interval $[a,b]$ into m equal subintervals whose endpoints are $a = c_0 < c_1 < \dots < c_{m-1} < c_m = b$.



For each $k = 0, 1, \dots, m$, $\lim_{n\to\infty} u_n(c_k) = u(c_k)$. Thus, there exists $N_k(\varepsilon) \in \mathbb{N}$ such that $|u_n(c_k) - u(c_k)| < \frac{\varepsilon}{3}$ for all $n > N_k$. Put $N = \max\{N_0, N_1, \dots, N_m\}$. Note that N depends only on ε. Then

$$|u_n(c_k) - u(c_k)| < \frac{\varepsilon}{3} \quad \forall n > N\ \forall k = 0, 1, \dots, m.$$

The length of each interval $[c_k, c_{k+1}]$ is $\frac{b-a}{m} < \delta$. Thus, for each $t \in [a,b]$, there exist $j \in \{0, 1, \dots, m\}$ such that $|t - c_j| < \delta$. We know that

$$|u_n(t) - u_n(c_j)| < \frac{\varepsilon}{3} \quad \forall n \in \mathbb{N},$$

$$|u(t) - u(c_j)| \leq \frac{\varepsilon}{3}.$$

Thus,
$$\begin{aligned}|u_n(t) - u(t)| &\leq |u_n(t) - u_n(c_j)| + |u_n(c_j) - u(c_j)| + |u(c_j) - u(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \quad \forall n \in \mathbb{N}.\end{aligned}$$

Because this is true for all $t \in [a,b]$, we have $\sup_{t\in[a,b]} |u_n(t) - u(t)| < \varepsilon$ for all $n > N$. Therefore, (u_n) converges to u uniformly on $[a,b]$.

(14) Let (Q_n) be a sequence of probability distributions on $\mathbb{R}^d$. Let β_n be the characteristic function of Q_n. Suppose (β_n) converges pointwise to a function f which is continuous at 0. We show that (β_n) converges to f uniformly on every bounded subset of $\mathbb{R}^d$.

Suppose (β_n) is equicontinuous on $\mathbb{R}^d$, i.e.

$$\forall \varepsilon > 0, \exists \delta > 0 : \ |\beta_n(\eta + h) - \beta_n(\eta)| < \varepsilon \quad \forall n \in \mathbb{N}, \forall \eta, h \in \mathbb{R}^d, |h| < \delta.$$

Let $\varepsilon > 0$. We have

$$|\beta_n(\eta + h) - \beta_n(\eta)| < \frac{\varepsilon}{3} \quad \forall n \in \mathbb{N} \ \forall \eta, h \in \mathbb{R}^d, \ |h| < \delta' = \delta\left(\frac{\varepsilon}{3}\right).$$

Let $n \to \infty$, we get

$$|f(\eta + h) - f(\eta)| \le \frac{\varepsilon}{3} \quad \forall \eta, h \in \mathbb{R}^d, \ |h| < \delta'.$$

Let B be a nonempty bounded subset of $\mathbb{R}^d$. Since $\overline{B}$ is compact in $\mathbb{R}^d$, there exist finitely many balls $B(a_1, \delta'), B(a_2, \delta'), \dots, B(a_m, \delta')$ such that

$$B \subset B(a_1, \delta') \cup \dots \cup B(a_m, \delta').$$

Because $\lim_{n\to\infty} \beta_n(a_k) = f(a_k)$ for all $1 \le k \le m$, there exists $N \in \mathbb{N}$ such that

$$|\beta_n(a_k) - f(a_k)| < \frac{\varepsilon}{3} \quad \forall 1 \le k \le m, \ \forall n > N \qquad (1)$$

Note that N depends only on ε. Take any $\eta \in B$. There exists $j \in \{1, 2, \dots, m\}$ such that $\eta \in B(a_j, \delta')$. Thus,

$$|\beta_n(a_j) - \beta_n(\eta)| < \frac{\varepsilon}{3} \quad \forall n \in \mathbb{N}, \qquad (2)$$

$$|f(a_j)-f(\eta)| < \frac{\varepsilon}{3} \quad \forall n \in \mathbb{N}. \qquad (3)$$

From (1), (2), (3), we get

$$|\beta_n(\eta)-f(\eta)| \leq |\beta_n(\eta)-\beta_n(a_j)| + |\beta_n(a_j)-f(a_j)| + |f(a_j)-f(\eta)|$$
$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$$
$$= \varepsilon \quad \forall n > N.$$

Since this is true for all $\eta \in B$, (β_n) converges to f uniformly on B.

The remaining problem is to show that (β_n) is equicontinuous in $\mathbb{R}$.

We have

$$|\beta_n(\eta+h)-\beta_n(\eta)| = \left|\int_{\mathbb{R}^d}\left(e^{i(\eta+h,x)}-e^{i(\eta,x)}\right)Q_n(dx)\right|$$
$$\leq \int_{\mathbb{R}^d}\left|e^{i(\eta,x)}\left(e^{i(h,x)}-1\right)\right| Q_n(dx)$$
$$= \int_{\mathbb{R}^d}\left|e^{i(h,x)}-1\right| Q_n(dx)$$
$$= \underbrace{\int_{[-M,M]^d}\left|e^{i(h,x)}-1\right| Q_n(dx)}_{\{1\}} + \underbrace{\int_{\mathbb{R}^d\setminus[-M,M]^d}\left|e^{i(h,x)}-1\right| Q_n(dx)}_{\{2\}}, \qquad (4)$$

where M is any positive number.

Put $(h,x)=y \in \mathbb{R}$. Then $|e^{iy}-1| = \left|2\sin\left(\frac{y}{2}\right)e^{i\left(\frac{y}{2}+\frac{\pi}{2}\right)}\right| = 2\left|\sin\left(\frac{y}{2}\right)\right| \leq |y|$.

For $x\in[-M,M]^d$, $|x|\leq\sqrt{d}M$. Then $|e^{i(h,x)}-1| \leq |(h,x)| \leq |h||x| \leq |h|\sqrt{d}M$.

$$\{1\} \leq \int_{[-M,M]^d} |h|\sqrt{d}M\, Q_n(dx) \leq |h|\sqrt{d}M.$$

$$\{2\} \leq \int_{\mathbb{R}^d\setminus[-M,M]^d} 2\, Q_n(dx) = 2\, Q_n\big(\mathbb{R}^d \setminus [-M,M]^d\big).$$

Then (4) implies

$$|\beta_n(\eta+h)-\beta_n(\eta)| \leq \{1\}+\{2\}$$
$$= |h|\sqrt{d}\,M + 2\,Q_n(\mathbb{R}^d\setminus[-M,M]^d) \quad \forall \eta, h\in\mathbb{R}^d, \forall n\in\mathbb{N}. \quad (5)$$

Suppose the sequence (Q_n) is tight, i.e. for each $\varepsilon>0$, there exists $M=M(\varepsilon)>0$ such that $Q_n(\mathbb{R}^d\setminus[-M,M]^d)<\varepsilon$. For $h\in\mathbb{R}^d$, $|h|<\frac{\varepsilon}{\sqrt{d}M}$,

$$|\beta_n(\eta+h)-\beta_n(\eta)| \overset{(5)}{\leq} \frac{\varepsilon}{\sqrt{d}M}\sqrt{d}\,M + 2\varepsilon = 3\varepsilon \quad \forall \eta\in\mathbb{R}^d, \forall n\in\mathbb{N}.$$

Thus, (β_n) is equicontinuous.

It remains to show that (Q_n) is tight. For each $1\leq k\leq d$, we define a function $\beta_n^{(k)}:\mathbb{R}\to\mathbb{C}$, $\beta_n^{(k)}(t)=\beta_n(0,\dots,\underset{\uparrow\atop k\text{'th}}{t},\dots,0)$. We show that $\beta_n^{(k)}$ is the characteristic function of a distribution on $\mathbb{R}$. Let $C(\mathbb{R})$ be the space of all continuous bounded function from $\mathbb{R}$ to $\mathbb{R}$. Define a functional $L: C(\mathbb{R})\to\mathbb{R}$,

$$L(g) = \int_{\mathbb{R}^d} \tilde{g}(x)\, Q_n(dx)$$

where $\tilde{g}(x)=\tilde{g}(x_1,\dots,x_d):=g(x_k)$. Then L is linear and $Lg\geq 0$ if $g\geq 0$. Let (g_m) be a decreasing sequence in $C(\mathbb{R})$ that converges pointwise to 0. Then $(\tilde{g}_m)$ is a decreasing sequence in $C(\mathbb{R}^d)$ that converges pointwise to 0. Then $(\tilde{g}_1-\tilde{g}_m)$ is an increasing nonnegative sequence that converges pointwise to $\tilde{g}_1$.

By the Monotone Convergence Theorem,

$$\lim_{m\to\infty} \int_{\mathbb{R}^d} (\tilde{g}_1(x) - \tilde{g}_m(x))\, Q_n(dx) = \int_{\mathbb{R}^d} \tilde{g}_1(x)\, Q_n(dx).$$

Thus, $$L g_m = \int_{\mathbb{R}^d} \tilde{g}_m(x)\, Q_n(dx) \longrightarrow 0 \quad \text{as } m\to\infty.$$

By Riesz-Markov-Kakutani's theorem, there exists a finite measure $\mu_n^{(k)}$ on $\mathbb{R}$ such that

$$Lg = \int_{\mathbb{R}} g(x)\, \mu_n^{(k)}(dx) \quad \forall g \in C(\mathbb{R}).$$

In particular, $$\mu_n^{(k)}(\mathbb{R}) = \int_{\mathbb{R}} \mu_n^{(k)}(dx) = L(1) = \int_{\mathbb{R}^d} 1\, Q_n(dx) = Q_n(\mathbb{R}^d) = 1.$$

Moreover, $$\beta_n^{(k)}(t) = \beta_n(0,\dots,\underset{\substack{\uparrow\\ k\text{'th}}}{t},\dots,0) = \int_{\mathbb{R}^d} e^{itx_k}\, Q_n(dx) = \int_{\mathbb{R}} e^{itx_k} \mu_n^{(k)}(dx_k).$$

Thus, $\beta_n^{(k)}$ is the characteristic function of $\mu_n^{(k)}$. For each Borel set $B \subset \mathbb{R}$,

$$\mu_n^{(k)}(B) = \int_{\mathbb{R}} I_B(x_k)\, \mu_n^{(k)}(dx_k) = \int_{\mathbb{R}^d} I_{\tilde{B}}(x)\, Q_n(dx) = Q_n(\tilde{B}),$$

where $\tilde{B} = \mathbb{R}\times\cdots\times \underset{\substack{\uparrow\\ k\text{'th}}}{B} \times\cdots\times\mathbb{R}$.

Suppose that for each $k=1,2,\dots,d$, the sequence $(\mu_n^{(k)})_{n\in\mathbb{N}}$ is tight. Then for each $\varepsilon>0$, there exist $M_1, M_2, \dots, M_d > 0$ such that

$$\mu_n^{(k)}(\mathbb{R}\setminus[-M_k, M_k]) < \frac{\varepsilon}{d} \quad \forall k=1,2,\dots,d,\ \forall n\in\mathbb{N}.$$

Put $M = \max\{M_1,\dots,M_k\}$. Then

$$\mu_n^{(k)}(\mathbb{R}\setminus[-M,M]) \le \mu_n^{(k)}(\mathbb{R}\setminus[-M_k, M_k]) < \frac{\varepsilon}{d} \quad \forall k=1,\dots,d,\ \forall n\in\mathbb{N}$$

In other words,

$$Q_n(\mathbb{R}\times\cdots\times\underbrace{(\mathbb{R}\setminus[-M,M])}_{\substack{\uparrow\\ k\text{'th}}}\times\cdots\times\mathbb{R}) < \frac{\varepsilon}{d} \quad \forall k=1,2,\dots,d \quad \forall n\in\mathbb{N}$$

Then

$$\begin{aligned} Q_n(\mathbb{R}^d\setminus[-M,M]^d) &= Q_n\Big(\bigcup_{k=1}^{d} \mathbb{R}\times\cdots\times\underset{\substack{\uparrow\\ k\text{'th}}}{(\mathbb{R}\setminus[-M,M])}\times\cdots\times\mathbb{R}\Big) \\ &\leq \sum_{k=1}^{d} Q_n(\mathbb{R}\times\cdots\times\underset{\substack{\uparrow\\ k\text{'th}}}{(\mathbb{R}\setminus[-M,M])}\times\cdots\times\mathbb{R}) \\ &< \sum_{k=1}^{d}\frac{\varepsilon}{d} \\ &= \varepsilon \qquad \forall n\in\mathbb{N}. \end{aligned}$$

Hence, (Q_n) is tight.

It remains to show that each sequence $(\mu_n^{(k)})_{n\in\mathbb{N}}$ is tight. We will only deal with $k=1$ because other values of k will be treated the same way. For simplicity, we write $\tilde{\mu}_n$ instead of $\mu_n^{(1)}$, and $\tilde{\beta}_n$ instead of $\beta_n^{(1)}$. By Theorem 13, Fristedt-Gray, page 258, it suffices to show that for every $\varepsilon>0$, there exist $\delta>0$ and $N\in\mathbb{N}$ such that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}(1-\operatorname{Re}\tilde{\beta}_n(t))\,dt \leq \varepsilon \qquad \forall n>N.$$

Put $\tilde{f}:\mathbb{R}\to\mathbb{C}$, $\tilde{f}(t)=f(t,0,\dots,0)$. By the hypotheses, $\lim_{n\to\infty}\tilde{\beta}_n(t)=\tilde{f}(t)$ for all $t\in\mathbb{R}$, and $\tilde{f}$ is continuous at 0. Since $\tilde{\beta}_n(0)=1$ for all $n\in\mathbb{N}$, $\tilde{f}(0)=1$. There exists $\delta>0$ such that $|\tilde{f}(t)-1|<\varepsilon$ for all $t\in(-\delta,\delta)$.

Then

$$\lim_{n\to\infty}\frac{1}{2\delta}\int_{-\delta}^{\delta}(1-\operatorname{Re}\tilde{\beta}_n(t))\,dt = \frac{1}{2\delta}\int_{-\delta}^{\delta}(1-\operatorname{Re}\tilde{f}(t))\,dt$$

$$\leq \frac{1}{2\delta} \int_{-\delta}^{\delta} (1-(1-\varepsilon))\, dt = \varepsilon .$$

Thus, there exists $N \in \mathbb{N}$ such that $\frac{1}{2\delta} \int_{-\delta}^{\delta} (1 - \mathrm{Re}\, \tilde{\beta}_n(t))\, dt \leq \varepsilon \quad \forall n > N$.