Theory of Probability and Measure Theory – Math 8651

Take-home final exam

1) Let ν be an outer measure on Ω. Prove that if $A, B \in \Sigma$ and $AB = \emptyset$ the $\nu(X(A \cup B)) = \nu(XA) + \nu(XB)$. In particular, when $X = \Omega$, $\nu(A \cup B) = \nu(A) + \nu(B)$.

2) Let F be a nondecreasing finite left-continuous function on $\mathbb{R}$. Define $\mathcal{E}$ as the collection of finite unions of disjoint intervals of the type $(a, b]$ with $-\infty \le a < b \le \infty$ as in the lecture notes. If $A \in \mathcal{E}$ is given by

$$A = \bigcup_{i=1}^{n} (a_i, b_i]$$

with disjoint $(a_i, b_i]$ then set

$$R(A) = \bigcup_{i=1}^{n} R((a_i, b_i]),$$

where $R((a, b]) = F(b) - F(a)$ if $a \le b$, $F(\infty) = \lim_{x\to\infty} F(x)$, $F(-\infty) = \lim_{x\to-\infty} F(x)$.

Show that R is an additive but not a σ-additive function on $\mathcal{E}$ if F has at least one point of discontinuity.

3) We know that $\beta(t) := (1 - |t|)_+$ is the characteristic of a distribution on $\mathbb{R}$. One can scale β and this will preserve the property. Prove that, if $a_1, ..., a_n$ and $c_1, ..., c_n$ are positive numbers with

$$\sum_{k=1}^{n} a_k = 1,$$

then

$$\sum_{k=1}^{n} a_k \beta(c_k t)$$

is a characteristic function.

4) Let numbers b_k, a_k^n and a_k be given for $n, k = 1, 2, ...$ Assume that $|a_k^n| \le b_k$ and $a_k^n \to a_k$ as $n \to \infty$ for any k. Also assume $\sum_k b_k < \infty$. Prove that

$$\lim_{n\to\infty} \sum_k a_k^n = \sum_k a_k.$$

5) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f \geq 0$ defined on Ω be measurable. Assume that

$$\int\limits_{\Omega} f(x)\mu(dx) < \infty.$$

Prove that if $A_n\mathcal{F}$ are such that $\mu(A_n) \to 0$ the

$$\int\limits_{\Omega} f(x)I_{A_n}(x)\mu(dx) \to 0.$$

6) Let $\{r_1, r_2, ...\}$ be the set of all rational numbers on (0,1) and X be a random variable such that $P(X = r_n) = cn^{-2}$, where the constant c is chosen so that

$$c\sum_{n=1}^{\infty} n^{-2} = 1.$$

Prove that the distribution function of X is discontinuous at any rational point in (0,1) and is continuous elsewhere.

7) Let X_n, $n = 1, 2, ...$, be pairwise independent random variables such that $P(X_n \in (a, b)) = (1 - 2^{-n})(b - a)$ for $0 \leq a \leq b \leq 1$ and $P(X_n = 2^n) = 2^{-n}$. Show that there exists a constant c such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = c \quad \text{a.s.}$$

and ind this constant.

8) If g is any function on a Polish space X, denote by Δ_g the set of points of discontinuity of g. Prove that Δ_g is a Borel set.

9) We know that if Γ_n is a decreasing sequence of closed sets in a Polish space such that $\text{diam}\Gamma_n \to 0$ as $n \to \infty$, then $\cap_n\Gamma_n$ is nonempty and consists of only one point.

Let $f(x) = \sin(1/x)$. Consider the sets

$$\Gamma_n = \left\{ x \in C([0, 1]) : \sup_{1\geq t\geq 1/n} |x(t) - f(t)| \leq \frac{1}{2} \text{ and } \|x\| \leq 2 \right\}$$

for $n \geq 1$. Show that the sets Γ_n are bounded, closed, nested, and $\cap_n\Gamma_n = \emptyset$.

10) (Problem 14.31 in the textbook) Let probability distributions Q and Q_n on a Polish space have densities f and f_n with respect to a common σ-finte measure μ. Assume that $f_n \to f$ μ-a.e. Prove that $Q_n \Rightarrow Q$.

11) (Problem 18.9 in the textbook) We know that the space $C([0,1])$ of real-valued continuous functions on [0,1] provided with the metric

$$\rho(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|$$

is a Polish space. Let X_n be $C([0,1])$-valued random variable converging to X in distribution. Prove that $\max_{[0,1]} X_n(t)$ converges to $\max_{[0,1]} X(t)$ in distribution.

12) (Similar to Problem 13.13 in the textbook) By using characteristic functions, prove that if ϵ_n, $n = 1, 2, ...$ are iid with $P(\epsilon_1 = 1) = P(\epsilon_1 = -1) = 1/2$ then

$$\xi = \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}$$

is uniformly distributed on [-1,1].

13) Prove that if $u_n(t)$, $n = 1, 2, ...$ are equicontinuous on $[a,b]$ and converge to $u(t)$ for each $t \in [a,b]$, then u is continuous on $[a,b]$ and the convergence is uniform.

14) Prove that if Q_n, $n \geq 1$, is a sequence of distributions on $\mathbb{R}^d$ such that the sequence of the corresponding characteristic functions converges pointwise to a function, say f, which is continuous at zero, then the convergence of the characteristic functions to f is uniform on every bounded subset of $\mathbb{R}^d$.