The final examination is scheduled on Wednesday, Dec. 17, 2014, same room, 8:00-10:00 am. Here are 14 problems, 7 of which will be given on final. If you solve all of them, you can just hand in your solutions of the seven chosen or all solutions before 10:00 am on Wednesday, Dec. 17, 2014 to me personally or put under the door of my office. In the latter case I want to have an email notification that you did that.

1. Let ν be an outer measure on Ω. prove that if $A, B \in \Sigma$ and $AB = \emptyset$, then $\nu(X(A \cup B)) = \nu(XA) + \nu(XB)$. In particular, when $X = \Omega$, $\nu(A \cup B) = \nu(A) + \nu(B)$.

2. Let F be a nondecreasing finite **left-continuous** function on $\mathbb{R}$. Define $\mathcal{E}$ as the collection of finite unions of disjoint intervals of the type $(a, b]$ with $-\infty \le a < b \le \infty$ as in the lecture notes. If $A \in \mathcal{E}$ is given by

$$A = \bigcup_{i=1}^{n} (a_i, b_i]$$

with dijoint $(a_i, b_i]$ then set

$$R(A) = \sum_{i=1}^{n} R((a_i, b_i]),$$

where $R((a, b]) = F(b) - F(a)$ if $a \le b$, $F(\infty) = \lim_{x\to\infty} F(x)$, $F(-\infty) = \lim_{x\to-\infty} F(x)$.

Show that R is an additive but not a σ-additive function on $\mathcal{E}$ if F has at least one point of discontinuity.

3. We know that $\beta(t) := (1 - |t|)_+$ is the characteristic function of a distribution on $\mathbb{R}$. One can scale β and this will preserve the property. Prove that, if $a_1, ..., a_n$ and $c_1, ..., c_n$ are positive numbers with

$$\sum_{k=1}^{n} a_k = 1,$$

then

$$\sum_{k=1}^{n} a_k \beta(c_k t)$$

is a characteristic function. By using this prove the following result of Polya: If $\gamma(t) \ge 0$ is an even, continuous function on $\mathbb{R}$ such that $\gamma(t)$ is convex and decreasing on $[0, \infty)$ and $\gamma(0) = 1$, then γ is the characteristic function of a probability distribution. (Hint: Approximate γ with broken lines and use the first part of the problem.)

4. Let numbers b_k, a_k^n and a_k be given for $n, k = 1, 2, ...$ Assume that $|a_k^n| \le b_k$ and $a_k^n \to a_k$ as $n \to \infty$ for any k. Also assume $\sum_k b_k < \infty$. Prove

then that

$$\lim_{n\to\infty}\sum_k a_k^n=\sum_k a_k.$$

5. Let $(\Omega,\mathcal{F},\mu)$ be a measure space and let $f\geq 0$ defined on Ω be measurable. Assume that

$$\int_\Omega f(x)\,\mu(dx)<\infty$$

and prove that if $A_n\in\mathcal{F}$ are such that $\mu(A_n)\to 0$, then

$$\int_\Omega f(x)I_{A_n}(x)\,\mu(dx)\to 0.$$

6. Let $\{r_1,r_2,...\}$ be the set of all rational numbers on $(0,1)$ and X be a random variable such that $P(X=r_n)=cn^{-2}$, where the constant c is chosen so that

$$c\sum_{n=1}^\infty n^{-2}=1.$$

Prove that the distribution function of X is discontinuous at any rational point in $(0,1)$ and is continuous elsewhere.

7. Let X_n, $n=1,2,...$, be pairwise independent random variables such that $P(X_n\in(a,b))=(1-2^{-n})(b-a)$ for $0\leq a\leq b\leq 1$ and $P(X_n=2^n)=2^{-n}$. Show that there exists a constant c such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n X_k=c\quad\text{(a.s.)}$$

and find this constant.

8. If g is any function on a Polish sopace X denote by Δ_g the set of points of discontinuity of g and prove that Δ_g is a Borel set. (Hint: Introduce

$$M_n(x)=\sup\{|f(y)-f(z)|:\rho(y,x),\rho(z,x)<1/n\},$$

$\Delta_{n,m}=\{x:M_n(x)>1/m\}$ and prove that the sets $\Delta_{n,m}$ are open and the set of discontinuity of f is

$$\bigcup_{m=1}^\infty\bigcap_{n=1}^\infty\Delta_{n,m}.)$$

9. We know that if Γ_n is a decreasing sequence of closed sets in a Polish space such that $\operatorname{diam}\Gamma_n\to 0$ as $n\to\infty$, then $\cap_n\Gamma_n$ is nonempty and consists of only one point.

Let $f(x)=\sin(1/x)$ and in $C([0,1])$ consider the sets

$$\Gamma_n=\{x(\cdot)\in C([0,1]):\sup_{1\geq t\geq 1/n}|x(t)-f(t)|\leq 1/2\},\sup_{[0,1]}|x(t)|\leq 2\}$$

for $n\geq 1$. Show that the sets Γ_n are bounded, closed, nested, and $\cap_n\Gamma_n=\emptyset$.

10. (Problem 14.31) Let probability distributions Q and Q_n on a Polish space have densities f and f_n with respect to a common σ-finite measure μ. Assume that $f_n \to f$ μ-a.e.. Prove that $Q_n \Longrightarrow Q$.

11. (Problem 18.9) We know that the space $C[0,1]$ of real-valued bounded continuous functions on $[0,1]$ provided with the metric

$$\rho(f,g) = \max_{t \in [0,1]} |f(t) - g(t)|$$

is a Polish space. Let X_n be $C[0,1]$-valued random variable converging to X in distribution. Prove that $\max_{[0,1]} X_n(t)$ converge to $\max_{[0,1]} X(t)$ in distribution.

12. ($\sim$ Problem 13.13) By using characteristic functions prove that, if $\varepsilon_n, n = 1, 2, ...$, are iid with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2$, then

$$\xi := \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$$

is uniformly distributed on $[-1, 1]$. (Hint: Use that $2 \cos x \sin x = \sin(2x)$.)

13. Prove that if $u_n(t)$, $n = 1, 2, ...$, are equicontinuous on $[a,b]$ and converge to $u(t)$ for each $t \in [a,b]$, then u is continuous on $[a,b]$ and the convergence is uniform.

14. Prove that if Q_n, $n \geq 1$, is a sequence of distributions on $\mathbb{R}^d$ such that the sequence of the corresponding characteristic functions converges pointwise to a function, say f, which is continuous at zero, then the convergence of the characteristic functions to f is uniform on any bounded subset of $\mathbb{R}^d$.