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Math 8651: Theory of Probability

Homework #1

① Problem 1, Fristedt-Gray, page 5.

Consider the experiment consisting of n successive flips of a coin:

$$\Omega = \{(\omega_1, \dots, \omega_n) : \omega_i \in \{0,1\}, 1 \le i \le n\}.$$

Here we denoted 'head' by 1 and 'tail' by 0. Let $\mathcal{F}$ be the collection of all subsets of Ω. We know that $\#\mathcal{F} = 2^{\#\Omega}$. Because each ω_i can assume two values, $\#\Omega = 2^n$. Thus, $\#\mathcal{F} = 2^{2^n}$.

② Problem 3, Fristedt-Gray, page 5.

Consider the experiment in Problem ① with $n \geq 2$. Let A be the event that the first two flips are both 0. Then

$$A = \{(0,0,\omega_3, \dots, \omega_n) : \omega_i \in \{0,1\}, 3 \le i \le n\}.$$

Thus, $\#A = 2^{n-2}$. The probability of A is

$$P(A) = \frac{\#A}{\#\Omega} = \frac{2^{n-2}}{2^n} = \frac{1}{4}.$$

This number is independent of n.

We can generalize this result as follows: let $r_1 < \dots < r_k$ and $s_1 < \dots < s_\ell$ be pairwise distinct natural numbers. Let A be the event that the r_i'th flip is head for all $1 \le i \le k$, and the s_j'th flip is tail for all $1 \le j \le \ell$. As long as $n \geq \max\{r_k, s_\ell\}$, the probability of A is independent of n.

Indeed, an n-dimensional vector in A consists of number 1 at positions $r_1, r_2, \dots, r_k$ and number 0 at position $s_1, \dots, s_\ell$. Each of the rest $n-k-\ell$ positions can be either 0 or 1. Thus, $\#A = 2^{n-k-\ell}$.

$$P(A) = \frac{\#A}{\#\Omega} = \frac{2^{n-k-\ell}}{2^n} = 2^{-k-\ell},$$

which is independent of n.

(3) Problem 4, Fristedt-Gray, page 6.

Consider the experiment consisting of an infinite sequence of coin flips:

$$\Omega = \{(\omega_1, \omega_2, \dots) : \omega_i \in \{0,1\} \ \forall i \in \mathbb{N}\}.$$

For each $j \in \mathbb{N}$, the event that the first head occurs on the flip number j occurs if and only if $(\omega_1, \dots, \omega_{j-1}, \omega_j) = (\underbrace{0, \dots, 0}_{j-1}, 1)$. Put

$$A_j = \{(\underbrace{0, \dots, 0}_{j-1}, 1)\}.$$

By the definition (1.1) in the textbook,

$$P(\{(\omega_1, \omega_2, \dots) : (\omega_1, \dots, \omega_j) \in A_j\}) = \frac{\#A_j}{2^j} = 2^{-j}.$$

(4) Problem 5, Fristedt-Gray, page 6.

Let $\mathcal{F}$ be a σ-field of subsets of Ω, i.e.

(i) $\phi \in \mathcal{F}$,

(ii) if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$,

(iii) if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Because $\phi \in \mathcal{F}$, $\Omega = \Omega \setminus \phi \in \mathcal{F}$. Let (A_n) be a sequence in $\mathcal{F}$. We show that $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.

By (ii), this is equivalent to showing that $\Omega \setminus (\bigcap_{n=1}^{\infty} A_n) \in \mathcal{F}$. We have

$$\Omega \setminus \left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} (\Omega \setminus A_n). \qquad (1)$$

Since $A_n \in \mathcal{F}$, $\Omega \setminus A_n \in \mathcal{F}$. By (iii), the set on the right hand side of (1) is in $\mathcal{F}$. Thus, $\Omega \setminus (\bigcap_{n=1}^{\infty} A_n) \in \mathcal{F}$.

A consequence of the above property is that $\mathcal{F}$ is closed under finite intersections. Indeed, by taking $A_{n+1} = A_{n+2} = \cdots = \Omega$, we get

$$\bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{\infty} A_k \in \mathcal{F},$$

provided that $A_1, A_2, \ldots, A_n \in \mathcal{F}$.

Now let $A, B \in \mathcal{F}$. We show that $A \setminus B \in \mathcal{F}$:

$$A \setminus B = A \cap \underbrace{(\Omega \setminus B)}_{\in \mathcal{F} \text{ by (ii)}} \quad \in \mathcal{F}.$$

(5) Additional problem A.

Consider the experiment consisting of n flips of a fair coin:

$$\Omega = \{(\omega_1, \ldots, \omega_n) : \omega_i \in \{0,1\} \ \forall i\},$$

where ones denote 'head' and zeros denote 'tail'. Let $\mathcal{F}$ be the collection of all subsets of Ω. The event that the number of heads is even is

$$A = \left\{(\omega_1, \ldots, \omega_n) \in \Omega : \sum_{i=1}^{n} \omega_i \text{ is even}\right\}.$$

For each $0 \le k \le n$, we put

$$A_k = \left\{(\omega_1, \ldots, \omega_n) \in \Omega : \sum_{i=1}^{n} \omega_i = k\right\}.$$

Then $A_0, A_1, \ldots, A_n$ are pairwise disjoint and $A = \bigcup_{k=0}^{[\frac{n}{2}]} A_{2k}$, where

$\lfloor \frac{n}{2} \rfloor$ denotes the largest integer less than or equal to $\frac{n}{2}$. The probability of A is

$$P(A) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P(A_{2k}) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\# A_{2k}}{\# \Omega} = 2^{-n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k},$$

where

$$\binom{n}{2k} = \frac{n!}{(n-2k)!(2k)!}.$$

⑥ Additional problem B.

Consider the experiment consisting of an infinite sequence of coin flips:

$$\Omega = \{(\omega_1, \omega_2, \dots) : \omega_i \in \{0,1\} \ \forall i \in \mathbb{N}\},$$

where ones denote 'head' and 'zeros' denote 'tail'. Let A be the event that the number of flips to get the first head is even. For each $k \in \mathbb{N}$, we put

$$A_k = \{(\omega_1, \omega_2, \dots) \in \Omega : (\omega_1, \dots, \omega_{2k-1}, \omega_{2k}) = (\underbrace{0, \dots, 0}_{2k-1}, 1)\}.$$

Then $A_1, A_2, \dots$ are pairwise disjoint and $A = \bigcup_{k=0}^{\infty} A_k$. Thus,

$$P(A) = \sum_{k=1}^{\infty} P(A_k).$$

By Definition (1.1), page 6 in the textbook, $P(A_k) = \frac{1}{2^{2k}}$. Thus,

$$P(A) = \sum_{k=1}^{\infty} 2^{-2k} = \frac{2^{-2}}{1-2^{-2}} = \frac{1}{3}.$$

⑦ Additional problem C.

$$\mathcal{E} = \{(r, \infty) : r \in \mathbb{Q}\}.$$

We show that $\sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R})$, where $\sigma(\mathcal{E})$ is the smallest σ-field containing $\mathcal{E}$ and $\mathcal{B}(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$. Let τ be the usual topology on $\mathbb{R}$.

By definition, $\mathcal{B}(\mathbb{R}) = \sigma(\tau)$. Because $\mathcal{E} \subset \tau$, $\sigma(\mathcal{E}) \subset \sigma(\tau)$.

Take any set $I \in \tau$. We show that $I \in \sigma(\mathcal{E})$. We know that I is an at-most countable union of disjoint open intervals in $\mathbb{R}$, each of which is a connected component of I. To show that $I \in \sigma(\mathcal{E})$, it suffices to show that each open interval belongs to $\sigma(\mathcal{E})$. An open interval in $\mathbb{R}$ is of one of the following types: $(a,b), (a,\infty), (-\infty, a), (-\infty, \infty)$.

For each $a \in \mathbb{R}$, there exists a decreasing sequence of rational numbers (r_n) such that $\lim r_n = a$. Then $(a,\infty) = \bigcup_{n=1}^{\infty} (r_n, \infty) \in \sigma(\mathcal{E})$. Also,

$$(-\infty, \infty) = \bigcup_{n=1}^{\infty} (-n, \infty) \in \sigma(\mathcal{E}).$$

Thus, $(-\infty, a] = (-\infty,\infty) \setminus (a,\infty) \in \sigma(\mathcal{E})$.

For each $a \in \mathbb{R}$,

$$(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - \tfrac{1}{n}] \in \sigma(\mathcal{E}).$$

For $a, b \in \mathbb{R}$, $(a,b) = (a,\infty) \cap (-\infty, b) \in \sigma(\mathcal{E})$.

Thus, we have showed that $\tau \subset \sigma(\mathcal{E})$. Since $\sigma(\tau)$ is the smallest σ-field containing τ, $\sigma(\tau) \subset \sigma(\mathcal{E})$. Therefore, $\sigma(\mathcal{E}) = \sigma(\tau) = \mathcal{B}(\mathbb{R})$.