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Math 8651: Theory of Probability
Homework #2

(1) Problem 2, Fristedt-Gray, page 86

Let Ω be a set and $\mathcal{A}$ be a Sierpinski class of subsets of Ω which is closed under pairwise intersections and contains Ω. We show that $\mathcal{A}$ is a σ-field.

Because $\mathcal{A}$ is closed under proper set differences, if $A \in \mathcal{A}$ then $\Omega \setminus A \in \mathcal{A}$. Let (A_n) be a sequence in $\mathcal{A}$, not necessarily pairwise disjoint, we show that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. For each $n \in \mathbb{N}$, put $B_n = \bigcup_{k=1}^{n} A_k$. Then (B_n) is an increasing sequence and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{k=1}^{\infty} A_k$. We show by induction in $n \in \mathbb{N}$ that $B_n \in \mathcal{A}$. For $n=1$, $B_1 = A_1 \in \mathcal{A}$. Suppose that $B_1, \dots, B_{n-1} \in \mathcal{A}$ for some $n \geq 2$. Then

$$B_n = \bigcup_{k=1}^{n} A_k = B_{n-1} \cup A_n = \Omega \setminus (\Omega \setminus (B_{n-1} \cup A_n))$$
$$= \Omega \setminus \underbrace{\Big(\underbrace{(\Omega \setminus B_{n-1})}_{\in \mathcal{A}} \cap \underbrace{(\Omega \setminus A_n)}_{\in \mathcal{A}} \Big)}_{\in \mathcal{A}}$$

(since $\mathcal{A}$ is closed under pairwise intersection)

Thus, $B_n \in \mathcal{A}$. We have showed that $B_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Because $\mathcal{A}$ is closed under limits of increasing sequences of sets,

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}.$$

(2) Problem 8, Fristedt-Gray, page 87.

Let $R: \mathcal{E} \to [0,\infty]$ be a countably additive function on a field $\mathcal{E}$ of subsets of Ω. Let (A_n) be a sequence in $\mathcal{E}$ such that $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{E}$.

We show that $R(A) \leq \sum_{n=1}^{\infty} R(A_n)$.

Put $B_1 = A_1$ and $B_n = A_n \setminus (B_1 \cup \dots \cup B_{n-1})$ for all $n \in \mathbb{N}, n \geq 2$. Then $B_1, B_2, B_3, \dots$ are pairwise disjoint. Moreover, $\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k$ for all $n \in \mathbb{N}$.

Thus,
$$B_n = A_n \setminus \underbrace{(A_1 \cup \dots \cup A_{n-1})}_{\in \mathcal{E}} \in \mathcal{E}.$$

Since R is countably additive, $R\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} R(B_n)$. $\quad (1)$

$$\text{LHS}(1) = R\left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} B_k\right) = R\left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} A_k\right) = R\left(\bigcup_{k=1}^{\infty} A_k\right).$$

Thus, (1) becomes
$$R\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} R(B_n). \qquad (2)$$

Since $B_n \subset A_n$, $R(A_n) = R(B_n) + R(A_n \setminus B_n) \geq R(B_n)$. Thus, (2) implies
$$R\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} R(A_n).$$

3 Problem 14, Fristedt-Gray, page 93.

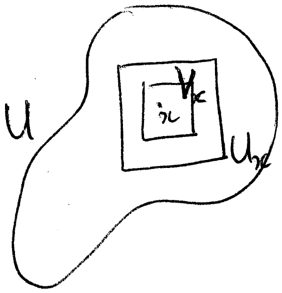
Let $\mathcal{B}$ be the Borel σ-field on $\mathbb{R}^d$ and Q be a probability measure on $(\mathbb{R}^d, \mathcal{B})$.

A set $B \subset \mathbb{R}^d$ is called regular for Q if for each $\varepsilon > 0$ there exist a compact set K and an open set O in $\mathbb{R}^d$ such that $K \subset B \subset O$ and $Q(O \setminus K) < \varepsilon$.

We show that every Borel set is regular.

Put $\mathcal{C} = \{(a_1 - r_1, a_1 + r_1) \times \dots \times (a_d - r_d, a_d + r_d) : r_i \in \mathbb{Q}, r_i > 0, a_i \in \mathbb{Q} \ \forall 1 \leq i \leq d\} \cup \{\emptyset\}$.

First we show that $\mathcal{B} = \sigma(\mathcal{A})$. Because $\mathcal{A} \subset \mathcal{B}$, $\sigma(\mathcal{A}) \subset \mathcal{B}$. Because $\mathcal{B}$ is generated by open subsets of $\mathbb{R}^d$, it suffices to show that each open subset of $\mathbb{R}^d$ belongs to $\sigma(\mathcal{A})$. Let U be an open subset of $\mathbb{R}^d$. For each $x \in U$, there exists a cube U_x centered at x and contained in U.



Write $x = (x_1, \dots, x_d)$ and $U_x = (x_1 - \delta, x_1 + \delta) \times \dots \times (x_d - \delta, x_d + \delta)$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $r \in \mathbb{Q}$ such that $0 < r < \frac{\delta}{2}$ and $a_1, a_2, \dots, a_d \in \mathbb{Q}$ such that $|a_i - x_i| < r$ for all $1 \le i \le d$. Then

$$x \in V_x := (a_1 - r, a_1 + r) \times \dots \times (a_d - r, a_d + r) \subset U_x \subset U.$$

We have $$\bigcup_{x \in U} V_x \subset U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} V_x.$$

Thus, $U = \bigcup_{x \in U} V_x$. Because $V_x \in \mathcal{A}$, which is a countable family, the union over $x \in U$ can be reduced to a countable union. Thus $U \in \sigma(\mathcal{A})$.

We have showed that $\mathcal{B} = \sigma(\mathcal{A})$.

Next, denote by $\mathcal{C}$ the family of all regular Borel sets in $\mathbb{R}^d$. We show that $\mathcal{C} = \mathcal{B}$. That is to show $\sigma(\mathcal{A}) \subset \mathcal{C}$. Because the intersection of two rectangular boxes is another rectangular box or the empty set, $\mathcal{A}$ is closed under pairwise unions. By the Sierpinski Class theorem, we need to show the following.

(i) $\mathcal{A} \subset \mathcal{C}$,

(ii) $\mathbb{R}^d \in \mathcal{C}$

(iii) If $A_1, A_2 \in \mathcal{C}$ and $A_1 \subset A_2$ then $A_2 \setminus A_1 \in \mathcal{C}$.

(iv) If $A_1 \subset A_2 \subset A_3 \subset \cdots$ is an increasing sequence in $\mathcal{C}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

Proof of (i) Take a rectangular box $U = (a_1 - r_1, a_1 + r_1) \times \cdots \times (a_d - r_d, a_d + r_d) \in \mathcal{A}$.

For each $n \in \mathbb{N}$, we put $K_n = [a_1 - r_1 + \frac{1}{n}, a_1 + r_1 - \frac{1}{n}] \times \cdots \times [a_d - r_d + \frac{1}{n}, a_d + r_d - \frac{1}{n}]$.

The sequence (K_n) is increasing and $\bigcup_{n=1}^{\infty} K_n = U$. By the continuity of measure Q, $\lim_{n\to\infty} Q(K_n) = Q(U)$. Since Q is a finite measure,

$$\lim_{n\to\infty} Q(U \setminus K_n) = Q(U) - \lim_{n\to\infty} Q(K_n) = 0.$$

Because K_n is compact, U is open in $\mathbb{R}^d$, $K_n \subset U \subset U$ and $Q(U \setminus K_n) \to 0$, we conclude that U is regular for Q. In other words, $U \in \mathcal{C}$.

Proof of (ii) Put $K_n = [-n, n]^d$ for all $n \in \mathbb{N}$. Then (K_n) is an increasing sequence and $\bigcup_{n=1}^{\infty} K_n = \mathbb{R}^d$. By the continuity of Q, $\lim_{n\to\infty} Q(K_n) = Q(\mathbb{R}^d) = 1$.

Thus, $\lim_{n\to\infty} Q(\mathbb{R}^d \setminus K_n) = Q(\mathbb{R}^d) - \lim_{n\to\infty} Q(K_n) = 0$.

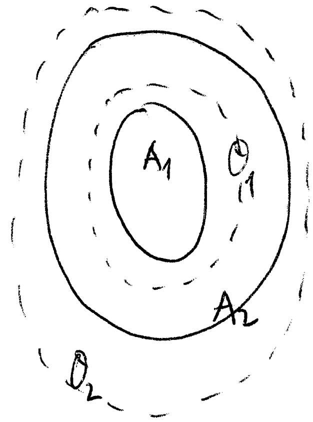
Because K_n is compact, $\mathbb{R}^d$ is open in $\mathbb{R}^d$, $K_n \subset \mathbb{R}^d \subset \mathbb{R}^d$ and $\lim_{n\to\infty} Q(\mathbb{R}^d \setminus K_n) = 0$, we conclude that $\mathbb{R}^d$ is regular for Q. In other words, $\mathbb{R}^d \in \mathcal{C}$.

Proof of (iii) For each $\varepsilon > 0$, we find a compact set K and an open set O in $\mathbb{R}^d$ such that $K \subset A_2 \setminus A_1 \subset O$ and $Q(O \setminus K) < \varepsilon$.

Because $A_1, A_2 \in \mathcal{C}$, there exist compact sets K_1, K_2 and open sets O_1, O_2

in $\mathbb{R}^d$ such that $K_i \subset A_i \subset O_i$ and $Q(O_i \setminus K_i) < \frac{\varepsilon}{2}$ for $i = 1, 2$.

Put $K = K_2 \setminus O_1$ and $O = O_2 \setminus K_1$. Then $K = K_2 \cap \underbrace{(\mathbb{R}^d \setminus O_1)}_{\text{closed}}$ is a closed



subset of K_2, and thus compact; $O_2 = O_2 \cap \underbrace{(\mathbb{R}^d \setminus K_1)}_{\text{open}}$ is open in $\mathbb{R}^d$. Moreover,

$$A_2 \setminus A_1 = A_2 \cap (\mathbb{R}^d \setminus A_1) \subset O_2 \cap (\mathbb{R}^d \setminus K_1) = O_2 \setminus K_1 = O,$$

$$A_2 \setminus A_1 = A_2 \cap (\mathbb{R}^d \setminus A_1) \supset K_2 \cap (\mathbb{R}^d \setminus O_1) = K_2 \setminus O_1 = K.$$

Finally,

$$\begin{aligned} Q(O \setminus K) &= Q(O) - Q(K) \\ &= \underbrace{Q(O_2 \setminus K_1)}_{\substack{= Q(O_2) - Q(K_1) \\ \text{because } K_1 \subset A_1 \subset A_2 \subset O_2}} - \underbrace{Q(K_2 \setminus O_1)}_{\geqslant Q(K_2) - Q(O_1)} \\ &\leqslant \big(Q(O_2) - Q(K_1)\big) - \big(Q(K_2) - Q(O_1)\big) \\ &= \big(Q(O_2) - Q(K_2)\big) + \big(Q(O_1) - Q(K_1)\big) \\ &= Q(O_2 \setminus K_2) + Q(O_1 \setminus K_1) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

<u>Proof of (iv)</u> For each $\varepsilon > 0$, we find a compact set K and an open set O in $\mathbb{R}^d$ such that $K \subset \bigcup_{n=1}^{\infty} A_n \subset O$ and $Q(O \setminus K) < \varepsilon$. Because each $A_n \in \mathcal{C}$, there exist a compact set K_n and an open set O_n in $\mathbb{R}^d$ such that $K_n \subset A_n \subset O_n$ and $Q(O_n \setminus K_n) < \varepsilon 2^{-n-1}$. Put

$$\mathcal{O} = \bigcup_{n=1}^{\infty} \mathcal{O}_n, \quad A = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \tilde{K} = \bigcup_{n=1}^{\infty} K_n.$$

Then $\mathcal{O}$ is an open subset of $\mathbb{R}^d$ and $\tilde{K} \subset A \subset \mathcal{O}$. We have

$$\mathcal{O} \setminus \tilde{K} = \left(\bigcup_{n=1}^{\infty} \mathcal{O}_n\right) \setminus \left(\bigcup_{n=1}^{\infty} K_n\right) \subset \bigcup_{n=1}^{\infty} (\mathcal{O}_n \setminus K_n).$$

Thus,
$$Q(\mathcal{O} \setminus \tilde{K}) \leq Q\left(\bigcup_{n=1}^{\infty} (\mathcal{O}_n \setminus K_n)\right)$$
$$\leq \sum_{n=1}^{\infty} Q(\mathcal{O}_n \setminus K_n) \quad (\text{by Problem 2})$$
$$< \sum_{n=1}^{\infty} \varepsilon 2^{-n-1} = \frac{\varepsilon}{2}.$$

For each $m \in \mathbb{N}$, put $\tilde{K}_m = \bigcup_{n=1}^{m} K_n$. Then $\tilde{K}_m \subset \tilde{K} \subset A \subset \mathcal{O}$ and $\tilde{K}_m$ is compact. Since $(\tilde{K}_m)$ is an increasing sequence and $\bigcup_{m=1}^{\infty} \tilde{K}_m = \tilde{K}$, we have $\lim_{m\to\infty} Q(\tilde{K}_m) = Q(\tilde{K})$. In other words, $\lim_{m\to\infty} Q(\tilde{K} \setminus \tilde{K}_m) = 0$.

There exists $m_0 \in \mathbb{N}$ such that $Q(\tilde{K} \setminus \tilde{K}_{m_0}) < \frac{\varepsilon}{2}$. Put $K = \tilde{K}_{m_0}$. Then K is compact, $K \subset \tilde{K} \subset A \subset \mathcal{O}$ and

$$Q(\mathcal{O} \setminus K) = Q((\mathcal{O} \setminus \tilde{K}) \cup (\tilde{K} \setminus K)) = Q(\mathcal{O} \setminus \tilde{K}) + Q(\tilde{K} \setminus K)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon.$$

4 Additional problem A.

~~We need~~ Let $\mathcal{A}$ be a family of subsets of Ω. Suppose that $\mathcal{A}$ is both a π-system and a λ-system. We show that $\mathcal{A}$ is a σ-field.

All we need to show is that $\mathcal{A}$ is closed under countable unions (not necessarily pairwise disjoint). Let (B_n) be a sequence in $\mathcal{A}$. Define a sequence (A_n) as follows.

$$\begin{aligned} A_1 &= B_1 \\ A_2 &= B_2 \setminus B_1 \\ A_3 &= B_3 \setminus (B_1 \cup B_2) \\ &\dots \\ A_n &= B_n \setminus (B_1 \cup B_2 \cup \dots \cup B_{n-1}) \\ &\dots \end{aligned}$$

Then $\bigcup_{i=1}^{k} A_i = \bigcup_{i=1}^{k} B_i$ and $A_i \cap A_j = \phi$ if $i \neq j$. We show by induction in $n \in \mathbb{N}$ that $A_n \in \mathcal{A}$. First, $A_1 = B_1 \in \mathcal{A}$ and $A_2 = B_2 \setminus B_1 = \underbrace{B_2 \cap \underbrace{(\Omega \setminus B_1)}_{\in \mathcal{A}}}_{\in \mathcal{A}} \in \mathcal{A}$.

Suppose that $A_1, A_2, \dots, A_{n-1} \in \mathcal{A}$ for some $n \geq 2$. Then

$$A_n = B_n \setminus \left(\bigcup_{k=1}^{n-1} B_k\right) = B_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right) = B_n \cap \underbrace{\left(\Omega \setminus \underbrace{\left(\bigcup_{k=1}^{n-1} A_k\right)}_{\in \mathcal{A} \ (\lambda\text{-system})}\right)}_{\in \mathcal{A} \ (\lambda\text{-system})} \in \mathcal{A}.$$

Now that $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ because $\mathcal{A}$ is a λ-system

Thus, $\bigcup_{k=1}^{\infty} B_k = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} B_k = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$.

⑤ Additional problem B.

Let $\mu, \nu : \mathcal{F} \to [0, \infty]$ be two σ-finite measures on a measurable space $(\Omega, \mathcal{F})$. Suppose that $\mathcal{F} = \sigma(\mathcal{E})$ and $\mu = \nu$ on $\mathcal{E}$, where $\mathcal{E}$ is a (nonempty) π-system. We are asked to show that $\mu = \nu$ on $\mathcal{F}$.

We will give a proof for the case <u>μ is σ-finite on $\mathcal{E}$</u>, i.e. there exists an increasing sequence (Ω_n) in $\mathcal{E}$ such that

(i) $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$,

(ii) $\mu(\Omega_n) < \infty$.

Then we point out a counterexample in case μ is not σ-finite on $\mathcal{E}$.

Now assume that μ is σ-finite on $\mathcal{E}$. Let (Ω_n) be the sequence in $\mathcal{E}$ as described above. For each $n \in \mathbb{N}$, we put $\mathcal{F}_n = \{A \in \mathcal{F} : \mu(A \cap \Omega_n) = \nu(A \cap \Omega_n)\}$. Because $\mathcal{E}$ is a π-system, $A \cap \Omega_n \in \mathcal{E}$ for all $A \in \mathcal{E}$. Thus, $\mathcal{E} \subset \mathcal{F}_n$. We show that $\mathcal{F}_n = \mathcal{F}$. If we can show that $\mathcal{F}$ is a λ-system then by the Sierpinski Class theorem, $\sigma(\mathcal{E}) \subset \mathcal{F}$; then $\mathcal{F} = \mathcal{F}_n$. Because

$$\mu(\Omega \cap \Omega_n) = \mu(\Omega_n) = \nu(\Omega_n) = \nu(\Omega \cap \Omega_n),$$

$\Omega \in \mathcal{F}$. For $A \in \mathcal{F}_n$,

$$\begin{aligned}
\mu((\Omega \setminus A) \cap \Omega_n) &= \mu(\Omega_n \setminus A) = \mu(\Omega_n \setminus (A \cap \Omega_n)) \\
&= \mu(\Omega_n) - \mu(A \cap \Omega_n) = \nu(\Omega_n) - \nu(A \cap \Omega_n) \\
&= \nu(\Omega_n \setminus (A \cap \Omega_n)) = \nu(\Omega_n \setminus A) \\
&= \nu((\Omega \setminus A) \cap \Omega_n).
\end{aligned}$$

Thus, $\Omega \setminus A \in \mathcal{F}_n$. Let (A_k) be a sequence of pairwise disjoint sets in $\mathcal{E}$. Then

$$\begin{aligned}
\mu\left(\left(\bigcup_{k=1}^{\infty} A_k\right) \cap \Omega_n\right) &= \mu\Big(\underbrace{\bigcup_{k=1}^{\infty} (A_k \cap \Omega_n)}_{\text{disjoint union}}\Big) = \sum_{k=1}^{\infty} \mu(A_k \cap \Omega_n) \\
&= \sum_{k=1}^{\infty} \nu(A_k \cap \Omega_n) = \nu\left(\bigcup_{k=1}^{\infty} (A_k \cap \Omega_n)\right) \\
&= \nu\left(\left(\bigcup_{k=1}^{\infty} A_k\right) \cap \Omega_n\right).
\end{aligned}$$

Thus, $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_n$. We have showed that $\mathcal{F}_n$ is a λ-system. Therefore, $\mathcal{F}_n = \mathcal{F}$.

In other words, $\mu(A\cap\Omega_n)=\nu(A\cap\Omega_n)$ for all $A\in\mathcal{F}$ and $n\in\mathbb{N}$.

For each $A\in\mathcal{F}$, $(A\cap\Omega_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{E}$ whose union is $A\cap\Omega=A$. By the continuity of μ and ν,

$$\mu(A)=\lim_{n\to\infty}\mu(A\cap\Omega_n),\quad \nu(A)=\lim_{n\to\infty}\nu(A\cap\Omega_n).$$

Thus, $\mu(A)=\nu(A)$.

Now consider the case that μ is <u>not</u> σ-finite on $\mathcal{E}$. Take $(\Omega,\mathcal{F})$ as the Borel measurable space $(\mathbb{R},\mathcal{B})$, μ as the Lebesgue measure, $\nu=2\mu$, and $\mathcal{E}=\{(a,\infty):a\in\mathbb{R}\}$. Then both μ and ν are σ-finite on $(\mathbb{R},\mathcal{B})$ because we can take $\Omega_n=(-n,n)$ for all $n\in\mathbb{N}$. $\mathcal{E}$ is a π-system generating $\mathcal{F}$ and $\mu((a,\infty))=\nu((a,\infty))=\infty$. However, $\mu\neq\nu$ because $\mu((0,1))=1<2$ $=\nu((0,1))$.

⑥ Additional problem C.

Let $\mathcal{B}$ be a collection of subsets of Ω containing ϕ and f be a nonnegative function on $\mathcal{B}$ such that $f(\phi)=0$. $A\subset\Omega$ is called an f-null set if $f^*(A)=0$, where $f^*:\mathcal{P}(\Omega)\to[0,\infty]$, $f^*(A')=\inf\left\{\sum_{n=1}^{\infty}f(B_n):B_n\in\mathcal{B},\ A'\subset\bigcup_{n=1}^{\infty}B_n\right\}$.
We show that an f-null set is also an f^*-null set.

Let A be an f-null set. We have

$$f^{**}(A)=\inf\left\{\sum_{n=1}^{\infty}f^*(D_n):D_n\in\mathcal{B},\ A\subset\bigcup_{n=1}^{\infty}D_n\right\}\geq 0. \qquad (1)$$

For each $\varepsilon>0$, there exists a sequence (B_n) in $\mathcal{B}$ such that $A\subset\bigcup_{n=1}^{\infty}B_n$

and $$0\leq\sum_{n=1}^{\infty}f(B_n)\leq f^*(A)+\varepsilon=\varepsilon. \qquad (2)$$

For each $B \in \mathcal{B}$, the sequence (C_n): $C_1 = B$, $C_n = \phi$ for all $n \geq 2$ is in $\mathcal{B}$ and $B = \bigcup_{n=1}^{\infty} C_n$. Thus, by the definition of f^*, $f^*(B) \leq \sum_{n=1}^{\infty} f(C_n) = f(B)$. We apply this fact to $B = B_n$ for each $n \in \mathbb{N}$. Then $f^*(B_n) \leq f(B_n)$. Then (2) implies

$$0 \leq \sum_{n=1}^{\infty} f^*(B_n) \leq \sum_{n=1}^{\infty} f(B_n) \leq \varepsilon.$$

By (1),

$$0 \leq f^{**}(A) \leq \sum_{n=1}^{\infty} f^*(B_n) \leq \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, we conclude that $f^{**}(A) = 0$. Thus, A is f^*-null.

(7) Additional problem D.

Let $\mathcal{B}$ be a σ-field on a set Ω and f be a nonnegative σ-additive function on $\mathcal{B}$ with $f(\phi) = 0$. Take any $A \subset \Omega$. We show that there exists $B \in \mathcal{B}$ such that $A \subset B$ and $f^*(A) = f(B)$. By definition,

$$f^*(A) = \inf \left\{ \sum_{n=1}^{\infty} f(B_n) : B_n \in \mathcal{B},\, A \subset \bigcup_{n=1}^{\infty} B_n \right\}.$$

Note that the set to be taken infimum is never empty because the sequence $(\Omega, \phi, \phi, \dots)$ is in $\mathcal{B}$ and is a cover of A.

For each $m \in \mathbb{N}$, there exists a sequence $(B_n^{(m)})_{n \in \mathbb{N}}$ in $\mathcal{B}$ such that

$$A \subset \bigcup_{n=1}^{\infty} B_n^{(m)} \text{ and } \sum_{n=1}^{\infty} f(B_n^{(m)}) \leq f^*(A) + \frac{1}{m}. \qquad (1)$$

Put $B = \bigcap_{m=1}^{\infty} \left(\bigcup_{n=1}^{\infty} B_n^{(m)} \right)$. Then $A \subset B$. Since $\mathcal{B}$ is a σ-field, $B \in \mathcal{B}$.

Because f is additive and nonnegative, $f(C) \leq f(D)$ if $C \subset D$. Indeed, $f(D) = f(C) + f(D \setminus C) \geq f(C)$. Thus,

$$f(B) \leq f\left(\bigcup_{n=1}^{\infty} B_n^{(m)}\right) \overset{\text{Problem ②}}{\leq} \sum_{n=1}^{\infty} f(B_n^{(m)}) \overset{(1)}{\leq} f^*(A) + \frac{1}{m} \quad \forall m \in \mathbb{N}.$$

Hence, $f(B) \leq f^*(A)$.

On the other hand, the sequence $(B, \phi, \phi, \ldots)$ is in $\mathcal{B}$ and is a cover of A. By ~~de~~ the definition of f^*, $f^*(A) \leq f(B) + f(\phi) + f(\phi) + \ldots =$ $= f(B)$. Therefore, $f^*(A) = f(B)$.