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Math 8651: Theory of Probability

Homework #3

① Problem 3, Fristedt-Gray, page 13.

Let X and Y be two random variables from a probability space $(\Omega, \mathcal{F}, P)$ to measurable space $(\Psi, \mathcal{G})$. Assume that the set $A = \{\omega : X(\omega) \neq Y(\omega)\}$ is an event having probability 0. We show that X and Y have the same distribution.

The distribution of X is defined as a measure Q_X on $(\Psi, \mathcal{G})$,

$$Q_X(B) = P(X^{-1}(B)) \quad \forall B \in \mathcal{G}.$$

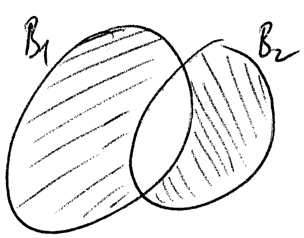
The distribution of Y is defined as a measure Q_Y on $(\Psi, \mathcal{G})$,

$$Q_Y(B) = P(\cancel{X} Y^{-1}(B)) \quad \forall B \in \mathcal{G}.$$

Take $B \in \mathcal{G}$. We want to show that $Q_X(B) = Q_Y(B)$.

$$Q_X(B) - Q_Y(B) = P(\underbrace{X^{-1}(B)}_{B_1}) - P(\underbrace{Y^{-1}(B)}_{B_2})$$

$$= P(B_1 \setminus B_2) - P(B_2 \setminus B_1).$$



For each $\omega \in B_1 \setminus B_2$, $Y(\omega) \notin B$ and $X(\omega) \in B$ and hence $\omega \in A$. This implies $B_1 \setminus B_2 < A$. Similarly, $B_2 \setminus B_1 \subset A$. Because $P(A) = 0$, $P(B_1 \setminus B_2) = P(B_2 \setminus B_1) = 0$. Therefore,

$$Q_X(B) = Q_Y(B).$$

② Problem 6, Fristedt-Gray, page 13.

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Let $(\Omega, \mathcal{F}), (\Psi, \mathcal{G}), (\Theta, \mathcal{H})$ be measurable spaces and $X: \Omega \to \Psi$, $Y: \Psi \to \Theta$ be measurable functions. We show that $Z = Y \circ X : \Omega \to \Theta$ is also a measurable function.

Take $B \in \mathcal{H}$. We need to show $Z^{-1}(B) \in \mathcal{F}$.

$$\begin{aligned} Z^{-1}(B) &= \{\omega \in \Omega : Z(\omega) = Y(X(\omega)) \in B\} \\ &= \{\omega \in \Omega : X(\omega) \in Y^{-1}(B)\} \\ &= X^{-1}(Y^{-1}(B)). \end{aligned}$$

Because Y is measurable, $Y^{-1}(B) \in \mathcal{G}$. Because X is measurable, $X^{-1}(Y^{-1}(B)) \in \mathcal{F}$.

Thus, $Z^{-1}(B) \in \mathcal{F}$.

3 Problem 19, Fristedt-Gray, page 98.

Let μ be a translation-invariant measure on $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-field, satisfying $\mu([0,1)) = 1$. We show that μ is the Lebesgue measure on $\mathcal{B}$. That is to show $\mu((a,b]) = b-a$ for all $a,b \in \mathbb{R}$, $a<b$.

Define a function $f: [0,\infty) \to [0,\infty]$, $f(r) = \mu([0,r))$. We want to show that $f(r) = r$ for all $r \geq 0$. Because $\mu(\phi) = 0$ ~~and~~ and $\mu([0,1)) = 1$, $f(0) = 0$ and $f(1) = 1$. For $r, s \geq 0$,

$$\begin{aligned} f(r+s) = \mu([0, r+s)) &= \mu([0,r) \cup [r, r+s)) \\ &= \mu([0,r)) + \underbrace{\mu([r,r+s))}_{=\mu([0,s)) \text{ since } \mu \text{ is translation-invariant}} \\ &= f(r) + f(s). \end{aligned}$$

Thus, f is finitely additive. For each $n \in \mathbb{N}$,

$$f(1) = f\Big(\underbrace{\tfrac{1}{n} + \tfrac{1}{n} + \dots + \tfrac{1}{n}}_{n \text{ times}}\Big) = f(\tfrac{1}{n}) + f(\tfrac{1}{n}) + \dots + f(\tfrac{1}{n}) = n f(\tfrac{1}{n}).$$

Thus, $f(\frac{1}{n}) = \frac{f(1)}{n} = \frac{1}{n}$.

For $m, n \in \mathbb{N}$, $f(\frac{m}{n}) = f\Big(\underbrace{\tfrac{1}{n} + \tfrac{1}{n} + \dots + \tfrac{1}{n}}_{m \text{ times}}\Big) = f(\tfrac{1}{n}) + f(\tfrac{1}{n}) + \dots + f(\tfrac{1}{n})$

$$= m f(\tfrac{1}{n}) = \frac{m}{n}.$$

Thus, $f(r) = r$ for all $r \in \mathbb{Q}$, $r \geq 0$. Because $\mathbb{Q} \cap [0, \infty)$ is dense in $[0, \infty)$, each $r \in [0, \infty)$ is the limit of a decreasing sequence of rational numbers (r_n). Then $[0, r) = \bigcap_{n=1}^{\infty} [0, r_n)$. Because of the continuity of measure μ, $\mu([0, r)) = \lim_{n \to \infty} \mu([0, r_n))$. In other words,

$$f(r) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} r_n = r.$$

Therefore, $f(r) = r$ for all $r \in [0, \infty)$, which means

$$\mu([0, r)) = r \quad \forall r \geq 0 \qquad (1)$$

Take $a, b \in \mathbb{R}$, $a < b$. We want to show $\mu([a, b)) = b - a$. Because $[a, b) = b - a + [0, b-a)$ and that μ is translation-invariant, $\mu([a,b)) = \mu([0, b-a)) \overset{(1)}{=} b - a$.

④ Additional problem A.

Let μ be a measure on a measurable space $(\Omega, \mathcal{F})$. Define

$$\overline{\mathcal{F}} = \{B \cup N : B \in \mathcal{F}, \exists\, C \in \mathcal{F} : \mu(C) = 0 \text{ and } N \subset C\}.$$

We show the following statements.

(i) $\overline{\mathcal{F}}$ is a σ-field.

(ii) Define $\bar{\mu} : \bar{\mathcal{F}} \to [0,\infty]$, $\bar{\mu}(B \cup N) = \mu(B)$. Then $\bar{\mu}$ is a measure on $(\Omega, \bar{\mathcal{F}})$ that coincides with μ on $\mathcal{F}$.

Proof of (i)

Because $\Omega = \Omega \cup \phi$, $\Omega \in \mathcal{F}$, $\mu(\phi) = 0$, we get $\Omega \in \bar{\mathcal{F}}$. Take $A \in \bar{\mathcal{F}}$. Write $A = B \cup N$ where $B \in \mathcal{F}$ and $N \subset C$ for some $C \in \mathcal{F}$, $\mu(C) = 0$. Then

$$\begin{aligned}
\Omega \setminus A = \Omega \setminus (B \cup N) &= (\Omega \setminus B) \cap (\Omega \setminus N) \\
&= (\Omega \setminus B) \cap [(\Omega \setminus C) \cup (C \setminus N)] \\
&= \underbrace{[(\Omega \setminus B) \cap (\Omega \setminus C)]}_{B_1} \cup \underbrace{[(\Omega \setminus B) \cap (C \setminus N)]}_{N_1} \\
&= B_1 \cup N_1
\end{aligned}$$

Because $B, C \in \mathcal{F}$, $B_1 \in \mathcal{F}$. Because $N_1 \subset C$ and $\Omega \setminus A = B_1 \cup N_1$, we conclude that $\Omega \setminus A \in \bar{\mathcal{F}}$.

Let (A_n) be a sequence in $\bar{\mathcal{F}}$. We want to show that $A = \bigcup_{n=1}^{\infty} A_n \in \bar{\mathcal{F}}$. Write $A_n = B_n \cup N_n$ where $B_n \in \mathcal{F}$ and $N_n \subset C_n$ for some $C_n \in \mathcal{F}$, $\mu(C_n) = 0$.

Then $$A = \bigcup_{n=1}^{\infty} (B_n \cup N_n) = \underbrace{\left(\bigcup_{n=1}^{\infty} B_n\right)}_{B} \cup \underbrace{\left(\bigcup_{n=1}^{\infty} N_n\right)}_{N}.$$

Since $\mathcal{F}$ is a σ-field, $B \in \mathcal{F}$. Put $C = \bigcup_{n=1}^{\infty} C_n \in \mathcal{F}$. Then $N \subset C$ because $N_n \subset C_n \subset C$ for each $n \in \mathbb{N}$. By Problem 2 of Homework #2,

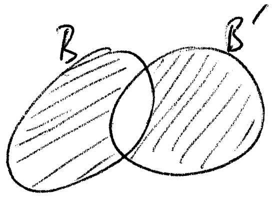
$$\mu(C) \leq \sum_{n=1}^{\infty} \mu(C_n) = 0.$$

Thus $\mu(C) = 0$. Because $A = B \cup N$, $B \in \mathcal{F}$, $N \subset C$ and $\mu(C) = 0$, we conclude that $A \in \bar{\mathcal{F}}$.

Proof of (ii)

First, we show that $\bar{\mu}$ is well-defined. Suppose that a set $A \subset \Omega$ is written as $A = B \cup N = B' \cup N'$ where $B, B' \in \mathcal{F}$, $N \subset C$, $N \subset C'$ for some $C, C' \in \mathcal{F}$, $\mu(C) = \mu(C') = 0$. We want to show that $\mu(B) = \mu(B')$.

Because B and B' play the same role, it suffices to show that $\mu(B) \leq \mu(B')$. This is true if $\mu(B') = \infty$. Consider the case $\mu(B') < \infty$.



$$\mu(B) - \mu(B') = \mu(B \setminus B') - \mu(B' \setminus B) \leq \mu(B \setminus B').$$

We have $B \setminus B' \subset (B \cup N) \setminus B' = (B' \cup N') \setminus B' \subset N' \subset C'$.

Thus, $\mu(B \setminus B') \leq \mu(C') = 0$. Hence, $\mu(B) - \mu(B') \leq 0$.

Next, we show that $\bar{\mu}$ coincides with μ on $\mathcal{F}$. For each $B \in \mathcal{F}$, $B = B \cup \phi$. Thus, $B \in \bar{\mathcal{F}}$ and $\bar{\mu}(B) = \bar{\mu}(B \cup \phi) = \mu(B)$. Thus, $\mathcal{F} \subset \bar{\mathcal{F}}$ and $\bar{\mu}|_{\mathcal{F}} = \mu$.

Now we show that $\bar{\mu}$ is a measure on $\bar{\mathcal{F}}$. Since $\phi \in \mathcal{F}$, $\bar{\mu}(\phi) = \mu(\phi) = 0$. Let (A_n) be a sequence of disjoint sets in $\bar{\mathcal{F}}$. Write $A_n = B_n \cup N_n$ where $B_n \in \mathcal{F}$ and $N_n \subset C_n$ for some $C_n \in \mathcal{F}$, $\mu(C_n) = 0$. We pointed out in Part (i) that the set $A = \bigcup_{n=1}^{\infty} A_n$ is of the form $A = B \cup N$ where $B = \bigcup_{n=1}^{\infty} B_n$ and N is contained in a set of measure 0 in $\mathcal{F}$. Thus, $\bar{\mu}(A) = \mu(B) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right)$. Because $B_i \cap B_j \subset A_i \cap A_j = \phi$ for $i \neq j$, (B_n) is a sequence of disjoint sets in $\mathcal{F}$. Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \bar{\mu}(A_n).$$

Therefore, $\bar{\mu}(A) = \sum_{n\geq 1}^{\infty} \bar{\mu}(A_n)$.

⑤ Additional problem B.

For $a, b \in \mathbb{R}$ and $B \subset \mathbb{R}$, we denote $aB+b := \{ax+b : x \in B\}$. Let $\mathcal{B}$ be the Borel σ-field on $\mathbb{R}$ and λ be the Lebesgue measure on $\mathcal{B}$. Take $a, b \in \mathbb{R}$. We show that

$$\text{(i) } aB+b \in \mathcal{B} \quad \forall B \in \mathcal{B},$$

$$\text{(ii) } \lambda(aB+b) = |a|\lambda(B) \quad \forall B \in \mathcal{B}.$$

Proof of (i)

If $a=0$,
$$aB+b = \begin{cases} \phi & \text{if } B = \phi \\ \{b\} & \text{if } B \neq \phi. \end{cases}$$

In either case, $aB+b$ is a closed subset of $\mathbb{R}$ and hence is in $\mathcal{B}$.

Consider the case $a \neq 0$. Define a function $\varphi : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$, $\varphi(x) = \frac{x-b}{a}$. Since φ is continuous, it is measurable. Thus, $\varphi^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$.

$$\varphi^{-1}(B) = \{x \in \mathbb{R} : \varphi(x) \in B\} = \{x \in \mathbb{R} : \tfrac{x-b}{a} = y \in B\}$$
$$= \{ay+b : y \in B\}$$
$$= aB+b.$$

Therefore, $aB+b \in \mathcal{B}$.

Proof of (ii)

If $a=0$ then $\lambda(aB+b) = \lambda(\{b\}) = \lim_{n\to\infty} \lambda([b, b+\frac{1}{n})) = \lim_{n\to\infty} \frac{1}{n} = 0 = |a|\lambda(B)$.

Consider the case $a \neq 0$. Define $\mu(B) = \dfrac{\lambda(aB+b)}{|a|}$ for each $B \in \mathcal{B}$.

We first show that μ is a measure on $(\mathbb{R}, \mathcal{B})$. It is clear that $\mu(B) \geq 0$ for all $B \in \mathcal{B}$.

$$\mu(\phi) = \frac{\lambda(a\phi + b)}{|a|} = \frac{\lambda(\phi)}{|a|} = 0.$$

Let (A_n) be a sequence of disjoint sets in $\mathcal{B}$. Put $B_n = aA_n + b$. Then $B_n \cap B_m = \phi$ for $m \neq n$. Indeed, suppose there exists $x \in B_n \cap B_m$. Then $\frac{x-b}{a} \in A_n \cap A_m$, which is a contradiction. We have

$$a\left(\bigcup_{n=1}^{\infty} A_n\right) + b = \left\{ax+b : x \in \bigcup_{n=1}^{\infty} A_n\right\} = \bigcup_{n=1}^{\infty}\{ax+b : x \in A_n\} = \bigcup_{n=1}^{\infty}(aA_n + b) = \bigcup_{n=1}^{\infty} B_n.$$

~~Thus~~ According to Part (i), $B_n \in \mathcal{B}$. Thus,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \frac{\lambda\left(a\left(\bigcup_{n=1}^{\infty} A_n\right)+b\right)}{|a|} = \frac{\lambda\left(\bigcup_{n=1}^{\infty} B_n\right)}{|a|} = \frac{\sum_{n=1}^{\infty}\lambda(B_n)}{|a|} = \sum_{n=1}^{\infty}\frac{\lambda(aA_n+b)}{|a|} = \sum_{n=1}^{\infty}\mu(A_n).$$

Therefore, μ is a measure on $(\mathbb{R}, \mathcal{B})$.

Because of the uniqueness of Lebesgue measure on $(\mathbb{R}, \mathcal{B})$, all we need to show next is that μ is a Lebesgue measure. Problem 3 gives us a method to do so. Accordingly, we have to show that μ is translation-invariant and $\mu([0,1)) = 1$. For $B \in \mathcal{B}$ and $c \in \mathbb{R}$,

$$\begin{aligned} a(B+c)+b = \{ay+b : y \in B+c\} &= \{a(z+c)+b : z \in B\} \\ &= \{az+ac+b : z \in B\} \\ &= aB + ac + b. \qquad (1) \end{aligned}$$

Because λ is translation-invariant, $\lambda(aB+ac+b) = \lambda(aB+b)$. $\qquad (2)$

Then
$$\mu(B+c) = \frac{\lambda(a(B+c)+b)}{|a|} \overset{(1)}{=\!=} \frac{\lambda(aB+ac+b)}{|a|} \overset{(2)}{=\!=} \frac{\lambda(aB+b)}{|a|} = \mu(B).$$

This means μ is translation-invariant. Moreover,
$$a[0,1)+b = \begin{cases} [b, a+b) & \text{if } a>0, \\ (a+b, b] & \text{if } a<0. \end{cases}$$

Thus,
$$\lambda(a[0,1)+b) = \begin{cases} (a+b)-b & \text{if } a>0, \\ b-(a+b) & \text{if } a<0 \end{cases}$$
$$= |a|.$$

Therefore,
$$\mu([0,1)) = \frac{\lambda(a[0,1)+b)}{|a|} = 1.$$

(6) Additional problem C.

We first repeat what is taught in class about the setting of this problem. We started with the observation that each $x \in [0,1)$ is written uniquely as
$$x = 0.a_1 a_2 a_3 \dots \quad \text{(in base 2)}$$
where each $a_i \in \{0,1\}$ infinitely many of which are 0. The expression simply means
$$x = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots = \sum_{i=1}^{\infty} \frac{a_i}{2^i}.$$

Because of the uniqueness of such an expression, we can denote the dependance of each a_i on x by writting $a_i = a_i(x)$. For $n \in \mathbb{N}$ and $\alpha \in \{0,1\}$,
$$\{x \in [0,1) : a_n(x) = \alpha\} = \bigcup_{\alpha_1,\dots,\alpha_{n-1} \in \{0,1\}} \{x \in [0,1) : a_1(x) = \alpha_1, \dots, a_{n-1}(x) = \alpha_{n-1}, a_n(x) = \alpha\}$$
$$= \bigcup_{\alpha_1,\dots,\alpha_{n-1} \in \{0,1\}} \left[\sum_{i=1}^{n-1} \frac{\alpha_i}{2^i} + \frac{\alpha}{2^n}, \sum_{i=1}^{n-1} \frac{\alpha_i}{2^i} + \frac{\alpha+1}{2^n}\right) \qquad (1)$$

The observation can be described in probabilistic language as follows. Let

$(\Omega, \mathcal{B}, P) = ([0,1), \mathcal{B}([0,1)), P)$ be the probability ~~measure~~ space with Lebesgue measure. For each $n \in \mathbb{N}$, define a function $\omega_n : \Omega \to \{0,1\}$, $\omega_n(\omega) = a_n(\omega)$. Because of (1), ω_n is a random variable.

For $a_1, \dots, a_n \in \{0,1\}$, $\{\omega : \omega_1(\omega) = a_1, \dots, \omega_n(\omega) = a_n\} = \left[\sum_{i=1}^{n} \frac{a_i}{2^i}, \sum_{i=1}^{n} \frac{a_i}{2^i} + \frac{1}{2^n}\right)$.

Thus, $P(\omega_1 = a_1, \dots, \omega_n = a_n) = \frac{1}{2^n}$. (2)

Now we return to the problem. Let $k_1 < k_2 < \dots < k_n$ be positive integers and $a_1, \dots, a_n \in \{0,1\}$. The set

$$\{\omega : \omega_{k_j}(\omega) = a_j \;\; \forall 1 \le j \le n\} = \bigcap_{j=1}^{n} \{\omega : \omega_{k_j}(\omega) = a_j\} = \bigcap_{j=1}^{n} \omega_{k_j}^{-1}(\{a_j\})$$

is an event because each $\omega_{k_j}^{-1}(\{a_j\})$ is an event. We want to compute its probability. Put $m = k_n - n \ge 0$. If $m = 0$ then $k_j = j$ for all $1 \le j \le n$. Thus,

$$P(\omega_{k_j}(\omega) = a_j \;\; \forall 1 \le j \le n) \overset{(2)}{=} \frac{1}{2^n}.$$

Consider the case $m \ge 1$. Let $s_1 < s_2 < \dots < s_m$ be the elements of the set $\{1, 2, \dots, k_n\} \setminus \{k_1, k_2, \dots, k_n\}$. Then

$$\{\omega : \omega_{k_j}(\omega) = a_j \;\; \forall 1 \le j \le n\} = \bigcup_{\alpha_1, \dots, \alpha_m \in \{0,1\}} \{\omega : \omega_{s_i}(\omega) = \alpha_i \;\; \forall 1 \le i \le m, \; \omega_{k_j}(\omega) = a_j \;\; \forall 1 \le j \le n\}$$

The sets in the union are pairwise disjoint, each of which has probability $\frac{1}{2^{k_n}}$ because of (2). By the additivity of P,

$$P(\omega_{k_j} = a_j \;\; \forall 1 \le j \le n) = \sum_{\alpha_1, \dots, \alpha_m \in \{0,1\}} \frac{1}{2^{k_n}} = \frac{2^m}{2^{k_n}} = \frac{1}{2^{k_n - m}} = \frac{1}{2^n}.$$

Therefore, in both the cases we get $P(\omega_{k_j} = a_j \;\; \forall 1 \le j \le n) = \frac{1}{2^n}$.

⑦ Additional problem D.

With the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and the random variables ω_n as in Problem ⑥, we define

$$X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k}(\omega)}{2^k}, \qquad Y(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k-1}(\omega)}{2^k}.$$

Because X is the (pointwise) limit of a sequence of random variables, it is also a ~~random variable~~. The same is true for Y. Take $a, b, c, d \in [0,1]$, $a \le b$, $c \le d$. We show that $\mathbb{P}((X,Y) \in (a,b] \times (c,d]) = (b-a)(d-c)$.

First, we compute $\mathbb{P}((X,Y) \in (\alpha,1] \times (\beta,1])$ for $\alpha, \beta \in [0,1]$. If $\alpha = 1$ or $\beta = 1$, the event $\{\omega : (X(\omega), Y(\omega)) \in (\alpha,1] \times (\beta,1]\}$ is empty and thus has probability 0. Consider the case $\alpha, \beta \in (0,1]$. Write

$$\alpha = 0.\alpha_1\alpha_2 \cdots \qquad \text{(in base 2)}$$
$$\beta = 0.\beta_1\beta_2 \cdots$$

such that there are infinitely many α_i's are 0 and infinitely many β_i's are 0.

Then
$$\{\omega : X(\omega) > \alpha, Y(\omega) > \beta\} = \bigcup_{m,n=1}^{\infty} \underbrace{\{\omega : \omega_{2i}(\omega) = \alpha_i \ \forall 1 \le i < m,\ \omega_{2j-1}(\omega) = \beta_j \ \forall 1 \le j < n,\ \omega_{2m}(\omega) > \alpha_m,\ \omega_{2n-1}(\omega) > \beta_n\}}_{A_{mn}}$$

If $\alpha_m = 1$ or $\beta_n = 1$ then $A_{mn} = \emptyset$. If $\alpha_m = \beta_n = 0$ then

$$A_{mn} = \{\omega : \omega_{2i}(\omega) = \alpha_i \ \forall 1 \le i < m,\ \omega_{2j-1}(\omega) = \beta_j \ \forall 1 \le j < n,\ \omega_{2m}(\omega) = 1,\ \omega_{2n-1}(\omega) = 1\}$$

According to Problem ⑥, $\mathbb{P}(A_{mn}) = \dfrac{1}{2^{m+n}}$. Combining two cases, we get

$$\mathbb{P}(A_{mn}) = \frac{(1-\alpha_m)(1-\beta_n)}{2^{m+n}}.$$

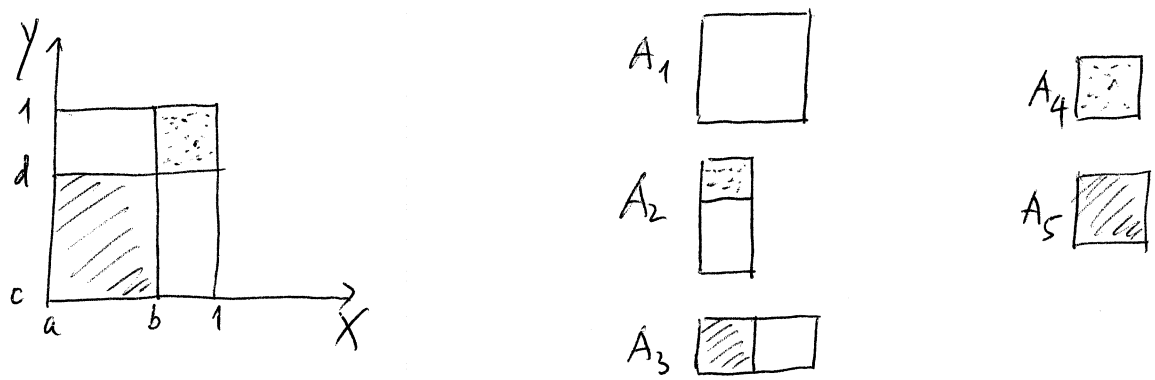
We see that $A_{mn} \cap A_{m'n'} = \phi$ if $(m,n) \neq (m',n')$. Thus,

$$\mathbb{P}(X>\alpha, Y>\beta) = \mathbb{P}\Big(\bigcup_{m,n=1}^{\infty} A_{mn}\Big) = \sum_{m,n=1}^{\infty} \mathbb{P}(A_{mn}) = \sum_{m,n=1}^{\infty} \frac{(1-\alpha_m)(1-\beta_n)}{2^{m+n}}$$

$$= \sum_{m,n=1}^{\infty} \Big(\frac{1-\alpha_m}{2^m}\Big)\Big(\frac{1-\beta_n}{2^n}\Big) = \Big(\sum_{m=1}^{\infty} \frac{1-\alpha_m}{2^m}\Big)\Big(\sum_{n=1}^{\infty} \frac{1-\beta_n}{2^n}\Big)$$

$$= \Big(\sum_{m=1}^{\infty} \frac{1}{2^m} - \sum_{m=1}^{\infty} \frac{\alpha_m}{2^m}\Big)\Big(\sum_{n=1}^{\infty} \frac{1}{2^n} - \sum_{n=1}^{\infty} \frac{\beta_n}{2^n}\Big) = (1-\alpha)(1-\beta).$$

Therefore, $\mathbb{P}(X>\alpha, Y>\beta) = (1-\alpha)(1-\beta) \quad \forall \alpha,\beta \in [0,1]. \qquad (1)$



Put $A_1 = \{\omega : X(\omega) \in (a,1],\ Y(\omega) \in (c,1]\}$

$A_2 = \{\omega : X(\omega) \in (b,1],\ Y(\omega) \in (c,1]\}$

$A_3 = \{\omega : X(\omega) \in (a,1],\ Y(\omega) \in (d,1]\}$

$A_4 = \{\omega : X(\omega) \in (b,1],\ Y(\omega) \in (d,1]\}$

$A_5 = \{\omega : X(\omega) \in (a,b],\ Y(\omega) \in (c,d]\}$

Then $A_1 = A_5 \cup (A_2 \cup A_3)$, $A_5 \cap (A_2 \cup A_3) = \phi$ and $A_2 \cap A_3 = A_4$. By (1),

$\mathbb{P}(A_1) = (1-a)(1-c)$, $\mathbb{P}(A_2) = (1-b)(1-c)$, $\mathbb{P}(A_3) = (1-a)(1-d)$, $\mathbb{P}(A_4) = (1-b)(1-d)$.

We have $\mathbb{P}(A_1) = \mathbb{P}(A_5) + \mathbb{P}(A_2 \cup A_3) = \mathbb{P}(A_5) + \mathbb{P}(A_2) + \mathbb{P}(A_3) - \mathbb{P}(\underbrace{A_2 \cap A_3}_{=A_4})$.

Thus,
$$\begin{aligned}
\mathbb{P}(A_5) &= \mathbb{P}(A_1) - \mathbb{P}(A_2) + \mathbb{P}(A_4) - \mathbb{P}(A_3) \\
&= (1-a)(1-c) - (1-b)(1-c) + (1-b)(1-d) - (1-a)(1-d) \\
&= [(1-a)-(1-b)](1-c) + [(1-b)-(1-a)](1-d) \\
&= (b-a)(1-c) + (a-b)(1-d) \\
&= (b-a)[(1-c)-(1-d)] \\
&= (b-a)(d-c).
\end{aligned}$$

Therefore, $\mathbb{P}((X,Y) \in (a,b] \times (c,d]) = (b-a)(d-c)$.