Name: Tuan Pham
ID: 4652218

Math 8651: Theory of Probability
Homework #4

(1) Problem 5, Fristedt-Gray, page 28.

Let X be an $\mathbb{R}$-valued random variable and $F: \mathbb{R} \to \mathbb{R}$, $F(x) = \mathbb{P}(X \le x)$, be the distribution function of X. For $a, b \in \mathbb{R}$, $a < b$, we determined $\mathbb{P}(a < X \le b)$, $\mathbb{P}(a < X < b)$, $\mathbb{P}(a \le X < b)$, $\mathbb{P}(a \le X \le b)$ and $\mathbb{P}(X = x)$ in terms of F.

$$\begin{aligned}
\mathbb{P}(a < X \le b) = \mathbb{P}(X^{-1}((a,b])) &= \mathbb{P}(X^{-1}((-\infty,b]) \setminus X^{-1}((-\infty,a])) \\
&= \mathbb{P}(X^{-1}((-\infty,b])) - \mathbb{P}(X^{-1}((-\infty,a])) \\
&= \mathbb{P}(X \le b) - \mathbb{P}(X \le a) \\
&= F(b) - F(a). \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}(X = x) = \mathbb{P}(X^{-1}(\{x\})) &= \mathbb{P}\left(\bigcap_{n=1}^{\infty} X^{-1}\left(\left(x - \tfrac{1}{n}, x\right]\right)\right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(X^{-1}\left(\left(x-\tfrac{1}{n}, x\right]\right)\right) \quad \text{(by the continuity of } \mathbb{P}) \\
&= \lim_{n\to\infty} \mathbb{P}\left(x - \tfrac{1}{n} < X \le x\right) \\
&\overset{(1)}{=} \lim_{n\to\infty} \left(F(x) - F\left(x - \tfrac{1}{n}\right)\right) \\
&= F(x) - F(x-), \qquad (2)
\end{aligned}$$

where $F(x-) = \lim_{y \to x^-} F(y)$. This limit exists because F is increasing and bounded.

$$\begin{aligned}
\mathbb{P}(a < X < b) = \mathbb{P}(X^{-1}((a,b))) &= \mathbb{P}(X^{-1}((a,b]) \setminus X^{-1}(\{b\})) \\
&= \mathbb{P}(X^{-1}((a,b])) - \mathbb{P}(X^{-1}(\{b\})) \\
&= \mathbb{P}(a < X \le b) - \mathbb{P}(X = b)
\end{aligned}$$

$$\overset{(1),(2)}{=} F(b) - F(a) - (F(b) - F(b-))$$
$$= F(b-) - F(a). \qquad (3)$$

$$P(a \le X < b) = P(a < X < b) + P(X = a)$$
$$\overset{(2),(3)}{=} F(b-) - F(a) + F(a) - F(a-)$$
$$= F(b-) - F(a-). \qquad (4)$$
$$P(a \le X \le b) = P(a \le X < b) + P(X = b)$$
$$\overset{(2),(4)}{=} F(b-) - F(a-) + F(b) - F(b-)$$
$$= F(b) - F(a-). \qquad (5)$$

A side remark is that if F is continuous in $\mathbb{R}$ then $F(x-) = \lim_{y \to x^-} F(y) = F(x)$ for all $x \in \mathbb{R}$. Thus (2) implies that the events $\{\omega : X(\omega) = x\}$ are all null events.

(2) Problem 19, Fristedt-Gray, page 50.

Let (X_n) be a sequence of $[0,\infty]$-valued random variables. We show that
$$E\Big(\sum_{k=1}^{\infty} X_k\Big) = \sum_{k=1}^{\infty} E(X_k),$$
with the understanding that both sides could be infinity. Put
$$f_n(\omega) = \sum_{k=1}^{n} X_k(\omega), \qquad f(\omega) = \sum_{k=1}^{\infty} X_k(\omega),$$
for all $n \in \mathbb{N}$ and $\omega \in \Omega$ (the probability space). Then (f_n) is an increasing sequence of nonnegative random variables. In addition, (f_n) converges to f pointwise in Ω. By the Monotone Convergence Theorem,
$$\lim_{n\to\infty} \int_\Omega f_n(\omega)\, P(d\omega) = \int_\Omega f(\omega)\, P(d\omega).$$

Thus, $\lim_{n\to\infty} \sum_{k=1}^{n} \int_{\Omega} X_k(\omega)\, P(d\omega) = Ef$.

Therefore, $\sum_{k=1}^{\infty} E(X_k) = E\left(\sum_{k=1}^{\infty} X_k\right)$.

(3) Additional problem A.

Let ξ be a binomially distributed random variable. That is, there are $n \in \mathbb{N}$ and $p \in (0,1)$ such that
$$P(\xi = k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k \in \{0,1,\dots,n\} \\ 0 & \text{otherwise.} \end{cases}$$

Fix a number $z > 0$. The function $x \in \mathbb{R} \mapsto z^x \in \mathbb{R}$ is continuous. Thus, z^{ξ} is also a random variable. We want to compute $E z^{\xi}$.

Because ξ can assume only finitely many values, it is a simple random variable. Write $\xi(\omega) = \sum_{k=0}^{n} k I_{A_k}(\omega)$, where $A_k = \{\omega : \xi(\omega) = k\}$. Because $A_0, A_1, \dots, A_n$ are pairwise disjoint, $z^{\xi(\omega)} = \sum_{k=0}^{n} z^k I_{A_k}(\omega)$. Thus, z^{ξ} is a simple random variable. By definition, the expectation of z^{ξ} is
$$E z^{\xi} = \int_{\Omega} z^{\xi(\omega)} P(d\omega) = \sum_{k=0}^{n} z^k P(A_k) = \sum_{k=0}^{n} z^k P(\xi = k)$$
$$= \sum_{k=0}^{n} \binom{n}{k} (zp)^k (1-p)^{n-k}$$
$$= (zp + 1 - p)^n \quad \text{(by Newton's binomial formula).}$$

Thus, $E z^{\xi} = (zp + 1 - p)^n$.

Now let k be a nonnegative integer. Put $X = \varphi(\xi)$ where $\varphi(x) = x(x-1)\cdots(x-k)$. Because $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function, X is a random variable. We want to compute EX. If $k \geq n$ then $X(\omega) = 0$ for all $\omega \in \Omega$ because $\xi(\omega) \in \{0,1,\dots,n\}$. In this case, the expectation of X is 0. Consider the case $0 \leq k \leq n-1$.

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Because $A_0, A_1, \ldots, A_n$ are disjoint, $X(\omega) = \varphi(\xi(\omega)) = \sum_{j=0}^{n} \varphi(j) I_{A_j}(\omega)$.

By definition, the ~~expection~~ expectation of X is

$$EX = \int_\Omega X(\omega) P(d\omega) = \sum_{j=0}^{n} \varphi(j) P(A_j) = \sum_{j=0}^{n} j(j-1)\cdots(j-k)\binom{n}{j} p^j (1-p)^{n-j}. \quad (1)$$

We know the formula $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$. Taking the partial derivative with respect to x $(k+1)$ times, we get

$$n(n-1)\cdots(n-k)(x+y)^{n-k-1} = \sum_{j=0}^{n} j(j-1)\cdots(j-k)\, x^{j-k-1} y^{n-j}.$$

Multiplying both the sides by x^{k+1}, we get

$$n(n-1)\cdots(n-k)(x+y)^{n-k-1} x^{k+1} = \sum_{j=0}^{n} j(j-1)\cdots(j-k)\, x^j y^{n-j}. \quad (2)$$

Applying (2) for $x = p$ and $y = 1-p$, we can rewrite (1) as

$$E\xi(\xi-1)\cdots(\xi-k) = n(n-1)\cdots(n-k)p^{k+1}. \quad (3)$$

Note that this formula also includes the case $k \geq n$. Now we want to compute $E\xi^2$. Substituting $k=0$ and $k=1$ into (3), we get

$$E\xi = np, \quad E\xi(\xi-1) = n(n-1)p^2.$$

Therefore, $E\xi^2 = E(\xi + \xi(\xi-1)) = E\xi + E\xi(\xi-1) = np + n(n-1)p^2$.

(4) Additional problem B.

Let a_k^n, for $n, k \in \mathbb{N}$, be nonnegative numbers. Suppose

$$a_k^1 \leq a_k^2 \leq a_k^3 \leq \cdots \quad \text{and} \quad \lim_{n\to\infty} a_k^n = a_k < \infty.$$

We show that $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k^n = \sum_{k=1}^{\infty} a_k$.

We want to bring the problem into a situation where the Monotone Convergence

Theorem can be applied. The set $\mathbb{N} = \{1,2,3,\dots\}$ with the σ-field $\mathcal{P}(\mathbb{N})$ is a measurable space. We introduce a measure, so called the count measure,

$$\mu(A) = \# A \quad \forall A \subset \mathbb{N}.$$

For each $n \in \mathbb{N}$, we define a function $f_n : \mathbb{N} \to \mathbb{R}$, $f_n(k) = a_k^n \geq 0$. Because every subset of $\mathbb{N}$ is measurable, f_n is measurable. By the hypotheses, $f_1(k) \leq f_2(k) \leq f_3(k) \leq \cdots$ for all $k \in \mathbb{N}$. Define $f : \mathbb{N} \to \mathbb{R}$, $f(k) = a_k$. Then $\lim_{n\to\infty} f_n(k) = f(k)$ for all $k \in \mathbb{N}$. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{N}} f_n(k)\mu(dk) = \int_{\mathbb{N}} f(k)\mu(dk). \qquad (1)$$

We now compute each integral. For each $m \in \mathbb{N}$, define $g_m(k) = f(k) I_{\{1,\dots,m\}}(k)$. Then $0 \leq g_1(k) \leq g_2(k) \leq g_3(k) \leq \cdots$ and $\lim_{m\to\infty} g_m(k) = f(k)$ for all $k \in \mathbb{N}$. By the Monotone Convergence Theorem,

$$\lim_{m\to\infty} \int_{\mathbb{N}} g_m(k)\mu(dk) = \int_{\mathbb{N}} f(k)\mu(dk). \qquad (2)$$

Because $g_m(k) = \sum_{j=1}^{m} f(j) I_{\{j\}}(k)$, which is a simple function, we have

$$\int_{\mathbb{N}} g_m(k)\mu(dk) = \sum_{j=1}^{m} f(j)\mu(\{j\}) = \sum_{j=1}^{m} f(j).$$

Thus, (2) becomes $\displaystyle\sum_{j=1}^{\infty} f(j) = \int_{\mathbb{N}} f(k)\mu(dk)$.

Likewise, $\displaystyle\sum_{j=1}^{\infty} f_n(j) = \int_{\mathbb{N}} f_n(k)\mu(dk)$.

Then (1) becomes $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{\infty} f_n(j) = \sum_{j=1}^{\infty} f(j)$,

which gives $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{\infty} a_j^n = \sum_{j=1}^{\infty} a_j$.

(5) Additional problem C.

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Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and (A_n) be a sequence of measurable sets. Put $A = \underline{\lim}_{n\to\infty} A_n = \{x \in \Omega : \exists\, n_0 \in \mathbb{N} \text{ such that } x \in A_n \ \forall\, n \geq n_0\}$.

First, we show $A = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$. Denote the latter set by B. For each $x \in A$, there is $n_0 \in \mathbb{N}$ such that $x \in A_n$ for all $n \geq n_0$. Then $x \in \bigcap_{k=n_0}^{\infty} A_k \subset B$. Thus, $A \subset B$. For each $x \in B$, there is $n \in \mathbb{N}$ so that $x \in \bigcap_{k=n}^{\infty} A_k$. This means $x \in A_k$ for all $k \geq n$. Thus $x \in A$. We have showed that $B \subset A$. Therefore, $A = B$.

As a consequence, $A \in \mathcal{F}$. Next, we show that $\underline{\lim}_{n\to\infty} I_{A_n} = I_A$. Take $x \in A$. There is $n_0 \in \mathbb{N}$ such that $x \in A_n$ for all $n \geq n_0$. Thus, $I_{A_n}(x) = 1$ for all $n \geq n_0$.

$$\underline{\lim}\, I_{A_n}(x) = \lim I_{A_n}(x) = 1 = I_A(x).$$

Now take $x \in \Omega \setminus A$. Then $x \notin \bigcap_{k=n}^{\infty} A_k$ for all $n \in \mathbb{N}$. This means for each $n \in \mathbb{N}$ there exists $k_n \geq n$ such that $x \notin A_{k_n}$. Because $\lim_{n\to\infty} k_n = \infty$, it has an increasing subsequence. By considering this subsequence instead of (k_n) itself, we can assume that (k_n) is increasing. We have $I_{A_{k_n}}(x) = 0$ for all $n \in \mathbb{N}$. Since $I_{A_n}(x) \geq 0$ for all $n \in \mathbb{N}$,

$$\underline{\lim}_{n\to\infty} I_{A_n}(x) = \lim_{n\to\infty} I_{A_{k_n}}(x) = 0,$$

which is equal to $I_A(x)$. Therefore, $\underline{\lim}\, I_{A_n}(x) = I_A(x)$ for all $x \in A$.

Next, we show that $\mu(A) \leq \underline{\lim}_{n\to\infty} \mu(A_n)$. For a sequence of nonnegative measurable functions (f_n), Fatou's lemma states that

$$\int_\Omega \underline{\lim}\, f_n(x)\,\mu(dx) \leq \underline{\lim} \int_\Omega f_n(x)\,\mu(dx).$$

Applying this lemma for $f_n = I_{A_n}$, we get

$$\int_\Omega \underbrace{\underline{\lim}\, I_{A_n}(x)}_{=I_A(x)}\,\mu(dx) \leq \underline{\lim} \int_\Omega I_{A_n}(x)\,\mu(dx).$$

This gives $\mu(A) \leq \underline{\lim}\, \mu(A_n)$.

(6) Additional problem D.

Let $(\Omega, \mathcal{F})$ be a measurable space, X be a set and $\xi : \Omega \to X$ be a map. Put $\mathcal{G} = \{B \subset X : \xi^{-1}(B) \in \mathcal{F}\}$. We show that $\mathcal{G}$ is a σ-field of subsets of X. That is to check of $\phi \in \mathcal{G}$, that $\mathcal{G}$ is closed under complements, and that $\mathcal{G}$ is closed under countable unions.

Because $\xi^{-1}(\phi) = \phi \in \mathcal{F}$, $\phi \in \mathcal{G}$. For $A \in \mathcal{G}$,

$$\xi^{-1}(X \setminus A) = \{\omega \in \Omega : \xi(\omega) \in X \setminus A\} = \Omega \setminus \{\omega \in \Omega : \xi(\omega) \in A\}$$
$$= \Omega \setminus \underbrace{\xi^{-1}(A)}_{\in \mathcal{F}}.$$

Thus $\xi^{-1}(X \setminus A) \in \mathcal{F}$. Hence, $X \setminus A \in \mathcal{G}$.

Let (A_n) be a sequence in $\mathcal{G}$ and $A = \bigcup_{n=1}^{\infty} A_n$. Then

$$\xi^{-1}(A) = \{\omega \in \Omega : \xi(\omega) \in \bigcup_{n=1}^{\infty} A_n\} = \bigcup_{n=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \in A_n\} = \bigcup_{n=1}^{\infty} \underbrace{\xi^{-1}(A_n)}_{\in \mathcal{F}} \in \mathcal{F}.$$

Thus $A \in \mathcal{G}$.

(7) Additional problem E.

Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and $f : \Omega \to \mathbb{R}$ be a measurable function. Suppose that $\mu(f \in B) = \mu(-f \in B)$ for all Borel set $B \subset \mathbb{R}$, and that $\int_\Omega f(\omega)\mu(d\omega)$ exists. We show $\int_\Omega f(\omega)\mu(d\omega) = 0$.

Put $g = -f$. Then $\int_\Omega g(\omega)\mu(d\omega)$ exists and $\int_\Omega g(\omega)\mu(d\omega) = -\int_\Omega f(\omega)\mu(d\omega)$. Let F and G be the distributions of f and g respectively. They are measures on $(\mathbb{R}, \mathcal{B})$

$$F(B) = \mu(f \in B), \quad G(B) = \mu(g \in B) = \mu(-f \in B) \quad \forall B \in \mathcal{B}.$$

By the hypothesis, $F(B) = G(B)$ for all $B \in \mathcal{B}$. Thus, $F = G$. The change-of-variable theorem states that for ~~every~~ measurable functions $X : \Omega \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\int_\Omega \varphi(X(\omega))\,\mu(d\omega) = \int_{\mathbb{R}} \varphi(x)\,F_X(dx) \qquad (1)$$

provided that at least one integral exists. Applying this result for $\varphi(x) = x$ and $X = f$, we get

$$\int_\Omega f(\omega)\,\mu(d\omega) = \int_{\mathbb{R}} x\,F(dx). \qquad (2)$$

Applying (1) for $\varphi(x) = x$ and $X = g$, we get

$$\int_\Omega g(\omega)\,\mu(d\omega) = \int_{\mathbb{R}} x\,G(dx). \qquad (3)$$

Because $F = G$, from (2) and (3) we get

$$\int_\Omega f(\omega)\,\mu(d\omega) = \int_\Omega g(\omega)\,\mu(d\omega) = -\int_\Omega f(\omega)\,\mu(d\omega).$$

Therefore, $\int_\Omega f(\omega)\,\mu(d\omega) = 0$.