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Math 8651: Theory of Probability
Homework #5

① Additional problem A.

Let $\xi, \xi_1, \xi_2, \ldots$ be random variables on a measure space $(\Omega, \mathcal{F}, P)$ and $r \in [1, \infty)$. Assume $\lim_{n\to\infty} \xi_n = \xi$ a.s. and $\lim_{n\to\infty} E|\xi_n|^r = E|\xi| < \infty$. We show that $\lim_{n\to\infty} E|\xi_n - \xi|^r = 0$.

$$E|\xi_n-\xi|^r = \int_\Omega |\xi_n-\xi|^r P(d\omega) = \underbrace{\int_\Omega |\xi_n-\xi|^r I_{|\xi_n-\xi|\le 3|\xi|}\, P(d\omega)}_{\{1\}}$$
$$+ \underbrace{\int_\Omega |\xi_n-\xi|^r I_{|\xi_n-\xi|>3|\xi|}\, P(d\omega)}_{\{2\}}.$$

It suffices to show $\{1\}\to 0$ and $\{2\}\to 0$ as $n\to\infty$. Put $f_n = |\xi_n-\xi|^r I_{|\xi_n-\xi|\le 3|\xi|}$. Then f_n is a random variable. Because $0\le f_n \le |\xi_n-\xi|^r$, $\lim_{n\to\infty} f_n = 0$ a.s. In addition,

$$|f_n| = |\xi_n-\xi|^r I_{|\xi_n-\xi|\le 3|\xi|} \le (3|\xi|)^r I_{|\xi_n-\xi|\le 3|\xi|} \le 3^r|\xi|^r \quad \forall \omega\in\Omega,$$

and $E(3^r|\xi|^r) = 3^r E|\xi|^r < \infty$. By the Dominate Convergence Theorem,

$$\{1\} = \int_\Omega f_n(\omega) P(d\omega) \longrightarrow \int_\Omega 0\, P(d\omega) = 0.$$

Put $g_n = |\xi_n-\xi|^r I_{|\xi_n-\xi|>3|\xi|}$. Then g_n is a random variable. For $\omega\in\Omega$ such that $|\xi_n(\omega)-\xi(\omega)| > 3|\xi(\omega)|$, we have $|\xi_n(\omega)| \geq |\xi_n(\omega)-\xi(\omega)| - |\xi(\omega)| > 2|\xi(\omega)|$. Thus, $|\xi_n(\omega)|^r > 2^r|\xi(\omega)|^r \geq 2|\xi(\omega)|^r$. This implies

$$|\xi_n(\omega)|^r + |\xi(\omega)|^r \leq 3\left(|\xi_n(\omega)|^r - |\xi(\omega)|^r\right) \qquad (*)$$

We get an estimation for $g_n(\omega)$, $\omega \in \Omega$, as follows.

$$\begin{aligned}
|g_n| \leq (|\xi_n| + |\xi|)^r I_{|\xi_n - \xi| > 3|\xi|} &\leq \left(2\max\{|\xi_n|, |\xi|\}\right)^r I_{|\xi_n - \xi| > 3|\xi|} \\
&\leq 2^r (|\xi_n| + |\xi|)^r I_{|\xi_n - \xi| > 3|\xi|} \\
&\overset{(*)}{\leq} 2^r \cdot 3 (|\xi_n|^r - |\xi|^r) I_{|\xi_n - \xi| > 3|\xi|} \\
&\leq 2^r \cdot 3 \, \left| |\xi_n|^r - |\xi|^r \right|.
\end{aligned}$$

Put $h_n = |\xi_n|^r$ and $h = |\xi|^r$. Then h_n and h are nonnegative random variables with $Eh < \infty$, $\lim_{n\to\infty} Eh_n = Eh$ and $\lim_{n\to\infty} h_n = h$ a.s. by the hypotheses. Scheffe's lemma concludes that $\lim_{n\to\infty} E|h_n - h| = 0$. Thus,

$$\{2\} = \int_\Omega g_n(\omega) P(d\omega) \leq 2^r \cdot 3 \int_\Omega |h_n - h| \, P(d\omega) = 2^r \cdot 3 \, E|h_n - h| \to 0$$

as $n \to \infty$.

② Additional problem B.

Let $(\mathbb{R}, \mathcal{B}, \lambda)$ be the Lebesgue measure and $f: \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. For $a, b \in \mathbb{R}$, $a \neq 0$, we show that $\int_{\mathbb{R}} f(x) \lambda(dx) = |a| \int_{\mathbb{R}} f(ax+b) \lambda(dx)$. Hereafter, we will simply write dx for $\lambda(dx)$.

Put $\varphi(x) = ax + b$. We need to show $\int_{\mathbb{R}} f(x)\,dx = |a| \int_{\mathbb{R}} f \circ \varphi(x)\,dx$.
Consider the case $f = I_A$ for some $A \in \mathcal{B}$. Then $\int_{\mathbb{R}} f\,dx = \lambda(A)$.

$$f \circ \varphi(x) = I_A(\varphi(x)) = \begin{cases} 1 & \text{if } \varphi(x) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in \Psi(A) \\ 0 & \text{otherwise} \end{cases} = I_{\Psi(A)}(x),$$

where $\Psi(x) = \varphi^{-1}(x) = a^{-1}x - a^{-1}b$. Thus,

$$\int_{\mathbb{R}} f\circ\varphi\, dx = \int_{\mathbb{R}} I_{\Psi(A)}(x)\, dx = \lambda(\Psi(A)) = \lambda(a^{-1}A - a^{-1}b).$$

By Problem ⑤ in Homework #3, $\lambda(a^{-1}A - a^{-1}b) = |a^{-1}|\,\lambda(A)$. Hence,

$$|a|\int_{\mathbb{R}} f\circ\varphi\, dx = |a|\left(|a^{-1}|\,\lambda(A)\right) = \lambda(A) = \int_{\mathbb{R}} f\, dx.$$

Consider the case f is a simple function. Write $f = \sum_{i=1}^{n} c_i I_{A_i}$ where $c_i \in \mathbb{R}$ and $\lambda(A_i) < \infty$. Then

$$\int_{\mathbb{R}} f\, dx = \sum_{i=1}^{n} c_i \int_{\mathbb{R}} I_{A_i}\, dx, \qquad (1)$$

$$|a|\int_{\mathbb{R}} f\circ\varphi\, dx = \sum_{i=1}^{n} c_i\, |a| \int_{\mathbb{R}} I_{A_i}\circ\varphi\, dx. \qquad (2)$$

We already proved that $\int_{\mathbb{R}} I_{A_i}\, dx = |a| \int_{\mathbb{R}} I_{A_i}\circ\varphi\, dx \quad \forall\, 1 \le i \le n.$

Each integral is finite because $\lambda(A_i) < \infty$. Then by (1) and (2) we get

$$\int_{\mathbb{R}} f\, dx = |a| \int_{\mathbb{R}} f\circ\varphi\, dx.$$

Now we consider the general case $f \ge 0$. Let (f_n) be a sequence of simple functions such that $0 \le f_1 \le f_2 \le f_3 \le \cdots$ a.s. and $\lim_{n\to\infty} f_n = f$ a.s. We showed that

$$\int_{\mathbb{R}} f_n\, dx = |a| \int_{\mathbb{R}} f_n\circ\varphi\, dx \quad \forall n \in \mathbb{N} \qquad (3)$$

By the Monotone Convergence Theorem, $\lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dx = \int_{\mathbb{R}} f\, dx. \qquad (4)$

Put $g_n = f_n\circ\varphi$ and $g = f\circ\varphi$. Then g and g_n are measurable functions. Moreover, $0 \le g_1 \le g_2 \le g_3 \le \cdots$

$$\lambda(\{y\in\mathbb{R}: \lim_{n\to\infty} g_n(y) \neq g(y)\}) = \lambda(\Psi(\{x\in\mathbb{R}: \lim_{n\to\infty} f_n(x) \neq f(x)\}))$$
$$= |a^{-1}|\,\lambda(\{x\in\mathbb{R}: \lim_{n\to\infty} f_n(x) \neq f(x)\})$$
$$= 0.$$

Thus, $\lim_{n\to\infty} g_n = g$ a.s. By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n \, dx = \int_{\mathbb{R}} g \, dx.$$

In other words, $$\lim_{n\to\infty} \int_{\mathbb{R}} f_n \circ \varphi \, dx = \int_{\mathbb{R}} f\circ\varphi \, dx. \qquad (5)$$

The limits of both sides of (3) as $n\to\infty$ are found in (4) and (5). Thus,

$$\int_{\mathbb{R}} f \, dx = |a| \int_{\mathbb{R}} f\circ\varphi \, dx.$$

③ Additional problem C.

Let $(\mathbb{R}, \mathcal{B}, \mu)$ be the Lebesgue measure space and A be a bounded measurable subset. Suppose there exists $\varepsilon \in (0,1)$ such that $\mu(AI) \leq \varepsilon \mu(I)$ for every interval $I \subset \mathbb{R}$. We show that $\mu(A) = 0$.

Because A is bounded, it is contained in some finite interval K of $\mathbb{R}$.

First, we assume the following property.

[For each $B\in\mathcal{B}$, there exists a sequence (B_n) in $\mathcal{B}$, where each B_n is a finite union of disjoint subintervals of K, such that $I_{BK} = \lim_{n\to\infty} I_{B_n}$ a.e.] (*)

Applying this property for $B = A$, we have $I_A = I_{AK} = \lim_{n\to\infty} I_{A_n}$ a.e. where each A_n is a finite union $\bigcup_j A_{n,j}$ of disjoint subintervals of K. By the hypothesis, $\mu(AA_{n,j}) \leq \varepsilon \mu(A_{n,j})$. Summing over j, we get

$$\mu\Big(A\big(\bigcup_j A_{n,j}\big)\Big) \le \varepsilon \mu\big(\bigcup_j A_{n,j}\big).$$

Thus, $\mu(AA_n) \le \varepsilon \mu(A_n)$. In terms of Lebesgue integral,

$$\int_{\mathbb{R}} I_A I_{A_n} \, dx \le \varepsilon \int_{\mathbb{R}} I_{A_n} \, dx \qquad \forall n \in \mathbb{N}. \qquad (1)$$

The notation dx stands for $\mu(dx)$. We have

$$\lim_{n\to\infty} I_{A_n} = I_A \text{ a.e.}, \quad \lim_{n\to\infty} I_A I_{A_n} = I_A \lim_{n\to\infty} I_{A_n} = I_A I_A = I_A \text{ a.e.}$$

In addition, $I_{A_n}, I_A I_{A_n} \le I_K$ which has finite integral. By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} I_A I_{A_n} \, dx = \int_{\mathbb{R}} I_A \, dx = \mu(A)$$

$$\lim_{n\to\infty} \int_{\mathbb{R}} I_{A_n} \, dx = \int_{\mathbb{R}} I_A \, dx = \mu(A).$$

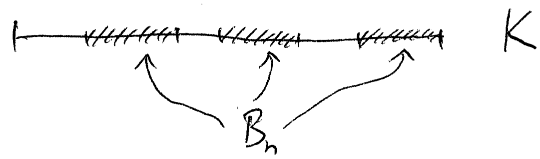
Then the inequality (1) as $n\to\infty$ gives $\mu(A) \le \varepsilon \mu(A)$. Since $\varepsilon \in (0,1)$, $\mu(A) = 0$.

For the rest of the problem, we prove the property $(*)$. Denote by $\mathcal{F}$ the family of all $B \in \mathcal{B}$ such that $(*)$ holds. We show $\mathcal{F} = \mathcal{B}$. First, $\mathcal{F}$ contains all intervals of $\mathbb{R}$. Indeed, let B be an interval of $\mathbb{R}$. Then BK is also an interval. We can ~~show~~ choose $B_n = BK$ for all $n \in \mathbb{N}$. Next, we show that $\mathcal{F}$ is a λ-system. Once this is done, $\mathcal{F}$ contains the σ-field generated by the family of all intervals of $\mathbb{R}$ by the Sierpinski class theorem. Thus, $\mathcal{B} \subset \mathcal{F}$ and so $\mathcal{B} = \mathcal{F}$.

$\phi \in \mathcal{F}$ because ϕ is an interval. Let $B \in \mathcal{F}$. We have $I_{BK} = \lim_{n\to\infty} I_{B_n}$ a.e. Denote J^c the complement of a set J in $\mathbb{R}$. We have

$$I_{B^c K} = I_K - I_{BK} = I_K - \lim_{n\to\infty} I_{B_n} = \lim_{n\to\infty}\left(I_K - I_{B_n}\right) = \lim_{n\to\infty} I_{K\setminus B_n} \quad \text{a.e.}$$

Because B_n is a finite union of disjoint intervals of K, so is $K\setminus B_n$. Thus $B^c \in \mathcal{F}$.



Let (B_n) be an increasing sequence in $\mathcal{F}$ and $B = \bigcup_{n=1}^{\infty} B_n$. We show that $B \in \mathcal{F}$.

Because $B_n \in \mathcal{F}$, $I_{B_n K} = \lim_{j\to\infty} I_{B_{n,j}}$ a.e. (2)

where $B_{n,j}$ is a finite union of disjoint intervals of K. Because $B_1 \subset B_2 \subset B_3 \subset \dots$

$$I_B = \lim_{n\to\infty} I_{B_n}.$$

Thus, $I_{BK} = I_B I_K = \lim_{n\to\infty} I_{B_n} I_K = \lim_{n\to\infty} I_{B_n K}$. (3)

We see that $I_{B_{n,j}}, I_{B_n K} \le I_K$ whose integral is finite. By (2), (3) and the Dominated Convergence Theorem,

$$\lim_{j\to\infty} \int_{\mathbb{R}} |I_{B_n K} - I_{B_{n,j}}|\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} |I_{B_n K} - I_{BK}|\,dx = 0.$$

For each $m \in \mathbb{N}$, there exists $n = n(m) \in \mathbb{N}$ such that $\int_{\mathbb{R}} |I_{B_n K} - I_{BK}|\,dx < \frac{1}{2^{m+2}}$.

There exists $j = j(m) \in \mathbb{N}$ such that $\int_{\mathbb{R}} |I_{B_n K} - I_{B_{n,j}}|\,dx < \frac{1}{2^{m+2}}$. Then

$$\int_{\mathbb{R}} |I_{BK} - I_{B_{n,j}}|\,dx \le \int_{\mathbb{R}} |I_{BK} - I_{B_n K}|\,dx + \int_{\mathbb{R}} |I_{B_n K} - I_{B_{n,j}}|\,dx < \frac{1}{2^{m+2}} + \frac{1}{2^{m+2}} = \frac{1}{2^{m+1}}.$$

~~Then~~ Put $C_m = B_{n,j}$, which is a finite union of disjoint intervals of K. Then

$$\int_{\mathbb{R}} |I_{BK} - I_{C_m}|\,dx < \frac{1}{2^{m+1}} \quad \forall m \in \mathbb{N}. \qquad (4)$$

Then $\int_{\mathbb{R}} |I_{C_{m+1}} - I_{C_m}|\,dx \le \int_{\mathbb{R}} |I_{BK} - I_{C_{m+1}}|\,dx + \int_{\mathbb{R}} |I_{BK} - I_{C_m}|\,dx$

$$< \frac{1}{2^{m+2}} + \frac{1}{2^{m+1}} < \frac{1}{2^m}.$$

We have $I_{C_m} = I_{C_1} + \sum_{n=2}^{m} (I_{C_n} - I_{C_{n-1}})$. (5)

$$\int_{\mathbb{R}} \Big(I_{C_1} + \sum_{n=2}^{\infty} |I_{C_n} - I_{C_{n-1}}| \Big) dx \overset{\text{Fatou}}{\leq} \int_{\mathbb{R}} I_{C_1} dx + \sum_{n=2}^{\infty} \int_{\mathbb{R}} |I_{C_n} - I_{C_{n-1}}| dx$$

$$< \mu(C_1) + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} < \infty.$$

Thus, $I_{C_1} + \sum_{n=2}^{\infty} |I_{C_n} - I_{C_{n-1}}| < \infty$ a.e. This implies that the series $I_{C_1} + \sum_{n=2}^{\infty} (I_{C_n} - I_{C_{n-1}})$ absolutely converges a.e. By (5), I_{C_m} converges a.e. as $m \to \infty$. Denote the limit by g. Because $I_{C_m} \leq I_K$ for all $m \in \mathbb{N}$, by the Dominated Convergence Theorem $\lim_{m\to\infty} \int_{\mathbb{R}} |I_{C_m} - g| dx = 0$.

Then $\int_{\mathbb{R}} |I_{BK} - g| dx \leq \underbrace{\int_{\mathbb{R}} |I_{BK} - I_{C_m}| dx}_{\leq \frac{1}{2^{m+1}} \text{ by } (4)} + \int_{\mathbb{R}} |I_{C_m} - g| dx \longrightarrow 0$ as $m \to \infty$.

Thus $\int_{\mathbb{R}} |I_{BK} - g| dx = 0$. Hence $I_{BK} = g$ a.e. This means $I_{BK} = \lim_{m\to\infty} I_{C_m}$ almost everywhere. Therefore, $B \in \mathcal{F}$.

4 Additional problem D.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \{a_1, a_2\}$, $Y : \Omega \to \{b_1, b_2\}$ be random variables with $a_1, a_2, b_1, b_2 \in \mathbb{R}$, $a_1 \neq a_2$, $b_1 \neq b_2$. Suppose X and Y are uncorrelated. We show that they are independent. That is to show $\mathbb{P}(X = a_i, Y = b_j) = \mathbb{P}(X = a_i)\mathbb{P}(Y = b_j)$ for all $1 \leq i, j \leq 2$.

We only show $\mathbb{P}(X = a_1, Y = b_1) = \mathbb{P}(X = a_1)\mathbb{P}(Y = b_1)$. Other cases follow in the same manner. Put $A=\{\omega : X(\omega)=a_1\}$ and $B=\{\omega : Y(\omega)=b_1\}$. We are going to show $P(AB)=P(A)P(B)$. Since X and Y are uncorrelated, $EXY=EXEY$. We have

$$X = a_1 I_A + a_2 I_{A^c} = a_1 I_A + a_2 (I_\Omega - I_A) = (a_1-a_2) I_A + a_2 I_\Omega.$$

Similarly, $Y=(b_1-b_2)I_B + b_2 I_\Omega$. Thus,

$$EX = (a_1-a_2)P(A)+a_2, \qquad EY=(b_1-b_2)P(B)+b_2.$$

Hence,
$$\begin{aligned} EXEY &= [(a_1-a_2)P(A)+a_2][(b_1-b_2)P(B)+b_2] \\ &= (a_1-a_2)(b_1-b_2)P(A)P(B)+b_2(a_1-a_2)P(A)+a_2(b_1-b_2)P(B)+a_2b_2. \quad (1) \end{aligned}$$

We have
$$\begin{aligned} XY &= [(a_1-a_2)I_A + a_2 I_\Omega][(b_1-b_2)I_B + b_2 I_\Omega] \\ &= (a_1-a_2)(b_1-b_2)\underbrace{I_A I_B}_{=I_{AB}} + b_2(a_1-a_2)\underbrace{I_A I_\Omega}_{=I_A} + a_2(b_1-b_2)\underbrace{I_B I_\Omega}_{=I_B} + a_2 b_2 \underbrace{I_\Omega I_\Omega}_{=I_\Omega}. \end{aligned}$$

Thus, $EXY = (a_1-a_2)(b_1-b_2)P(AB) + b_2(a_1-a_2)P(A) + a_2(b_1-b_2)P(B) + a_2b_2$. (2)

Substituting (1) and (2) into the equality $EXEY=EXY$, we get

$$(a_1-a_2)(b_1-b_2)P(A)P(B) = (a_1-a_2)(b_1-b_2)P(AB).$$

Because $a_1 \neq a_2$, and $b_1 \neq b_2$, $P(A)P(B)=P(AB)$.

5 Additional problem E.

Let $X:\Omega \to \mathbb{R}$ be a random variable with density $f: R \to [0,1]$. Let $a,b \in \mathbb{R}$, $a\neq 0$. We find the density function of $Y=\varphi \circ X$ where $\varphi(x)=ax+b$.

Denote by F_X and F_Y the distribution functions of X and Y respectively. By definition,

$$F_X(x) = \mathbb{P}(X \le x) = \int_{\mathbb{R}} f(t) I_{t \le x}\, dt,$$

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\varphi \circ X \le y).$$

The inverse function of φ is $\Psi(x) = a^{-1}x - a^{-1}b$. Consider two cases $a > 0$ and $a < 0$.

- $\underline{a > 0}$

Ψ is strictly increasing. Then $\varphi \circ X \le y$ if and only if $X \le \Psi(y)$. Thus,

$$F_Y(y) = \mathbb{P}(X \le \Psi(y)) = \int_{\mathbb{R}} \underbrace{f(t) I_{t \le \Psi(y)}}_{g(t)}\, dt.$$

By Problem ②, $\int_{\mathbb{R}} g(t)\, dt = |a^{-1}| \int_{\mathbb{R}} g(\Psi(t))\, dt = a^{-1} \int_{\mathbb{R}} f(\Psi(t)) I_{\Psi(t) \le \Psi(y)}\, dt$

$$= a^{-1} \int_{\mathbb{R}} f(a^{-1}t - a^{-1}b) I_{t \le y}\, dt.$$

Define $\tilde{f}(t) = a^{-1} f(a^{-1}t - a^{-1}b)$ for all $t \in \mathbb{R}$. Then

$$F_Y(y) = \int_{\mathbb{R}} g(t)\, dt = \int_{\mathbb{R}} \tilde{f}(t) I_{t \le y}\, dt \quad \forall y \in \mathbb{R}.$$

Therefore, $\tilde{f}$ is the density function of Y.

- $\underline{a < 0}$

Ψ is strictly decreasing. Then $\varphi \circ X \le y$ if and only if $X \ge \Psi(y)$. Thus

$$F_Y(y) = \mathbb{P}(X \ge \Psi(y)) = 1 - \mathbb{P}(X < \Psi(y))$$

$$= \int_{\mathbb{R}} f(t)\, dt - F_X(\Psi(y)-)$$

Because F_X has a density function, it is continuous. Thus, $F_X(\Psi(y)-) = F_X(\Psi(y))$.

$$F_Y(y) = \int_{\mathbb{R}} f(t)\, dt - \int_{\mathbb{R}} f(t) I_{t \le \Psi(y)}\, dt = \int_{\mathbb{R}} \underbrace{f(t) I_{t > \Psi(y)}}_{h(t)}\, dt.$$

By Problem ②, $\int_{\mathbb{R}} h(t)\,dt = |a^{-1}| \int_{\mathbb{R}} h(\Psi(t))\,dt$

$$= -a^{-1} \int_{\mathbb{R}} f(\Psi(t))\, I_{\Psi(t) > \Psi(y)}\,dt$$

$$= -a^{-1} \int_{\mathbb{R}} f(a^{-1}x - a^{-1}b)\, I_{t<y}\,dt.$$

Define $\tilde{\tilde{f}}(t) = -a^{-1} f(a^{-1}t - a^{-1}b)$ for all $t \in \mathbb{R}$. Then

$$F_Y(y) = \int_{\mathbb{R}} h(t)\,dt = \int_{\mathbb{R}} \tilde{\tilde{f}}(t)\, I_{t<y}\,dt = \int_{\mathbb{R}} \tilde{\tilde{f}}(t)\, I_{t\le y}\,dt - \underbrace{\int_{\mathbb{R}} \tilde{f}(t)\, I_{\{y\}}\,dt}_{=0 \text{ because } \{y\} \text{ has measure zero.}}$$

Therefore, $\tilde{\tilde{f}}$ is the density function of Y.

For both cases, the density function of Y is $t \mapsto \frac{1}{|a|} f\left(\frac{t-b}{a}\right)$.

⑥ Additional problem F.

Let $X : \Omega \to \mathbb{R}$ be a random variable. For $0<p<q<\infty$, we show that $(E|X|^p)^{1/p} \le (E|X|^q)^{1/q}$.

Put $Y = |X|^p \geq 0$ and $r = q/p > 1$. We want to show $(EY)^{1/p} \le (EY^r)^{1/q}$. This is to show $(EY)^r \le EY^r$. Put $\varphi(x) = x^r$ for all $x \in [0,\infty)$. Then φ is convex because $\varphi''(x) = r(r-1)x^{r-1} > 0$ for all $x \in (0,\infty)$. If $EY < \infty$, by Jensen's inequality we have

$$(EY)^r = \varphi(EY) \le E\varphi(Y) = EY^r.$$

Consider the case $EY = \infty$. We show $EY^r = \infty$. Let (Y_n) be a sequence of simple functions such that $0 \le Y_1 \le Y_2 \le Y_3 \le \cdots$ and $\lim_{n\to\infty} Y_n = Y$ a.s.

Because $EY_n < \infty$, $(EY_n)^r \le EY_n^r \le EY^r \quad \forall n \in \mathbb{N}$. $\quad (*)$

By the Monotone Convergence Theorem, $\lim_{n\to\infty} EY_n = EY = \infty$. Taking the limit of $(*)$ as $n\to\infty$, we get $EY^r = \infty$.

⑦ Additional problem G.

Let $a_k^n \geq 0$, for $n, k \in \mathbb{N}$, and $a_k \geq 0$, for $k \in \mathbb{N}$, be numbers such that $\lim_{n\to\infty} a_k^n = a_k$ and $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_k^n = \sum_{k=1}^{\infty} a_k < \infty$. We show that $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_k^n - a_k| = 0$.

Recall Scheffé's lemma (Lecture 10/3/2014).

[Let $f, f_n : \Omega \to [0,\infty)$ be measurable functions such that $f_n \to f$ a.e. Assume

$$\lim_{n\to\infty} \int_\Omega f_n \,\mu(dx) = \int_\Omega f \mu(dx) < \infty.$$

Then $\lim_{n\to\infty} \int_\Omega |f_n - f| \,\mu(dx) = 0$.]

We want to model the problem so that this lemma can be applied. In solving Problem ④ of Homework #4, we introduced the counting measure on the set $\mathbb{N}$.

$$\mu(A) = \# A \quad \forall A \subset \mathbb{N}.$$

We pointed out that the measurable functions are sequences in $\mathbb{R}$, and that the integrals over $\mathbb{N}$ are the infinite sums. Define the functions $f, f_n : \mathbb{N} \to \mathbb{R}$,

$$f(k) = a_k, \qquad f_n(k) = a_k^n.$$

They are measurable nonnegative functions. We have $\lim_{n\to\infty} f_n(k) = f(k)$ for all $k \in \mathbb{N}$.

$$\lim_{n\to\infty} \int_{\mathbb{N}} f_n(k)\,\mu(dk) = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_k^n = \sum_{k=1}^{\infty} a_k = \int_{\mathbb{N}} f(k)\,\mu(dk) < \infty.$$

By Scheffé's lemma, $\lim_{n\to\infty} \int_{\mathbb{N}} |f_n(k) - f(k)| \,\mu(dk) = 0$.

Therefore, $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_n^k - a_k| = 0$.