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Math 8651: Theory of Probability

Homework #6

(1) Problem 31, Fristedt - Gray, page 72.

We show that the function $f:[0,1)\to\mathbb{R}$, $f(s)=\dfrac{2}{(2-s)(3-s)}$ is a probability generating function of a $\{0,1,2,\dots,\infty\}$-valued random variable.

By Theorem 14, Fristedt - Gray, page 73, we need to show that

(i) f is infinitely differentiable in $[0,1)$ and $f^{(n)}(s)\geq 0$ for all $s\in[0,1)$,

(ii) $f(1-)\leq 1$.

We have
$$f(s)=\frac{2}{2-s}-\frac{2}{3-s}=\frac{1}{1-\frac{s}{2}}-\frac{2}{3}\,\frac{1}{1-\frac{s}{3}}. \qquad (1)$$

We know that the function $t\mapsto\frac{1}{1-t}$ has the Taylor expansion
$$\frac{1}{1-t}=1+t+t^2+t^3+\cdots \qquad \forall t\in[0,1).$$

Thus,
$$\frac{1}{1-\frac{s}{2}}=1+\frac{s}{2}+\left(\frac{s}{2}\right)^2+\left(\frac{s}{2}\right)^3+\cdots \qquad \forall s\in[0,2),$$
$$\frac{1}{1-\frac{s}{3}}=1+\frac{s}{3}+\left(\frac{s}{3}\right)^2+\left(\frac{s}{3}\right)^3+\cdots \qquad \forall s\in[0,3).$$

Then (1) becomes
$$\begin{aligned} f(s)&=\sum_{k=0}^{\infty}\left(\frac{s}{2}\right)^k-\frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{s}{3}\right)^k\\ &=\frac{1}{3}+\sum_{k=1}^{\infty}\left[\left(\frac{s}{2}\right)^k-\frac{2}{3}\left(\frac{s}{3}\right)^k\right]\\ &=\frac{1}{3}+\sum_{k=1}^{\infty}\left(\frac{1}{2^k}-\frac{2}{3^{k+1}}\right)s^k \qquad \forall s\in[0,2).\end{aligned}$$

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This is the Taylor expansion of g about $s=0$. Thus, g is infinitely differentiable in $[0,1]$. Moreover,

$$\frac{g^{(k)}(0)}{k!}=\frac{1}{2^k}-\frac{2}{3^{k+1}}=\frac{3^{k+1}-2^{k+1}}{2^k\,3^{k+1}}>0 \quad \forall k=0,1,2,\dots \qquad (2)$$

Thus, (i) is proved. Because $g(1-)=g(1)=1$, (ii) is proved. Therefore, g is a probability generating function of a $\{0,1,2,\dots,\infty\}$- ~~random~~ valued random variable X. By definition,

$$g(s)=\sum_{k=0}^{\infty}P(X=k)s^k. \qquad (3)$$

Thus, $$P(X=k)=\frac{g^{(k)}(0)}{k!}\overset{(2)}{=\!=}\frac{1}{2^k}-\frac{2}{3^{k+1}}.$$

$$P(X=\infty)=1-g(1-)=0.$$

The mean of X is $$EX=\sum_{k=0}^{\infty}P(X=k)k\overset{(3)}{=\!=}g'(1).$$

By (1), $$g'(s)=\frac{2}{(2-s)^2}-\frac{2}{(3-s)^2}. \qquad (4)$$

Thus, $$EX=g'(1)=\frac{2}{1^2}-\frac{2}{2^2}=\frac{3}{2}.$$

We have $$EX(X-1)=\sum_{k=0}^{\infty}P(X=k)\,k(k-1)\overset{(3)}{=\!=}g''(1).$$

By (4), $$g''(s)=\frac{4}{(2-s)^3}-\frac{4}{(3-s)^3}.$$

Thus, $EX(X-1)=g''(1)=\frac{4}{1^3}-\frac{4}{2^3}=\frac{7}{2}.$

The variance of X is $\operatorname{Var}X=EX^2-(EX)^2=EX(X-1)+EX-(EX)^2$

$$= \frac{7}{2} + \frac{3}{2} - \left(\frac{3}{2}\right)^2 = \frac{11}{4}.$$

The standard deviation of X is $\sigma(X) = \sqrt{\mathrm{Var} X} = \frac{\sqrt{11}}{2}$.

(2) Problem 36, Fristedt-Gray, page 140.

Let X and Y be independent random variables. Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be an independent sequence of random vectors each of which has the same distribution as (X, Y). Let A and B be Borel subsets of $\mathbb{R}$ with $P(X \in A) > 0$. Define the random variable

$$S_n = \sum_{i=1}^{n} I_{A \times B}(X_i, Y_i),$$

$$M_n = \sum_{i=1}^{n} I_A(X_i).$$

We show that $\frac{S_n}{M_n} \to P(Y \in B)$ a.s.

Put $Z_i = I_{A\times B}(X_i, Y_i)$ and $T_i = I_A(X_i)$. Because $I_{A\times B}$ is a measurable function, $Z_1, Z_2, Z_3, \ldots$ is an independent sequence of random variables. Denote $\pi : \mathbb{R}^2 \to \mathbb{R}$, $\pi(x, y) = x$. Then $T_i = I_A \circ \pi(X_i, Y_i)$. Since π is continuous, $I_A \circ \pi$ is a measurable function. Thus, $T_1, T_2, T_3, \ldots$ is an independent sequence of random variables. We have

$$EZ_i = \int_\Omega I_{A\times B}(X_i(\omega), Y_i(\omega))\, P(d\omega) = P((X_i, Y_i) \in A \times B).$$

Because (X_i, Y_i) has the same distribution as (X, Y), $P((X_i, Y_i) \in A\times B) = P((X, Y) \in A \times B)$. Since X and Y are stochastically independent, $P((X, Y) \in A \times B) = P(X \in A)\, P(Y \in B)$.

$$ET_i = \int_A I_A(X_i(\omega))\, P(d\omega) = P(X_i \in A) = P((X_i, Y_i) \in A \times \Omega)$$

$$= \mathbb{P}((X,Y)\in A\times\Omega) = \mathbb{P}(X\in A)\,\mathbb{P}(Y\in\Omega) = \mathbb{P}(X\in A).$$

By the Strong Law of Large Numbers,

$$\lim_{n\to\infty} \frac{S_n}{n} = EZ_1 = \mathbb{P}(X\in A)\,\mathbb{P}(Y\in B) \quad \text{a.s.}$$

$$\lim_{n\to\infty} \frac{M_n}{n} = ET_1 = \mathbb{P}(X\in A) > 0 \quad \text{a.s.}$$

Therefore,
$$\lim_{n\to\infty} \frac{S_n}{M_n} = \lim_{n\to\infty} \frac{\frac{S_n}{n}}{\frac{M_n}{n}} = \mathbb{P}(Y\in B) \quad \text{a.s.}$$

3 Problem 45, Fristedt-Gray, page 143.

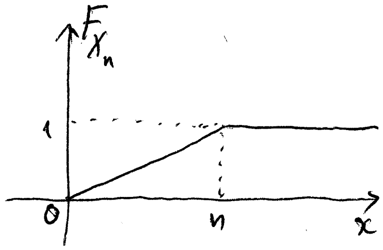
Let $X_1, X_2, \dots$ be an independent sequence of random variables where for each $n\in\mathbb{N}$, X_n is uniformly distributed on $[0,n]$. We show that $\mathbb{P}(\{\omega: \lim_{n\to\infty} X_n(\omega)=\infty\})=0$.

We have $\{\omega: \lim X_n(\omega)=\infty\} = \{\omega: \underline{\lim}\, X_n(\omega)=\infty\} = \Omega \setminus \underbrace{\{\omega: \underline{\lim}\, X_n(\omega)<\infty\}}_{A}$.

Because $\underline{\lim}\, X_n$ is a $[0,\infty]$-valued random variable, A is an event. We want to show $\mathbb{P}(A)=1$. For each $n\in\mathbb{N}$, put $A_n=\{\omega: X_n(\omega)\in[0,1]\}$ and $B=\overline{\lim}\, A_n$.

For $\omega\in B$, there is an increasing sequence (n_k) in $\mathbb{N}$ such that $\omega\in A_{n_k}$. Thus, $X_{n_k}(\omega)\in[0,1]$ for every $k\in\mathbb{N}$. This implies $\underline{\lim}\, X_n(\omega)\le 1<\infty$. Thus $\omega\in A$. We get $B\subset A$. Now it suffices to show $\mathbb{P}(B)=1$.

Because the random variables $X_1, X_2, X_3, \dots$ are independent, the events $A_1, A_2, A_3, \dots$ are also independent. We know that X_n has the distribution function



$$F_{X_n}(x) = \begin{cases} 0 & \text{if } x\le 0, \\ \frac{x}{n} & \text{if } 0\le x\le n, \\ 1 & \text{if } x\ge n. \end{cases}$$

Thus, $\mathbb{P}(A_n) = \mathbb{P}(X_n \le 1) = F_{X_n}(1) = \frac{1}{n}$. Then

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Borel-Cantelli's lemma says that $\mathbb{P}(\overline{\lim} A_n) = 1$. Hence, $\mathbb{P}(B) = 1$.

(4) Problem 9, Fristedt-Gray, page 189.

Let $X_1, X_2, X_3, \dots$ be a sequence of identically distributed random variables having finite mean. Take a number $c > 0$, and denote $A_n = \{\omega : |X_n(\omega)| > cn\}$.

First, we show that $\mathbb{P}(\overline{\lim} A_n) = 0$. Borel's lemma says that if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(\overline{\lim} A_n) = 0$. Thus, it suffices to show $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Because $X_1, X_2, X_3, \dots$ have the same distribution, $|X_1|, |X_2|, |X_3|, \dots$ also have the same distribution. Let $F : \mathbb{R} \to \mathbb{R}$ be the distribution function of $|X_n|$. Since F is nondecreasing, $F^{-1}((a, \infty))$ is an interval in $\mathbb{R}$ for any $a \in \mathbb{R}$. Thus, F is a measurable function. We have

$$E|X_n| = \int_{\Omega} |X_n(\omega)| \, \mathbb{P}(d\omega) = \int_{\Omega} \int_0^{\infty} I_{[0, |X_n(\omega)|)}(t) \, dt \, \mathbb{P}(d\omega)$$

$$\overset{\text{Fubini}}{=} \int_0^{\infty} \int_{\Omega} \underbrace{I_{[0, |X_n(\omega)|)}(t)}_{= I_{(t, \infty)}(|X_n(\omega)|)} \, \mathbb{P}(d\omega) \, dt$$

$$= \int_0^{\infty} \mathbb{P}(|X_n| > t) \, dt$$

$$= \int_0^{\infty} (1 - F(t)) \, dt.$$

Because X_n has finite mean, $E|X_n| < \infty$. Thus, $\int_0^{\infty} (1 - F(t)) \, dt < \infty$.

$\mathbb{P}(A_n) = 1 - F(cn) = \tilde{F}(n)$ where $\tilde{F} : \mathbb{R} \to [0,1]$, $\tilde{F}(x) = 1 - F(cx)$. We see that $\tilde{F}$ is nonincreasing and measurable. Thus,

$$\tilde{F}(n) \leq \int_{n-1}^{n} \tilde{F}(x)\, dx$$

and
$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq \sum_{n=1}^{\infty} \int_{n-1}^{n} \tilde{F}(x)\, dx = \int_0^\infty \tilde{F}(x)\, dx = \int_0^\infty (1 - F(cx))\, dx.$$

By changing variable $y = cx$, we get

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) \leq c^{-1} \int_0^\infty (1 - F(y))\, dy < \infty.$$

We have showed that $\mathbb{P}(\overline{\lim} A_n) = 0$.

Next, we show $\lim_{n\to\infty} \frac{X_n}{n} = 0$ a.s. Put $B_{n,k} = \{\omega : |X_n(\omega)| > \frac{k}{n}\}$ and $B_n = \overline{\lim_{k\to\infty}} B_{n,k}$. By the previous part, $\mathbb{P}(B_n) = 0$. Then by Problem ② of Homework #2, $\mathbb{P}(\bigcup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \mathbb{P}(B_n) = 0$. Put $B = \Omega \setminus (\bigcup_{n=1}^{\infty} B_n)$. Then $\mathbb{P}(B) = 1$. We show that

$$\lim_{k\to\infty} \frac{X_k(\omega)}{k} = 0 \qquad \forall \omega \in B.$$

Take $\omega \in B$. For each $n \in \mathbb{N}$, $\omega \notin B_n$. Thus there is $m_n \in \mathbb{N}$ such that $\omega \notin B_{n,k}$ for all $k \geq m_n$. Then $|X_k(\omega)| \leq \frac{k}{n}$ for all $k \geq m_n$.

For each number $\varepsilon > 0$, we take $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n_\varepsilon} < \varepsilon$. Then

$$\frac{|X_k(\omega)|}{k} \leq \frac{1}{n_\varepsilon} < \varepsilon \qquad \forall k \geq m_{n_\varepsilon}.$$

Therefore, $\lim_{k\to\infty} \frac{X_k(\omega)}{k} = 0$.

⑤ Additional problem A.

Let $X_1, X_2, X_3, \ldots$ be a sequence of nonnegative random variables. Suppose $\mu_n = EX_n < \infty$, $\operatorname{Var} X_n \leq M\mu_n$ where $M>0$ is a number independent of n, and $\operatorname{Cov}(X_i, X_j) \leq 0$ for all $i,j \in \mathbb{N}$, $i \neq j$. We show that

$$\sum_{n=1}^{\infty} \mu_n = \infty \Rightarrow \sum_{n=1}^{\infty} X_n = \infty \quad \text{a.s.}$$

Define $S_n = \sum_{k=1}^{n} X_k$ and $\nu_n = \sum_{k=1}^{n} \mu_k \geq 0$. Then $0 \leq S_1 \leq S_2 \leq S_3 \leq \ldots$ and $ES_n = \nu_n$. Put $A = \{\omega : \sup_{n \in \mathbb{N}} S_n(\omega) < \infty\}$. Suppose $\lim_{n\to\infty} \nu_n = \infty$. We want to show $P(A) = 0$. By Chebychev's inequality,

$$P\left(|S_n - \nu_n| \geq \frac{\nu_n}{2}\right) \leq \left(\frac{\nu_n}{2}\right)^{-2} \operatorname{Var} S_n \qquad (1)$$

We have
$$\begin{aligned}
\operatorname{Var} S_n = \operatorname{Cov}(S_n, S_n) &= \sum_{i,j=1}^{n} \operatorname{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \operatorname{Cov}(X_i, X_i) + 2 \sum_{1 \leq i < j \leq n} \underbrace{\operatorname{Cov}(X_i, X_j)}_{\leq 0} \\
&\leq \sum_{i=1}^{n} \operatorname{Var} X_i \\
&\leq \sum_{i=1}^{n} M\mu_i = M\nu_n .
\end{aligned}$$

Then the inequality (1) implies

$$P\left(|S_n - \nu_n| \geq \frac{\nu_n}{2}\right) \leq \left(\frac{\nu_n}{2}\right)^{-2} M\nu_n = \frac{4M}{\nu_n}.$$

Put $B_n = \{\omega : |S_n(\omega) - \nu_n| \geq \frac{\nu_n}{2}\}$. Then $P(B_n) \leq \frac{4M}{\nu_n}$. Since $\lim \nu_n = \infty$, $\lim P(B_n) = 0$.

We have
$$A = \bigcup_{k=1}^{\infty} \underbrace{\{\omega : |S_n(\omega)| \leq k \ \forall n \in \mathbb{N}\}}_{A_k}.$$

To show $P(A) = 0$, it suffices to show $P(A_k) = 0$ for every $k \in \mathbb{N}$. Indeed, once this is done, by Problem (2) of Homework #2 we get $P(A) \leq \sum_{k=1}^{\infty} P(A_k) = 0$.

Now fix $k \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that $v_n \geq 2k$ for all $n \geq N$.

Consider $n \geq N$ and $\omega \in A_k$. We have $|S_n(\omega)| \leq k \leq \frac{v_n}{2}$. Then

$$|S_n(\omega) - v_n| \geq v_n - |S_n(\omega)| \geq v_n - k \geq \frac{v_n}{2}.$$

Thus, $\omega \in B_n$. This implies $A_k \subset B_n$ for all $n > N$. Hence,

$$P(A_k) \leq P(B_n) \quad \forall n > N.$$

Because $\lim_{n\to\infty} P(B_n) = 0$, we get $P(A_k) = 0$.

⑥ Additional problem B.

Let $(E, \mathcal{F}, \mu)$ be a measure space with σ-finite measure. Let $f, g : [a,b] \times E \to \mathbb{R}$ be measurable functions with respect to $\mathcal{B}([a,b]) \otimes \mathcal{F}$ such that for every $x \in E$, $f'(t,x) = g(t,x)$ in the following sense

$$f(t,x) = f(a,x) + \int_a^t g(s,x)\,ds \qquad \text{a.e. } t \in [a,b].$$

Suppose $\int_E \int_a^b |g(t,x)|\,dt\,\mu(dx) < \infty$ and $\int_E |f(a,x)|\,\mu(dx) < \infty$. We show

$$\left(\int_E f(t,x)\,\mu(dx)\right)' = \int_E f'(t,x)\,\mu(dx).$$

This is equivalent to showing that ~~for every~~

$$\int_E f(t,x)\,\mu(dx) = \int_E f(a,x)\,\mu(dx) + \int_a^t \left(\int_E g(s,x)\,\mu(dx)\right) ds \qquad \text{a.e. } t \in [a,b].$$

For every $t \in [a,b]$, $\int_E \left(\int_a^t |g(s,x)|\,ds\right) \mu(dx) \leq \int_E \left(\int_a^b |g(s,x)|\,ds\right) \mu(dx) < \infty$.

Then Fubini's theorem concludes that

(i) the function $x \mapsto \int_a^t g(s,x)\,ds$ is measurable,

(ii) $$\int_E \left(\int_a^t g(s,x)\,ds \right) \mu(dx) = \int_a^t \left(\int_E g(s,x)\,\mu(dx) \right) ds \in \mathbb{R}.$$

By a hypothesis, the function $x \mapsto f(a,x)$ is integrable. Integrating both sides of the equation $f(t,x) = f(a,x) + \int_a^t g(s,x)\,ds$ with respect to $x \in E$, we get

$$\int_E f(t,x)\,\mu(dx) = \int_E f(a,x)\,\mu(dx) + \int_E \left(\int_a^t g(s,x)\,ds \right) \mu(dx)$$

$$= \int_E f(a,x)\,\mu(dx) + \int_a^t \left(\int_E g(s,x)\,\mu(dx) \right) ds \qquad \text{a.e. } t \in [a,b].$$

This is what we wanted to show.

(7) Additional problem C.

Let $F, F_1, F_2, F_3, \ldots$ be distribution functions. Let D be the set of all points at which F is discontinuous, and D' be a countable dense subset of $\mathbb{R}$ containing D. Suppose

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \forall x \in D',$$

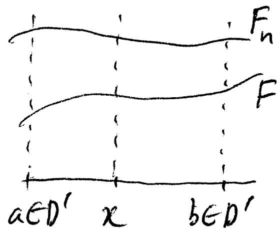
$$\lim_{n\to\infty} F_n(x-) = F(x-) \quad \forall x \in D.$$

We show that $\lim_{n\to\infty} \left(\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \right) = 0.$

First, we show that (F_n) converges pointwise to F. Since $D \subset D'$, $\lim F_n(x) = F(x)$ for all $x \in D$. Now take $x \in \mathbb{R} \setminus D$ and $\varepsilon > 0$. Because F is continuous at x, there exists $\delta > 0$ such that

$$|F(y) - F(x)| < \frac{\varepsilon}{2} \quad \forall y \in \mathbb{R},\ |y - x| < \delta.$$

Because D' is dense in $\mathbb{R}$, there exist $a, b \in D'$ such that $a < x < b$, $x - a < \delta$ and $b - x < \delta$. Then



$$0 \leq F(x) - F(a),\ F(b) - F(x) < \frac{\varepsilon}{2} \qquad (1)$$

We know that $\lim F_n(a) = F(a)$ and $\lim F_n(b) = F(b)$.
Thus, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|F_n(a) - F(a)|,\ |F_n(b) - F(b)| < \frac{\varepsilon}{2} \quad \forall n > N \qquad (2)$$

Then
$$F_n(x) - F(x) \leq F_n(b) - F(x) = (F_n(b) - F(b)) + (F(b) - F(x)) \overset{(1),(2)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n > N.$$

$$F_n(x) - F(x) \geq F_n(a) - F(x) = (F_n(a) - F(a)) + (F(a) - F(x)) \overset{(1),(2)}{>} -\frac{\varepsilon}{2} + \left(-\frac{\varepsilon}{2}\right) = \varepsilon \quad \forall n > N.$$

Thus, $|F_n(x) - F(x)| < \varepsilon$ for all $n \in \mathbb{N}$, $n > N$. This implies $\lim F_n(x) = F(x)$.
We have showed that (F_n) converges to F pointwise.

To show $\lim_{n \to \infty} \left(\sup_{x \in \mathbb{R}} |F_n(x) - F(x)|\right) = 0$, it suffices to show that for each $\varepsilon > 0$, $\overline{\lim} \left(\sup_{x \in \mathbb{R}} |F_n(x) - F(x)|\right) \leq \varepsilon$. Suppose this is not true for some $\varepsilon > 0$. Then there is a subsequence (F_{n_k}) such that $\sup_{x \in \mathbb{R}} |F_{n_k}(x) - F(x)| > \varepsilon$ for all $k \in \mathbb{N}$. By considering (F_{n_k}) instead of (F_n), we can assume $\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| > \varepsilon$ for all $n \in \mathbb{N}$. Then there exists $x_n \in \mathbb{R}$ such that

$$|F_n(x_n) - F(x_n)| > \varepsilon \quad \forall n \in \mathbb{N} \qquad (*)$$

Suppose by contradiction that (x_n) has a subsequence that tends to ∞. By replacing (x_n) by a suitable subsequence, we can assume (x_n) is increasing and $x_n \to \infty$ as $n \to \infty$. We have

$$F_n(x_n) - F(x_n) < -\varepsilon \quad \text{or} \quad F_n(x_n) - F(x_n) > \varepsilon \qquad \forall n \in \mathbb{N}.$$

One of these two cases must hold for infinitely many $n \in \mathbb{N}$. Suppose the first case holds for infinitely many $n \in \mathbb{N}$. By replacing (x_n) by a suitable subsequence, we can assume $F_n(x_n) - F(x_n) < -\varepsilon$ for all $n \in \mathbb{N}$. Then

$$F_n(x_m) \leq F_n(x_n) < F(x_n) - \varepsilon \qquad \forall m \leq n.$$

Letting $n \to \infty$, we get $F(x_m) \leq 1 - \varepsilon$. This is true for all $m \in \mathbb{N}$. Letting $m \to \infty$ we get $1 \leq 1 - \varepsilon$ which is a contradiction. Suppose the second case holds for infinitely many $n \in \mathbb{N}$. By replacing (x_n) by a suitable subsequence, we can assume $F_n(x_n) - F(x_n) > \varepsilon$. Then

$$F(x_n) < F_n(x_n) - \varepsilon \leq 1 - \varepsilon \qquad \forall n \in \mathbb{N}.$$

This is a contradiction because $\lim F(x_n) = 1$. We have showed that (x_n) has no subsequence that tends to ∞.

Now suppose by contradiction that (x_n) has a subsequence that tends to $-\infty$. By replacing (x_n) by a suitable subsequence, we can assume (x_n) is decreasing and $x_n \to -\infty$ as $n \to \infty$. We have

$$F_n(x_n) - F(x_n) < -\varepsilon \quad \text{or} \quad F_n(x_n) - F(x_n) > \varepsilon \qquad \forall n \in \mathbb{N}.$$

Suppose the first case holds for infinitely many $n \in \mathbb{N}$. By replacing (x_n) by a suitable subsequence, we can assume $F_n(x_n) - F(x_n) < -\varepsilon$ for all $n \in \mathbb{N}$. Then

$$F(x_n) > F_n(x_n) + \varepsilon \geq \varepsilon \qquad \forall n \in \mathbb{N}$$

This is a contradiction because $\lim F(x_n) = 0$. Suppose the second case holds for infinitely many $n \in \mathbb{N}$. By replacing (x_n) by a suitable subsequence, we can assume $F_n(x_n) - F(x_n) > \varepsilon$ for all $n \in \mathbb{N}$. Then

$$F_n(x_m) \geq F_n(x_n) > F(x_n) + \varepsilon \quad \forall m \leq n.$$

Letting $n \to \infty$, we get $F(x_m) \geq \varepsilon$. This is true for all $m \in \mathbb{N}$. Letting $m \to \infty$ we get $0 \geq \varepsilon$, which is a contradiction. We have showed that (x_n) has no subsequence that tends to $-\infty$.

Therefore, (x_n) is a bounded sequence. Thus, it has a convergent subsequence. By replacing (x_n) by this subsequence, we can assume $\lim x_n = x_0 \in \mathbb{R}$. Suppose there are infinitely many $n \in \mathbb{N}$ such that $x_n \geq x_0$. By replacing (x_n) by a suitable subsequence, we can assume (x_n) is decreasing and $\lim x_n = x_0$. Then

$$F_n(x_n) - F(x_n) = \underbrace{\left(F_n(x_n) - F_n(x_0)\right)}_{\geq 0} + \underbrace{\left(F_n(x_0) - F(x_0)\right)}_{\to 0} + \underbrace{\left(F(x_0) - F(x_n)\right)}_{\to 0 \text{ since } F \text{ is right-continuous}}$$

Thus, $\underline{\lim}\,(F_n(x_n) - F(x_n)) \geq 0$. Then because of $(*)$, $F_n(x_n) - F(x_n) > 0$ for all n sufficiently large. For those n's, $F_n(x_n) > F(x_n) + \varepsilon$. Taking liminf of both sides, we get

$$\underline{\lim}\, F_n(x_n) \geq F(x_0) + \varepsilon. \qquad (3)$$

On the other hand, $F_n(x_m) \geq F_n(x_n)$ for all $m \leq n$. Taking liminf of both sides as $n \to \infty$ we get $F(x_m) \geq \underline{\lim}\, F_n(x_n)$. Taking $m \to \infty$ and noting that F is right continuous, we get

$$F(x_0) \geq \underline{\lim}\, F_n(x_n) \qquad (4)$$

Then (3) and (4) contradict each other.

Therefore, there are only finitely many $n \in \mathbb{N}$ such that $x_n \geqslant x_0$. By replacing (x_n) by a suitable subsequence, we can assume (x_n) is increasing and $\lim x_n = x_0$. Then $F_n(x_n) - F(x_n) \leq F_n(x_0-) - F(x_n)$. Taking limsup of both sides, we get

$$\overline{\lim}\,(F_n(x_n) - F(x_n)) \leq \lim F_n(x_0-) - F(x_0-) = 0.$$

Then because of (*), $F_n(x_n) - F(x_n) < -\varepsilon$ for all n sufficiently large. For those n's, $F_n(x_n) < F(x_n) - \varepsilon$. Taking limsup of both sides, we get

$$\overline{\lim}\, F_n(x_n) \leq F(x_0-) - \varepsilon. \qquad (5)$$

On the other hand, $F_n(x_m) \leq F_n(x_n)$ for $m \leq n$. Taking limsup of both sides as $n \to \infty$, we get $F(x_m) \leq \overline{\lim}\, F_n(x_n)$. Taking $m \to \infty$ we get

$$F(x_0-) \leq \overline{\lim}\, F_n(x_n). \qquad (6)$$

Then (5) and (6) contradict each other.