Name: Tuan Pham
ID: 4652218

Math 8651: Theory of Probability
Homework #7

① Problem 20, Fristedt-Gray, page 50.

Let (A_n) be a sequence of events in a probability space (Ω, F, P), and let (a_n) be a sequence of real numbers. Suppose $\sum_{n=1}^{\infty} |a_n| P(A_n) < \infty$. First, we show that the series $X(\omega) = \sum_{j=1}^{\infty} a_j I_{A_j}(\omega)$ absolutely converges for almost every $\omega \in \Omega$.

For each $n \in \mathbb{N}$, we define $\tilde{X}_n(\omega) = \sum_{j=1}^{n} |a_j| I_{A_j}(\omega)$. Then $\tilde{X}_n$ is a nonnegative random variable. Also, the sequence $(\tilde{X}_n)$ is increasing and

$$\lim_{n\to\infty} \tilde{X}_n(\omega) = \tilde{X}(\omega) := \sum_{j=1}^{\infty} |a_j| I_{A_j}(\omega) \leq \infty.$$

By Monotone Convergence Theorem, $E\tilde{X} = \lim_{n\to\infty} E\tilde{X}_n = \lim_{n\to\infty} \sum_{j=1}^{n} |a_j| P(A_j) = \sum_{j=1}^{\infty} |a_j| P(A_j) < \infty$. Thus, $\{\omega : \tilde{X}(\omega) = \infty\}$ is a null-event. This implies

$$\sum_{j=1}^{\infty} |a_j| I_{A_j}(\omega) < \infty \quad \text{almost surely.}$$

Thus, $X(\omega) \in \mathbb{R}$ for almost every $\omega \in \Omega$. For each $n \in \mathbb{N}$, we define $X_n(\omega) = \sum_{j=1}^{n} a_j I_{A_j}(\omega)$. Then X_n is a random variable and $\lim X_n(\omega) = X(\omega)$. Thus, X is an almost surely defined random variable.

Next, we compute EX. We have $|X_n| \leq \tilde{X}$ and $E\tilde{X} < \infty$. By Dominated Convergence Theorem, $EX = \lim_{n\to\infty} EX_n = \lim_{n\to\infty} \sum_{j=1}^{n} a_j P(A_j) = \sum_{j=1}^{\infty} a_j P(A_j)$.

② Problem 7, Fristedt-Gray, page 124.

Let (X_1, X_2) be an independent pair of exponentially distributed random variables with means λ_1 and λ_2 respectively. Denote $\min\{X_1, X_2\}$ by $X_1 \wedge X_2$. We calculate the distribution function of $X_1 \wedge X_2$.

By the definition of exponentially distributed random variables, the distribution functions of X_1 and X_2 are respectively

$$F_1(x) = \begin{cases} 1 - e^{-\frac{x}{\lambda_1}} & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

$$F_2(x) = \begin{cases} 1 - e^{-\frac{x}{\lambda_2}} & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let $F : \mathbb{R} \to \mathbb{R}$ be the distribution function of $X_1 \wedge X_2$. We have

$$\{\omega : (X_1 \wedge X_2)(\omega) \leq x\} = \{\omega : X_1(\omega) \leq x, X_2(\omega) > x\} \cup \{\omega : X_1(\omega) > x, X_2(\omega) \leq x\}$$
$$\cup \{\omega : X_1(\omega) \leq x, X_2(\omega) \leq x\}.$$

The three sets on the right hand side are pairwise disjoint. Thus,

$$\begin{aligned} F(x) = P(X_1 \wedge X_2 \leq x) &= P(X_1 \leq x, X_2 > x) + P(X_1 > x, X_2 \leq x) + P(X_1 \leq x, X_2 \leq x) \\ &= P(X_1 \leq x) P(X_2 > x) + P(X_1 > x) P(X_2 \leq x) + P(X_1 \leq x) P(X_2 \leq x) \\ &= F_1(x)(1 - F_2(x)) + (1 - F_1(x)) F_2(x) + F_1(x) F_2(x) \\ &= F_1(x) + F_2(x) - F_1(x) F_2(x) \\ &= 1 - (1 - F_1(x))(1 - F_2(x)) \\ &= \begin{cases} 1 - e^{-\frac{x}{\lambda_1}} e^{-\frac{x}{\lambda_2}} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases} \end{aligned}$$

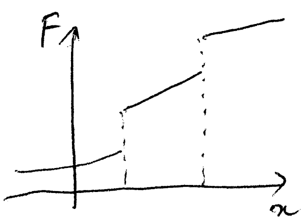
$$= \begin{cases} 1 - e^{-\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)x} & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Therefore, $X_1 \wedge X_2$ is also exponentially distributed. Its expectation is λ satisfying $\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$.

③ Problem 1, Fristedt-Gray, page 245.

Let F and F_n, $n \in \mathbb{N}$, be distribution functions for $\mathbb{R}$. We show that (F_n) weakly converges to F, denoted by $F_n \Rightarrow F$, if and only if there is a dense subset D of $\mathbb{R}$ such that $\lim F_n(x) = F(x)$ for every $x \in D$.

First, suppose (F_n) weakly converges to F. Let D' be the set of all points $x \in \mathbb{R}$ at which F is discontinuous. Because F is increasing, $F(x) - F(x-) > 0$ for every $x \in D'$. We could assign to each $x \in D'$ a number $r \in (F(x-), F(x)) \cap \mathbb{Q}$.



Then the map $x \in D' \mapsto r \in \mathbb{Q}$ is injective. Thus, D' is countable.

We show that $\mathbb{R} \setminus D'$ is dense in $\mathbb{R}$. Suppose otherwise. Then there exists an interval $(a,b) \subset \mathbb{R}$, $a < b$, such that $(\mathbb{R} \setminus D') \cap (a,b) = \phi$. Then $(a,b) \subset D'$, which is a contradiction because D' is countable. Thus, $\mathbb{R} \setminus D'$ is dense in $\mathbb{R}$. Put $D = \mathbb{R} \setminus D'$. Since $F_n \Rightarrow F$, $\lim_{n \to \infty} F_n(x) = F(x)$ for all $x \in D$.

Next, suppose there is a dense subset D of $\mathbb{R}$ such that $\lim F_n(x) = F(x)$ for every $x \in D$. Suppose by contradiction that (F_n) does not weakly converge to F. Then there exists $x_0 \in \mathbb{R}$, at which F is continuous, such that $F_n(x_0) \not\to F(x_0)$ as $n \to \infty$. Then there exists ~~$x_0 \in$~~ a number $\varepsilon > 0$ and a subsequence (F_{n_k}) such that $|F_{n_k}(x_0) - F(x_0)| > \varepsilon$ for all $k \in \mathbb{N}$. By replacing (F_n) with (F_{n_k}), we can assume

$$|F_n(x_0) - F(x_0)| > \varepsilon \quad \forall n \in \mathbb{N}.$$

There are only two following cases.

(i) There are infinitely many $n \in \mathbb{N}$ such that $F_n(x_0) - F(x_0) > \varepsilon$.

(ii) There are finitely many $n \in \mathbb{N}$ such that $F_n(x_0) - F(x_0) > \varepsilon$.

Consider case (i)

By replacing (F_n) with a suitable subsequence, we can assume $F_n(x_0) - F(x_0) > \varepsilon$ for all $n \in \mathbb{N}$. Because D is dense in $\mathbb{R}$, there is a sequence (x_m) in D such that $x_m \downarrow x$. Because F_n is increasing, $F_n(x_m) \geq F_n(x_0) > F(x_0) + \varepsilon$. Letting $n \to \infty$, we get $F(x_m) \geq F(x_0) + \varepsilon$. This is true for all $m \in \mathbb{N}$. On the other hand, $\lim_{m\to\infty} F(x_m) = F(x_0)$ because F is continuous at x_0. This is a contradiction.

Consider case (ii)

Then there are infinitely many $n \in \mathbb{N}$ such that $F_n(x_0) - F(x_0) < -\varepsilon$. By replacing (F_n) with a suitable subsequence, we can assume $F_n(x_0) - F(x_0) < -\varepsilon$ for all $n \in \mathbb{N}$. Because D is dense in $\mathbb{R}$, there is a sequence (x_m) in D such that $x_m \uparrow x$. Because F_n is increasing, $F_n(x_m) \leq F_n(x_0) < F(x_0) - \varepsilon$. Letting $n \to \infty$, we get $F(x_m) \leq F(x_0) - \varepsilon$. This is true for all $m \in \mathbb{N}$. On the other hand, $\lim_{m\to\infty} F(x_m) = F(x_0)$ because F is continuous at x_0. This is a contradiction.

4 Problem 15, Fristedt - Gray, page 247.

Let (X_n) be an independent and identically distributed (i.i.d.) sequence. Suppose each X_n is exponentially distributed with mean 1. Its distribution function is thus

$$F(x) = \begin{cases} 1-e^{-x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Put $M_n = \max\{X_1, X_2, \dots, X_n\}$. First we compute the distribution function of M_n.

$$\begin{aligned} F_{M_n}(x) = P(M_n \leq x) = P(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x) &= P(X_1 \leq x)P(X_2 \leq x)\dots P(X_n \leq x) \\ &= F(x)^n \\ &= \begin{cases} (1-e^{-x})^n & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \end{aligned}$$

Next, put $\tilde{M}_n = M_n - \log n$. We compute the distribution function $G_n : \mathbb{R} \to \mathbb{R}$ of $\tilde{M}_n$.

$$\begin{aligned} G_n(x) = P(\tilde{M}_n \leq x) = P(M_n \leq x + \log n) &= F_{M_n}(x + \log n) \\ &= \begin{cases} \left(1 - e^{-x-\log n}\right)^n & \text{if } x + \log n \geq 0, \\ 0 & \text{if } x + \log n < 0. \end{cases} \end{aligned}$$

Thus,
$$G_n(x) = \begin{cases} \left(1 - \frac{e^{-x}}{n}\right)^n & \text{if } x \geq -\log n, \\ 0 & \text{if } x < -\log n. \end{cases}$$

Let G be the standard Gumbel distribution function, i.e. $G(x) = e^{-e^{-x}}$. We show that the sequence (G_n) weakly converges to G. Because G is continuous weverywhere in $\mathbb{R}$, we are to show that $\lim G_n(x) = G(x)$ for every $x \in \mathbb{R}$. Let us fix $x \in \mathbb{R}$. The condition $x \geqslant -\log n$ is equivalent to $n \geqslant e^{-x}$.

Thus,
$$G_n(x) = \left(1 - \frac{e^{-x}}{n}\right)^n \quad \forall n \geqslant e^{-x}.$$

We know the identity
$$\lim_{t \to 0} (1+t)^{\frac{1}{t}} = e.$$

Taking $t = -\frac{e^{-x}}{n}$, we get
$$\lim_{n \to \infty} \left(1 - \frac{e^{-x}}{n}\right)^{\frac{-n}{e^{-x}}} = e.$$

Taking both sides to the power $-e^{-x}$, we get
$$\lim_{n \to \infty} G_n(x) = e^{-e^{-x}} = G(x).$$

⑤ Problem 42, Fristedt-Gray, page 259.

Let (Q_n) be a sequence of probability distributions on the extended real line $\overline{\mathbb{R}}$. That is, the distribution function $F_n : \mathbb{R} \to \mathbb{R}$, $F_n(x) = Q_n((-\infty, x])$ is increasing, right continuous and $0 \leq F_n(x) \leq 1$. Let Q be another probability distribution and $F : \mathbb{R} \to \mathbb{R}$ be its distribution function. Suppose that every subsequence of (Q_n) has a further subsequence that weakly converges to Q. Then every subsequence of (F_n) has a further subsequence that weakly converges to F. We show that (Q_n) weakly converges to Q. That is to show (F_n) weakly converges to F.

Suppose by contradiction that this is not true. Then there exists $x_0 \in \mathbb{R}$, at which F is continuous, such that $F_n(x_0) \not\to F(x_0)$ as $n \to \infty$. Then there are a number $\varepsilon > 0$ and a subsequence (F_{n_k}) such that $|F_{n_k}(x_0) - F(x_0)| > \varepsilon$ for all $k \in \mathbb{N}$. By the hypothesis, (F_{n_k}) has a subsequence $(F_{n_{k_\ell}})$ that weakly converges to F. That implies $\lim_{\ell \to \infty} F_{n_{k_\ell}}(x_0) = F(x_0)$. This is a contradiction because $|F_{n_{k_\ell}}(x_0) - F(x_0)| > \varepsilon$ for all $\ell \in \mathbb{N}$.

6) Additional problem A.

Let (X_n) be an i.i.d. sequence. Suppose each X_n is uniformly distributed on $[0,1]$. Put $Y_n = (X_1 X_2 \cdots X_n)^{\frac{1}{n}}$. We show that $Y_n \to e^{-1}$ almost surely.

Each X_n has the distribution function $F : \mathbb{R} \to \mathbb{R}$,
$$F(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

Because $\mathbb{P}(X_n \le 0) = F(0) = 0$, $\tilde{X}_n := -\log X_n$ is an almost surely defined random variable. Because $\mathbb{P}(X_n > 1) = 1 - F(1) = 0$, $\tilde{X}_n$ is nonnegative almost surely. Because (X_n) is an independent sequence, $(\tilde{X}_n)$ is also an independent sequence. The distribution function of each $\tilde{X}_n$ is

$$\tilde{F}(x) = \mathbb{P}(\tilde{X}_n \le x) = \mathbb{P}(\log X_n \ge -x) = \mathbb{P}(X_n \ge e^{-x}) = 1 - F(e^{-x})$$

$$= 1 - \begin{cases} e^{-x} & \text{if } x \ge 0 \\ 1 & \text{if } x \le 0 \end{cases}$$

$$= \begin{cases} 1 - e^{-x} & \text{if } x \ge 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

In Problem 4, Homework #6, we verified the formula

$$E\tilde{X}_1 = \int_0^\infty (1-\tilde{F}(x))\,dx.$$

This is also Corollary 20, Fristedt-Gray, page 58. Then we get

$$E\tilde{X}_1 = \int_0^\infty [1-(1-e^{-x})]\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1.$$

Put $\tilde{Y}_n = \dfrac{\tilde{X}_1+\tilde{X}_2+\dots+\tilde{X}_n}{n}$. Then by the Strong Law of Large Numbers, we get $\tilde{Y}_n \to E\tilde{X}_1 = 1$ almost surely.

$$\tilde{Y}_n = \frac{-\log X_1 - \log X_2 - \dots - \log X_n}{n} = -\frac{1}{n}\log(X_1X_2\cdots X_n) = -\log Y_n.$$

Thus, $Y_n = e^{-\tilde{Y}_n} \to e^{-1}$ almost surely.

(7) Additional problem B.

Let X be a Polish space and $K \subset X$. We show that the following statements are equivalent.

(i) K is closed and totally bounded.

(ii) Every sequence (x_n) in K has a subsequence that converges to a point in K.

These two statements are equivalent (both are true) for $K=\emptyset$. Thus we only consider the case $K \neq \emptyset$.

<u>Proof of (i) $\Rightarrow$ (ii)</u>

Let (x_n) be a sequence in K. Because K is totally bounded, it can be covered by finitely many balls of radius $\frac{1}{2}$. There is at least one of them, called B_1, that contains infinitely many terms of (x_n). These terms form a subsequence

$(x_n^{(1)})$ of (x_n). Suppose we have a ball $B_m \subset X$ with radius 2^{-m} that contains a sequence $(x_n^{(m)})_{n\in\mathbb{N}}$. Because K is totally bounded, it can be covered by finitely many balls of radius 2^{-m-1}. There is at least one of them, called B_{m+1}, that contains infinitely many terms of $(x_n^{(m)})_{n\in\mathbb{N}}$. These terms form a subsequence $(x_n^{(m+1)})_{n\in\mathbb{N}}$ of $(x_n^{(m)})_{n\in\mathbb{N}}$. The procedure is repeated as m takes values $1, 2, 3, \dots$ Put $a_n = x_n^{(n)}$. Then (a_n) is a subsequence of (x_n). By the construction, $a_m, a_{m+1}, a_{m+2}, \dots$ are terms in the sequence $(x_n^{(m)})_{n\in\mathbb{N}}$. Thus, $a_k \in B_m$ for every $k \geq m$. This implies

$$\rho(a_n, a_k) \leq \operatorname{diam}(B_m) \leq 2^{-m+1} \quad \forall k, n \geq m.$$

Here ρ denotes the metric on X. Hence, (a_n) is a Cauchy sequence in X. Because X is complete, (a_n) converges to a point in X.

<u>Proof of (ii) $\Rightarrow$ (i)</u>

First, we show that K is closed. Let (x_n) be a sequence in K that converges to some $x_0 \in K$. By the hypothesis, (x_n) has a subsequence (x_{n_k}) that converges to a point in K. This limit has to be x_0. Thus, $x_0 \in K$.

Next, we show that K is totally bounded. Suppose otherwise. Then there exists a number $\varepsilon > 0$ such that K cannot be covered by finitely many balls of radius ε. Take $x_1 \in K$. Because $K \not\subset B(x_1, \varepsilon)$, there exists $x_2 \in K \setminus B(x_1, \varepsilon)$

Suppose we have chosen $x_n \in K \setminus \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$. Note that $K \not\subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$ because otherwise K would be covered by finitely many balls of radius ε. Take an element $x_{n+1} \in K \setminus \bigcup_{i=1}^{n} B(x_i, \varepsilon)$. By following this procedure, we have constructed a sequence (x_n) in K. Accordingly, $x_n \notin B(x_m, \varepsilon)$ if $n > m$. Thus, $\rho(x_n, x_m) \geq \varepsilon$ if $n > m$. This implies (x_n) does not have any Cauchy subsequence. Hence, it does not have any convergent subsequence. This is a contradiction.