Homework 1 due on Monday September 8, 2014.

Chapter 1, Problems 1, 3, 4, 5. Additional problems:

A) Find the probability that in n flips of a fair coin the number of heads is even.

B) Find the probability that the number of flips to get the first head is even.

C) Let $\mathcal{E}$ be the collection of intervals in $\mathbb{R} = (-\infty, \infty)$ of type (r, ∞) where r is an arbitrary rational number. Prove that $\sigma(\mathcal{E}) = \mathfrak{B}(\mathbb{R})$ ($\sigma(\mathcal{E})$ is the smallest σ-field containing $\mathcal{E}$).

Ordered results for HW1, each problem is worth 7pts

49, 48, 48, 47, 46, 46, 45, 45, 44, 44, 43, 39, 36, 35, 34, 33, 31, 30, 29, 28, 27, 20.

Homework 2 due on Wednesday September 17, 2014.

Problems 7.2, 7.8. In Problem 7.14 (show not what is required there but that any Borel set is regular, by the way, $\mathcal{B}$ there is our $\mathfrak{B}(\mathbb{R}^d)$)

Also solve the following problems.

A) Prove that if $\mathfrak{A}$ is both a λ-system and a π-system, then it is a σ-field.

B) Prove that if μ and ν are σ-finite measures on a measurable space $(\Omega, \mathcal{F})$, $\mathcal{F} = \sigma(\mathcal{E})$, $\mathcal{E}$ is a π-system, and $\mu = \nu$ on $\mathcal{E}$, then $\mu = \nu$ on $\mathcal{F}$. In this situation we call μ σ-finite (relative to $\mathcal{E}$) if there is an expanding sequence of $\Omega_n \in \mathcal{E}$ such that $\Omega = \bigcup_n \Omega_n$ and $\mu(\Omega_n) < \infty$. The same definition applies to ν.

In the following two problems $\mathcal{B}$ is a collection of subsets of Ω such that $\emptyset \in \mathcal{B}$ and f is a nonnegative function on $\mathcal{B}$ such that $f(\emptyset) = 0$.

C) We call $A \subset \Omega$ an f-null set if $f^*(A) = 0$. Prove that f-null sets are f^*-sets.

D) Prove that if $\mathcal{B}$ is a σ-field and f is σ-additive on $\mathcal{B}$, then for any $A \subset \Omega$ there exists $B \in \mathcal{B}$ such that $A \subset B$ and $f^*(A) = f(B)$.

Ordered results for HW2, each problem is worth 7pts

49, 49, 49, 49, 49, 49, 49, 49, 48, 48, 48, 47, 46, 46, 46, 45, 43, 43, 42, 41, 41, 39, 31.

Ordered results for HW1+HW2

97, 97, 97, 96, 95, 95, 93, 92, 92, 88, 88, 83, 82, 79, 78, 76, 76, 76, 72, 69, 64, 46 (no HW1), 42 (no HW1).

Homework 3 due on Monday September 29, 2014

Problems 2.3 (a word of caution: You do not need to show that $\{\omega : X(\omega) \neq Y(\omega)\}$ is an event, this is given. The values of X, Y lie in an arbitrary measurable space), 2.6, 7.19 (Hint: Consider the function $f(t) = \mu([0,t))$, $t \geq 0$, and prove that $f(t+s) = f(t) + f(s)$, $t, s \geq 0$.)

Also do the following problems:

A) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\bar{\mathcal{F}} = \bar{\mathcal{F}}^{\mu}$ as the collection of sets of the type $B \cup N$, where $B \in \mathcal{F}$ and there is a $C \in \mathcal{F}$ such that $\mu(C) = 0$ and $N \subset C$. Prove that $\bar{\mathcal{F}}$ is a σ-field and if we define $\mu(B \cup N) = \mu(B)$, then this is a well-defined function on $\bar{\mathcal{F}}$, which is σ-additive, and, of course, agrees with μ on $\mathcal{F}$.

B) For $a \geq 0, b \in \mathbb{R}$ and Borel set $B \subset \mathbb{R}$ define $aB + b = \{ax + b : x \in B\}$. Prove that $aB + b$ is Borel and $\lambda(aB + b) = a\lambda(B)$, where λ is Lebesgue measure.

In the following problems we assume the setting of Example 1.4 presented in class and in lecture notes.

C) Take a finite subset $\{k_1, k_2, ..., k_n\}$ of integers $\{1, 2, 3, ...\}$ such that $k_1 < k_2 < ... < k_n$ and take some numbers $a_1, a_2, ...$ such that $a_j = 0$ or 1. Show that

$$\{\omega : \omega_{k_j}(\omega) = a_j \quad \forall j = 1, ..., n\}$$

is an event and compute its probability.

D) For $\omega = (\omega_1, \omega_2, ...)$ define

$$X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k}}{2^k}, \quad Y(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k-1}}{2^k}.$$

Show that for any numbers a, b, c, d such that $0 \leq a \leq b \leq 1$ and $0 \leq c \leq d \leq 1$ we have

$$P(\{\omega : (X(\omega), Y(\omega)) \in (a, b] \times (c, d]\}) = (b - a)(d - c).$$

(Hint: Start with $b = d = 1$ and use an argument given in class.)

Ordered results for HW3, each problem is worth 7pts

49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 48, 48, 48, 46, 46, 44, 43, 43, 43, 43, 43, 28, 25.

Ordered results for HW1+HW2+HW3

144, 144, 143, 143, 141, 140, 140, 140, 137, 137, 136, 132, 131, 128, 125, 125, 124, 121, 112, 107, 103, 94, 85, 28.

Homework 4 due on Monday October 13, 2014, each problem is worth 7pts.

Problems 3.5, 4.19

Also do the following.

A) ($\sim$ Problem 4.7) Let ξ be binomially distributed. For each z, find Ez^{ξ}. Then, for each integer k, find $E\xi(\xi - 1) \cdot ... \cdot (\xi - k)$. By comparing $E\xi$ with $E\xi(\xi - 1)$ find $E\xi^2$.

B) Let numbers $a_k^n \geq 0$ and a_k be given for $n, k = 1, 2, ...$ Assume that $a_k^n \uparrow a_k$ as $n \to \infty$ for any k. Prove then that

$$\lim_{n\to\infty} \sum_k a_k^n = \sum_k a_k.$$

C) ($\sim$ Problem 7.10) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $A_n \in \mathcal{F}$, $n = 1, 2, ...$ Introduce

$$\underline{\lim}_{n\to\infty} A_n$$

as the set of $x \in \Omega$ for each of which there exists an n_0 such that $x \in A_n$ for all $n \geq n_0$. Prove that

$$\underline{\lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

and

$$\underline{\lim}_{n\to\infty} I_{A_n} = I_{\underline{\lim}_{n\to\infty} A_n}.$$

Also prove that

$$\mu(\underline{\lim}_{n\to\infty} A_n) \leq \underline{\lim}_{n\to\infty} \mu(A_n).$$

D) Let two sets Ω, X and a function $\xi(\omega)$ defined on Ω with values in X be given. For an arbitrary set $B \subset X$ denote $\xi^{-1}(B) = \{\omega : \ \xi(\omega) \in B\}$ (the inverse image of B under the mapping $\xi : \ \Omega \to X$). Prove that if $\mathcal{F}$ is a σ-field in Ω, then $\{B : \ B \subset X, \ \xi^{-1}(B) \in \mathcal{F}\}$ is a σ-field in X.

E) ($\sim$ Problem 4.26) Assume that $\mu(f \in B) = \mu(-f \in B)$ for all Borel $B \subset \mathbb{R}$. Prove that the integral

$$\int_\Omega f \, \mu(dx)$$

equals zero if it exists.

Ordered results for HW4, each problem is worth 7pts

49, 49, 49, 49, 49, 49, 49, 49, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 48, 46, 46, 46, 46, 46.

Ordered results for HW1+...4

193, 191, 190, 190, 189, 188, 188, 186, 186, 185, 184, 181, 179, 177, 173, 173, 170, 169, 160, 156, 152, 142, 131, 77.

Homework 5 due on Monday October 27, 2014, each problem is worth 7pts.

A) Let $\xi, \xi_1, \xi_2, ...$ be a sequence of random variables, $r \in [1, \infty)$. Assume that $\lim_{n\to\infty} \xi_n$ exists (a.s.) and equals ξ (a.s.) Also assume that

$$\lim_{n\to\infty} E|\xi_n|^r = E|\xi|^r < \infty.$$

Prove that

$$\lim_{n\to\infty} E|\xi_n - \xi|^r = 0.$$

(Hint: Repeat with some modifications the proof of this fact given in Lecture Notes for $r = 1$.)

B) ($\sim$ Problem 8.36) Let f be a nonnegative Borel measurable function on $(-\infty, \infty)$, $a, b \in (-\infty, \infty), a \neq 0$. Show that (all integrals are Lebesgue integrals)

$$\int_{-\infty}^{\infty} f(x)\, dx = |a| \int_{-\infty}^{\infty} f(ax + b)\, dx.$$

C) Let μ be Lebesgue measure on $\mathbb{R}$, $\varepsilon \in (0, 1)$ and suppose that A is a Borel bounded set such that $\mu(AI) \le \varepsilon\mu(I)$ for any interval I. Prove that $\mu(A) = 0$.

D) Let X and Y be real-valued random variables each taking only two values: $a_1 \neq a_2$ for X and $b_1 \neq b_2$ for Y. Assume that X and Y are uncorrelated and prove that they are independent. (Hint: Start with the case $X = I_A, Y = I_B$.)

E) (Problem 3.32) Let a random variable X have a density $f(x)$. Let a, b be real numbers with $a \neq 0$. Prove that $aX + b$ has a density and find it.

F) (Problem 5.28) Prove that $\big(E|X|^p\big)^{1/p}$ is an increasing function of p on $(0, \infty)$.

G) Let numbers $a_k^n \ge 0$ and a_k be given for $n, k = 1, 2, ...$ Assume that $a_k^n \to a_k$ as $n \to \infty$ for any k. Also assume $\sum_k a_k^n \to \sum_k a_k < \infty$. Prove then that

$$\lim_{n\to\infty} \sum_k |a_k^n - a_k| = 0.$$

Ordered results for HW5, each problem is worth 7pts

49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 48, 48, 48, 48, 47, 44, 44, 42, 42, 42, 39, 37.

Ordered results for HW1+...+5

242, 239, 239, 238, 237, 236, 235, 235, 234, 234, 228, 226, 225, 223, 222, 221, 218, 217, 202, 201, 195, 191, 168, 121.

Homework 6 due on Monday November 10, 2014, each problem is worth 7pts.

Problem 5.31, Problem 9.36 (Prove not what is required but that

$$\lim_{n\to\infty} \frac{S_n}{M_n} = P(Y \in B) \quad \text{(a.s.).)}$$

Problems 9.45, 12.9.

Also do the following

A) Let $X_1, X_2, ...$ be a sequence of nonnegative pairwise nonpositively correlated random variables such that $\mu_n = EX_n < \infty$ and $\mathrm{Var}\, X_n \le M\mu_n$ for all n, where M is a constant. By generalizing an argument given in class, prove that

$$\sum_{n=1}^{\infty} \mu_n = \infty \quad \Longrightarrow \quad \sum_{n=1}^{\infty} X_n = \infty \quad \text{(a.s.)}.$$

B) Let $(E, \mathcal{F}, \mu)$ be a measure space with σ-finite measure. Let $f(t,x)$ be a function on $[a,b] \times E$ measurable with respect to $\mathfrak{B}([a,b]) \otimes \mathcal{F}$. Assume that there exists a measurable function $g(t,x)$ on $[a,b] \times E$ such that for any x we have $f'(t,x) = g(t,x)$ in the sense explained in class. Finally, assume that

$$\int_E \int_a^b |f'(t,x)|\, dt\mu(dx) < \infty, \qquad \int_E |f(a,x)|\, \mu(dx) < \infty.$$

Prove that

$$\Big(\int_E f(t,x)\, \mu(dx) \Big)' = \int_E f'(t,x)\, \mu(dx).$$

C) Prove the following. **Theorem** (Polya) Let F_n, $n = 1, 2, ...$, be a sequence of distribution functions and let F be a distribution function. Let ρ be a countable dense subset of $\mathbb{R}$ containing all points of discontinuity of F. Assume that for any $x \in \rho$

$$\lim_{n\to\infty} F_n(x) = F(x),$$

and for any point x of discontinuity of F

$$\lim_{n\to\infty} F_n(x-) = F(x-).$$

Then

$$\lim_{n\to\infty} \sup_x |F_n(x) - F(x)| = 0.$$

(Hint: First prove that $F_n(x) \to F(x)$ at each point of continuity of F and thus at any point in $\mathbb{R}$. Also prove that $F_n(x-) \to F(x-)$ at any point in $\mathbb{R}$. Then notice that to prove the theorem it suffices to show that for any converging sequence $x_n \in \mathbb{R}$ we have $F_n(x_n) - F(x_n) \to 0$. Assume the contrary and consider two cases 1) infinitely many points of the sequence are on the right of the limit point and 2) on the left.)

Ordered results for HW6, each problem is worth 7pts

49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 48, 48, 48, 48, 45, 42, 40, 21.

Ordered results for HW1+...+6

291, 288, 288, 287, 286, 285, 283, 283, 282, 277, 274, 272, 271, 271, 270, 267, 256, 249, 243, 242, 240, 217, 168, 163.

Homework 7 due on Monday November 24, 2014, each problem is worth 7pts.

Problems 4.20, 9.7, 14.1, 14.15, 14.42.

Also do the following.

A) (Like 12.17) Find the limit of $\left(\prod_{k=1}^{n} X_k\right)^{1/n}$, where X_k's are iid uniformly distributed on $[0, 1]$.

B) Let X be a Polish space and $K \subset X$. Prove that the following are equivalent:

(i) K is a closed, totally bounded set (totally bounded in the sense that for every $\varepsilon > 0$, there exists a finite set $A = \{x_1, ..., x_n\}$, called an ε-net, such that every point of K is in the ε-neighborhood of at least one point in A).

(ii) For every sequence of points $x_n \in K$, there is a subsequence $x_{n'}$ which converges to an element of K.

Ordered results for HW7, each problem is worth 7pts

49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 49, 48, 47, 47, 46, 46, 46, 45, 45, 43.

Ordered results for the sums of scores for the best 6 HWs

294, 293, 293, 292, 291, 290, 289, 289, 289, 289, 288, 288, 287, 286, 285, 284, 282, 278, 273, 264, 263, 261, 211, 208.