Theory of Probability and Measure Theory – Math 8651

Homework #3

Problems 2.3, 2.6, 7.19. Additional problems:

A) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\overline{\mathcal{F}} = \overline{\mathcal{F}}^{\mu}$ as the collection of sets of the type $B \cup N$, where $B \in \mathcal{F}$ and there is $C \in \mathcal{F}$ such that $\mu(C) = 0$ and $N \subset C$. Prove that $\overline{\mathcal{F}}$ is a σ-field and if we define $\mu(B \cup N = \mu(B)$ then this is a well-defined function on $\overline{\mathcal{F}}$, which is σ-additive and agrees with μ on $\mathcal{F}$.

B) For $a \geq 0$, $b \in \mathbb{R}$ and Borel set $B \subset \mathbb{R}$ define $aB + b = \{ax + b : x \in B\}$. Prove that $aB + b$ is Borel and $\lambda(aB + b) = a\lambda(B)$, where λ is Lebesgue measure.

In the following problems, we assume the setting of Example 1.4 presented in class and in lecture notes.

C) Take a finite subset $\{k_1, k_2, ..., k_n\}$ of integers $\{1, 2, 3, ...\}$ such that $k_1 < k_2 < ... < k_n$ and take some numbers $a_1, a_2, ...$ such that $a_j = 0$ or 1. Show that

$$\{\omega : \omega_{k_j}(\omega) = a_j \quad \forall j = 1, ..., n\}$$

is an event and compute its probability.

D) For $\omega = (\omega_1, \omega_2, ...)$ define

$$X(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k}}{2^k}, \quad Y(\omega) = \sum_{k=1}^{\infty} \frac{\omega_{2k-1}}{2^k}.$$

Show that for any numbers a, b, c, d such that $0 \leq a \leq b \leq 1$ and $0 \leq c \leq d \leq 1$ we have

$$P\left(\{\omega : (X(\omega), Y(\omega)) \in (a, b] \times (c, d]\}\right) = (b - a)(d - c).$$