Theory of Probability and Measure Theory – Math 8651

Homework #4

Problems 3.5, 4.19. Additional problems:

A) (Similar to Problem 4.7) Let ξ be binomially distributed. For each z, find Ez^{ξ}. Then, for each integer k, find $E\xi(\xi-1)...(\xi-k)$. By comparing $E\xi$ with $E\xi(\xi-1)$ find $E\xi^2$.

B) Let numbers $a_k^n \geq 0$ and a_k be given for $n, k = 1, 2, ...$ Assume that $a_k^n \uparrow a_k$ as $n \to \infty$ for any k. Prove then that

$$\lim_{n\to\infty} \sum_k a_k^n = \sum_k a_k.$$

C) (Similar to Problem 7.10) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $A_n \in \mathcal{F}$, $n = 1, 2, ...$ Introduce

$$\varliminf_{n\to\infty} A_n$$

as the set of $x \in \Omega$ for each of which there exists an n_0 such that $x \in A_n$ for all $n \geq n_0$. Prove that

$$\varliminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

and

$$\varliminf_{n\to\infty} I_{A_n} = I_{\varliminf_{n\to\infty} A_n}.$$

Also prove that

$$\mu\left(\varliminf_{n\to\infty} A_n\right) \leq \varliminf_{n\to\infty} \mu(A_n).$$

D) Let two sets Ω, X, and a function $\xi(\omega)$ defined on Ω with values in X be given. For an arbitrary set $B \subset X$ denote $\xi^{-1}(B) = \{\omega : \xi(\omega) \in B\}$. Prove that if $\mathcal{F}$ is a σ-field in Ω then $\{B : B \subset X,\ \xi^{-1}(B) \in \mathcal{F}\}$ is a σ-field in X.

E) (Similar to Problem 4.26) Assume that $\mu(f \in B) = \mu(-f \in B)$ for all Borel $B \subset \mathbb{R}$. Prove that the integral

$$\int_{\Omega} f \mu(dx)$$

equals to zero if it exists.