Theory of Probability and Measure Theory – Math 8651

Homework #5

A) Let $\xi, \xi_1, \xi_2, \ldots$ be a sequence of random variables, $r \in [1, \infty)$. Assume that $\lim_{n\to\infty} \xi_n$ exists (a.s.) and equals ξ (a.s.) Also assume that

$$\lim_{n\to\infty} E|\xi_n|^r = E|\xi|^r < \infty.$$

Prove that

$$\lim_{n\to\infty} E|\xi_n - \xi|^r = 0.$$

B) (Similar to Problem 8.36) Let f be a nonnegative Borel measurable function on $(-\infty, \infty)$, $a, b \in (-\infty, \infty)$, $a \neq 0$. Show that (all integrals are Lebesgue integrals)

$$\int_{-\infty}^{\infty} f(x)dx = |a| \int_{-\infty}^{\infty} f(ax + b)dx.$$

C) Let μ be Lebesgue measure on $\mathbb{R}$, $\varepsilon \in (0, 1)$ and suppose that A is a Borel bounded set such that $\mu(AI) \leq \varepsilon\mu(I)$ for any interval I. Prove that $\mu(A) = 0$.

D) Let X and Y be real-valued random variables each taking only two values: $a_1 \neq a_2$ for X, and $b_1 \neq b_2$ for Y. Assume that X and Y are uncorrelated and prove that they are independent.

E) (Problem 3.32) Let X be a random variable having density $f(x)$. Let a, b be real numbers with $a \neq 0$. Prove that $aX + b$ has a density and find it.

F) (Problem 5.28) Prove that $(E|X|^p)^{1/p}$ is an increasing function of p on $(0, \infty)$.

G) Let numbers $a_k^n \geq 0$ and a_k be given for $n, k = 1, 2, \ldots$ Assume that $a_k^n \to a_k$ as $n \to \infty$ for any k. Also assume $\sum_k a_k^n \to \sum_k a_k < \infty$. Prove that

$$\lim_{n\to\infty} \sum_k |a_k^n - a_k| = 0.$$