Theory of Probability and Measure Theory – Math 8651

Homework #6

Problems 5.31, 9.36, 9.45, 12.9. In Problem 9.36, prove not what is asked but that

$$\lim_{n\to\infty} \frac{S_n}{M_n} = P(Y \in B) \quad \text{(a.s.)}$$

Additional problems:

A) Let $X_1, X_2, ...$ be a sequence of nonegative pairwise nonpositively correlated random variables such that $\mu_n = EX_n < \infty$ and $\mathrm{Var} X_n \le M\mu_n$ for all n, where M is a constant. By generalizing an argument given in class, prove that

$$\sum_{n=1}^{\infty} \mu_n = \infty \ \Rightarrow \ \sum_{n=1}^{\infty} X_n = \infty \quad \text{(a.s.)}$$

B) Let $(E, \mathcal{F}, \mu)$ be a measure space with σ-finite measure. Let $f(t,x)$ be a function on $[a,b] \times E$ measurable with respect to $\mathcal{B}([a,b]) \otimes \mathcal{F}$. Assume that there exists a measurable function $g(t,x)$ on $[a,b] \times E$ such that for any x we have $f'(t,x) = f(t,x)$ in the sense explained in class. Finally, assume that

$$\int_E \int_a^b |f'(t,x)| dt \mu(dx) < \infty, \qquad \int_E |f(a,x)| \mu(dx) < \infty.$$

Prove that

$$\left(\int_E f(t,x) \mu(dx) \right)' = \int_E f'(t,x) \mu(dx).$$

C) (Polya's theorem) Let F_n, $n = 1, 2, ...$ be a sequence of distribution functions and let F be a distribution function. Let ρ be a countable dense subset of $\mathbb{R}$ containing all points of discontinuity of F. Assume that for any $x \in \rho$

$$\lim_{n\to\infty} F_n(x) = F(x),$$

and for any point x of discontinuity of F

$$\lim_{n\to\infty} F_n(x-) = F(x-).$$

Then

$$\lim_{n\to\infty} \sup_x |F_n(x) - F(x)| = 0.$$