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Math 8652: Theory of Probability

Problems for Final Exam

① Let $(w_t)_{t\geq 0}$ be a Wiener process. We find the average time the process spends in $[-1,1]$ before exiting $(-2,2)$. The following theorem is needed (Theorem 28.7 in Professor Krylov's lecture notes on 05/01/2015).

> Let $-\infty \leq \alpha < \beta \leq \infty$, $b \in \mathbb{R}$ and $u \in C^{1,1}_{loc}(\mathbb{R})$. Let $c : \mathbb{R} \to \mathbb{R}$ be a bounded Borel function such that $c > \delta$, where $\delta > 0$ is a constant. Denote
>
> $$f = cu - \frac{1}{2}u'' - bu'$$
>
> and assume that f and u are bounded. Then
>
> $$u(0) = E\int_0^\tau e^{-\phi_t} f(\eta_t)\,dt + E e^{-\phi_\tau} u(\eta_\tau),$$
>
> where $\eta_t = w_t + bt,$
>
> $$\phi_t = \int_0^t c(\eta_s)\,ds,$$
>
> $$\tau = \inf\{t \geq 0 : \eta_t \notin (\alpha,\beta)\}.$$

Take $\alpha = -2$, $\beta = 2$, $b = 0$, $c(x) \equiv \lambda > 0$ and $f(x) = I_{[-1,1]}(x)$. Then

$$\eta_t = w_t,$$

$$\phi_t = \int_0^t \lambda\,ds = \lambda t,$$

$$\tau = \inf\{t \geq 0 : w_t \notin (-2,2)\},$$

$$f(\eta_t) = I_{[-1,1]}(w_t).$$

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The theorem asserts that

$$u(0) = E\int_0^\tau e^{-\lambda t} I_{[-1,1]}(w_t)\,dt + E\, e^{-\lambda\tau} u(w_\tau) \qquad (1)$$

where $u \in C^{1,1}_{loc}(\mathbb{R})$ is a bounded solution to the differential equation

$$I_{[-1,1]}(x) = \lambda u(x) - \frac{1}{2}u''(x). \qquad (2)$$

The average time the process spends in $[-1,1]$ before exiting $(-2,2)$ is $E\int_0^\tau I_{[-1,1]}(w_t)\,dt$. Thus, we need to calculate both sides of (1) as $\lambda \to 0^+$.

Equation (2) becomes a system

$$u'' - 2\lambda u = \begin{cases} -2 & \text{if } x \in (-1,1), \\ 0 & \text{if } x < -1 \text{ or } x > 1. \end{cases}$$

The general solution to this system is

$$u(x) = \begin{cases} \frac{1}{\lambda} + c_1 e^{-x\sqrt{2\lambda}} + c_2 e^{x\sqrt{2\lambda}} & \text{if } x \in (-1,1), \\ c_3 e^{-x\sqrt{2\lambda}} + c_4 e^{x\sqrt{2\lambda}} & \text{if } x > 1, \\ c_5 e^{-x\sqrt{2\lambda}} + c_6 e^{x\sqrt{2\lambda}} & \text{if } x < -1, \end{cases}$$

where $c_1, c_2, \ldots, c_6$ are constants to be determined by the fact that u is bounded, and u and u' are continuous. Because u is bounded, $c_4 = c_5 = 0$. The conditions $u(1^-) = u(1^+)$ and $u'(1^-) = u'(1^+)$ yield

$$\begin{cases} c_2 = -\frac{1}{2\lambda} e^{-\sqrt{2\lambda}} & (3) \\ c_1 - c_3 = -\frac{1}{2\lambda} e^{\sqrt{2\lambda}}. & (4) \end{cases}$$

The conditions $u(-1^-) = u(-1^+)$ and $u'(-1^-) = u'(-1^+)$ yield

$$\begin{cases} c_1 = -\frac{1}{2\lambda} e^{-\sqrt{2\lambda}}, & (5) \\ c_6 - c_2 = \frac{1}{2\lambda} e^{\sqrt{2\lambda}}. & (6) \end{cases}$$

From (4) and (5), we get c_3. From (3) and (6), we get c_6. Therefore,

$$c_1 = c_2 = -\frac{1}{2\lambda} e^{-\sqrt{2\lambda}},$$

$$c_3 = c_6 = \frac{1}{2\lambda} \left(e^{\sqrt{2\lambda}} - e^{-\sqrt{2\lambda}}\right).$$

Then

$$u(x) = \begin{cases} \frac{1}{\lambda} - \frac{e^{-\sqrt{2\lambda}}}{2\lambda}\left(e^{-x\sqrt{2\lambda}} + e^{x\sqrt{2\lambda}}\right) & \text{if } |x| < 1, \\ \frac{e^{\sqrt{2\lambda}} - e^{-\sqrt{2\lambda}}}{2\lambda} e^{-|x|\sqrt{2\lambda}} & \text{if } |x| > 1. \end{cases}$$

This is an even function. We get

$$u(0) = \frac{1}{\lambda} - \frac{1}{\lambda} e^{-\sqrt{2\lambda}}, \qquad (7)$$

$$u(-2) = u(2) = \frac{e^{-\sqrt{2\lambda}} - e^{-3\sqrt{2\lambda}}}{2\lambda}. \qquad (8)$$

By the definition of τ, we get $w_\tau = \pm 2$. Then (1) becomes

$$u(0) = E \int_0^\tau e^{-\lambda t} I_{[-1,1]}(w_t)\, dt + u(\pm 2)\, E e^{-\lambda\tau}.$$

Thus,
$$E \int_0^\tau e^{-\lambda t} I_{[-1,1]}(w_t)\, dt = u(0) - u(2)\, E e^{-\lambda\tau}. \qquad (9)$$

The function $\lambda \in (0,\infty) \mapsto E e^{-\lambda\tau} \in \mathbb{R}$ is the moment generating function of τ. By Theorem 28, Fristedt-Gray page 389,

$$E e^{-\lambda\tau} = \frac{2\sinh(2\sqrt{2\lambda})}{\sinh(4\sqrt{2\lambda})} = \frac{1}{\cosh(2\sqrt{2\lambda})} = \frac{2}{e^{2\sqrt{2\lambda}}+e^{-2\sqrt{2\lambda}}}. \quad (10)$$

Substituting (7), (8), (10) into (9), we get

$$E\int_0^\tau e^{-\lambda t} I_{[-1,1]}(w_t)\,dt = \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\sqrt{2\lambda}} - \frac{e^{-\sqrt{2\lambda}} - e^{-3\sqrt{2\lambda}}}{2\lambda}\,\frac{2}{e^{2\sqrt{2\lambda}}+e^{-2\sqrt{2\lambda}}}$$

$$= \frac{1}{\lambda} - \frac{1}{\lambda}\alpha - \frac{\alpha-\alpha^3}{2\lambda}\,\frac{2}{\alpha^{-2}+\alpha^2}$$

$$(\text{where } \alpha = e^{-\sqrt{2\lambda}})$$

$$= \frac{(1-\alpha)^2(\alpha^{-2}+\alpha^{-1}+1)}{\lambda(\alpha^2+\alpha^{-2})}$$

$$= \underbrace{\left(\frac{1-\alpha}{-\sqrt{2\lambda}}\right)^2}_{\{1\}}\ \underbrace{\frac{2(\alpha^{-2}+\alpha^{-1}+1)}{\alpha^2+\alpha^{-2}}}_{\{2\}}.$$

Because $\lim_{x\to 0}\frac{e^x-1}{x} = 1$, $\{1\}\to 1$ as $\lambda\to 0^+$.

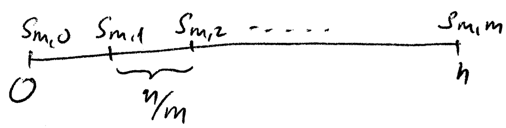
Because $\alpha\to 1$ as $\lambda\to 0^+$, $\{2\}\to 3$ as $\lambda\to 0^+$. Therefore,

$$\lim_{\lambda\to 0^+} E\int_0^\tau e^{-\lambda t} I_{[-1,1]}(w_t)\,dt = 3.$$

Because $0 \le e^{-\lambda t} I_{[-1,1]}(w_t) \le 1$ for all $\lambda > 0$ and $t \ge 0$, by the Dominated Convergence theorem, we get

$$E\int_0^\tau I_{[-1,1]}(w_t)\,dt = 3.$$

(2) Let $(w_t)_{t\geq 0}$ be a Wiener process. We determine the distribution of
$$X = \sup_{t\geq 0}(w_t - t).$$
First, we show that X is a random variable. For $t\geq 0$ and $n\in\mathbb{N}$, we denote $\eta_t = w_t - t$ and $X_n = \sup_{0\leq t\leq n}\eta_t$. Then
$$X_1(\omega)\leq X_2(\omega)\leq\cdots\leq X_n(\omega)\leq\cdots \quad \forall\omega\in\Omega,$$
$$\lim_{n\to\infty}X_n(\omega) = X(\omega).$$
It suffices to show that each X_n is a random variable. Fix $n\in\mathbb{N}$. For each $m\in\mathbb{N}$, we divide the interval $[0,n]$ into m equal subintervals by the points $0 = s_{m,0} < s_{m,1} < \cdots < s_{m,m} = n$.



Define $Y_{n,m}(\omega) = \max\{\eta_{s_{m,0}}(\omega), \eta_{s_{m,1}}(\omega), \dots, \eta_{s_{m,m}}(\omega)\}$ for all $\omega\in\Omega$. Fix $\omega\in\Omega$. The function $t\in[0,n]\mapsto\eta_t(\omega)\in\mathbb{R}$ is continuous. Thus, it attains maximum at some point $t_0\in[0,n]$. Then $X_n(\omega) = \eta_{t_0}(\omega)$. For each $\varepsilon>0$, there exists $\delta>0$ such that $0\leq\eta_{t_0}(\omega)-\eta_t(\omega)<\varepsilon$ for all $t\in[0,n]$, $|t-t_0|<\delta$. For $m>n\delta^{-1}$, the intervals $[s_{m,i}, s_{m,i+1})$ have length $\frac{n}{m}<\delta$. Thus, there exists $i\in\{0,1,\dots,m-1\}$ such that $|s_{m,i}-t_0|<\delta$. Then $\eta_{t_0}(\omega)-\eta_{s_{m,i}}(\omega)<\varepsilon$.

We get
$$Y_{n,m}(\omega)\geq\eta_{s_{m,i}}(\omega) > \eta_{t_0}(\omega)-\varepsilon = X_n(\omega)-\varepsilon \quad \forall m>n\delta^{-1}.$$
Thus,

$$X_n(\omega)-\varepsilon < Y_{n,m}(\omega) \leq X_n(\omega) \qquad \forall m > n\delta^{-1}.$$

This shows $\lim\limits_{m\to\infty} Y_{n,m}(\omega) = X_n(\omega)$. For each $m \in \mathbb{N}$, the map $Y_{n,m} : \Omega \to \mathbb{R}$ is a random variable. Thus, X_n is a random variable. We conclude that X is a random variable.

Because $X_n \to X$ a.s, $F_n \Rightarrow F$ where F_n and F are the distribution functions of X_n and X respectively (Theorem 3, Fristedt-Gray page 249).

$$F_n(x) = P(X_n \leq x),$$

$$F(x) = P(X \leq x).$$

For $a, b, T > 0$, the following identity was established in Lecture 04/24/2015 (the last line on page 126 of Professor Krylov's online notes).

$$P\left(\max_{t\leq T}(w_t - bt) \geq a\right) = \int_0^T \frac{a}{t\sqrt{2\pi t}}\, e^{-\frac{a^2}{2t} - ab - \frac{b^2 t}{2}}\, dt.$$

Applying this identity for $b=1$, $T=n$ and $a=x$, we get

$$P(X_n \geq x) = \int_0^n \frac{x}{t\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t} - x - \frac{t}{2}}\, dt \qquad \forall x > 0.$$

This is a continuous function in $x \in (0,\infty)$. Thus, $P(X_n = x) = 0$ and

$$F_n(x) = 1 - P(X_n \geq x) = 1 - \int_0^n \frac{x}{t\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t} - x - \frac{t}{2}}\, dt \qquad \forall x > 0.$$

Then

$$\lim_{n\to\infty} F_n(x) = 1 - \int_0^\infty \frac{x}{t\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t} - x - \frac{t}{2}}\, dt \qquad \forall x > 0.$$

This is a continuous function on $x \in (0,\infty)$. Thus,

$$F(x) = 1 - \int_0^\infty \frac{x}{t\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t} - x - \frac{t}{2}}\, dt \qquad \forall x > 0.$$

Now we simplify $F(x)$.

$$F(x) = 1 - \int_0^\infty \frac{x}{t\sqrt{2\pi t}}\, e^{-\frac{1}{2}\left(\frac{x}{\sqrt{t}} + \sqrt{t}\right)^2} dt$$

$$\underline{\underline{s=\sqrt{t}}} \quad 1 - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x}{s^2}\, e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} ds. \qquad (1)$$

Put
$$I(x) = \int_0^\infty e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} ds \qquad \forall x > 0.$$

We show that $I(x) < \infty$.

$$I(x) = \int_0^1 e^{-\frac{1}{2}\frac{x^2}{s^2} - x - \frac{s^2}{2}} ds + \int_1^\infty e^{-\frac{1}{2}\frac{x^2}{s^2} - x - \frac{s^2}{2}} ds$$

$$\leq \underbrace{\int_0^1 e^{-\frac{1}{2}\frac{x^2}{s^2}} ds}_{\{1\}} + \underbrace{\int_1^\infty e^{-\frac{s^2}{2}} ds}_{<\infty}.$$

$$\{1\} \underline{\underline{t=\frac{x}{s}}} \quad x\int_x^\infty \frac{1}{t^2} e^{-\frac{1}{2}t^2} dt \leq \frac{1}{x}\int_x^\infty e^{-\frac{1}{2}t^2} dt < \infty.$$

Thus, $I(x) < \infty$ for all $x > 0$.

For each $x_0 > 0$, there exists $\varepsilon > 0$ such that $x_0 - \varepsilon > 0$. For every $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$,

$$\left|\frac{d}{dx} e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2}\right| = \left(\frac{x}{s^2}+1\right) e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} \leq \left(\frac{x_0+\varepsilon}{s^2}+1\right) e^{-\frac{1}{2}\left(\frac{x_0-\varepsilon}{s}+s\right)^2}$$

$$\underline{\underline{a=x_0-\varepsilon}} \quad \underbrace{\left(\frac{a+2\varepsilon}{s^2}+1\right) e^{-\frac{1}{2}\left(\frac{a}{s}+s\right)^2}}_{f(s)}.$$

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We show that $\int_0^\infty f(s)ds < \infty$.

$$\int_0^\infty f(s)ds = (a+2\varepsilon)\underbrace{\int_0^\infty \frac{1}{s^2} e^{-\frac{1}{2}\left(\frac{a}{s}+s\right)^2} ds}_{\{2\}} + \underbrace{\int_0^\infty e^{-\frac{1}{2}\left(\frac{a}{s}+s\right)^2} ds}_{=I(a)<\infty} .$$

$$\{2\} = \int_0^1 \frac{1}{s^2} e^{-\frac{1}{2}\frac{a^2}{s^2}-a-\frac{s^2}{2}} ds + \int_1^\infty \frac{1}{s^2} e^{-\frac{1}{2}\left(\frac{a}{s}+s\right)^2} ds$$

$$\leq \underbrace{\int_0^1 \frac{1}{s^2} e^{-\frac{1}{2}\frac{a^2}{s^2}} ds}_{\{3\}} + \underbrace{\int_1^\infty e^{-\frac{1}{2}\left(\frac{a}{s}+s\right)^2} ds}_{\leq I(a)<\infty} .$$

By the change of variable $t = \frac{a}{s}$,

$$\{3\} = \frac{1}{a^3}\int_a^\infty t^2 e^{-\frac{1}{2}t^2} dt < \infty .$$

Thus, $\{2\} < \infty$ and hence $\int_0^\infty f(s)\,ds < \infty$. This allows us to differentiate $I(x)$ by differentiating the integrand of $I(x)$.

$$I'(x) = \int_0^\infty \frac{d}{dx} e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} ds = \int_0^\infty \left(-\frac{x}{s^2}-1\right) e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} ds \qquad (2)$$

$$\underset{y=\frac{x}{s}+s}{=\!=} \int_0^\infty \left(\frac{dy}{ds}-2\right) e^{-\frac{1}{2}y^2} ds$$

$$= \lim_{n\to\infty} \int_{\frac{\sqrt{x}}{n}}^{n\sqrt{x}} \frac{dy}{ds} e^{-\frac{1}{2}y^2} ds - \int_0^\infty 2 e^{-\frac{1}{2}y^2} ds$$

$$= \underbrace{\lim_{n\to\infty} \int_{n\sqrt{x}+\frac{\sqrt{x}}{n}}^{n\sqrt{x}+\frac{\sqrt{x}}{n}} e^{-\frac{1}{2}y^2} dy}_{=0} - 2I(x) = -2I(x).$$

The differential equation $I'(x) = -2I(x)$ yields $I(x) = Ce^{-2x}$, where $C > 0$ is some constant. To find C, we observe that

$$0 < e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} \leq 1 \quad \forall x > 0,$$

$$\lim_{x \to 0^+} e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} = e^{-\frac{1}{2}s^2}.$$

By the Dominated Convergence Theorem,

$$\lim_{x \to 0^+} I(x) = \int_0^\infty e^{-\frac{1}{2}s^2} ds = \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{1}{2}s^2} ds$$

$$= \frac{1}{2}\sqrt{2\pi} \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds}$$

$= 1$ because the integrand is the density function of the distribution $N(0,1)$.

$$= \sqrt{\frac{\pi}{2}}.$$

Therefore, $C = \lim_{x \to 0^+} I(x) = \sqrt{\frac{\pi}{2}}$. From (2), we get

$$I'(x) = -\int_0^\infty \frac{x}{s^2} e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} ds - I(x).$$

Then

$$\int_0^\infty \frac{x}{s^2} e^{-\frac{1}{2}\left(\frac{x}{s}+s\right)^2} ds = -I'(x) - I(x) = I(x).$$

Substituting this result into (1), we get

$$F(x) = 1 - \sqrt{\frac{2}{\pi}}\, I(x) = 1 - \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}}\, e^{-2x} = 1 - e^{-2x} \quad \forall x > 0.$$

Because a distribution function is right-continuous, $F(0) = \lim_{x \to 0^+} F(x) = 0$.

Because $X(\omega) = \sup_{t \geq 0} (w_t(\omega) - t) \geq w_0(\omega) - 0 = 0$, we have $F(x) = 0$

for all $x<0$. Therefore, the distribution function of X is

$$F(x) = \begin{cases} 1-e^{-2x} & \text{if } x \geq 0, \\ 0 & \text{if } x<0. \end{cases}$$

In other words, X is exponentially distributed with parameter $a=2$.

③ Let $(X_n)_{n\geq 0}$ be a Markov chain on $\mathbb{Z}^d$ starting at the origin, i.e. $X_0 \equiv 0$, and having transition kernel

$$p(x,B) = \sum_{y\in B} p(y-x) \quad \forall B\subset \mathbb{Z}^d, \forall x\in\mathbb{Z}^d,$$

where p is a probability distribution on $\mathbb{Z}^d$!

Suppose $p(0)<1$. We show that $(X_n)_{n\geq 0}$ is either transient or null recurrent. First, we show that $Y_n = I_{X_n=0}$ is a renewal sequence. (This turns out to be a general property of Markov chains, as stated in Lecture notes 04/10/2015)

Introduce

$$T_0 \equiv 0,$$
$$T_1 = \inf\{n>0 : Y_n = 1\},$$
$$T_m = \inf\{n>T_{m-1} : Y_n=1\} \quad \forall m\geq 2.$$

For any $n\geq 2$, $k_1, k_2, \ldots, k_n \in \mathbb{N}$, we denote $l_0=0$, $l_j = k_1+\cdots+k_j$ for $1\leq j\leq n$.

$$\{\omega : T_1(\omega)-T_0(\omega)=k_1, \ldots, T_n(\omega)-T_{n-1}(\omega)=k_n\} = (T_1-T_0=k_1, \ldots, T_n-T_{n-1}=k_n)$$

$$= (Y_0=1, Y_1=\cdots=Y_{l_1-1}=0, Y_{l_1}=1, Y_{l_1+1}=\cdots=Y_{l_2-1}=0, Y_{l_2}=1, \ldots, Y_{l_{n-1}}=1,$$
$$Y_{l_{n-1}+1}=\cdots=Y_{l_n-1}=0, Y_{l_n}=1)$$

$$= (X_0 = 0, X_1, \dots, X_{\ell_1 - 1} \neq 0, X_{\ell_1} = 0, \dots, X_{\ell_{n-1}} = 0, X_{\ell_{n-1}+1}, \dots, X_{\ell_n - 1} \neq 0, X_{\ell_n} = 0)$$

$$= \bigcup_{\substack{i_1, \dots, i_{\ell_1 - 1} \neq 0 \\ \cdots \\ i_{\ell_{n-1}+1}, \dots, i_{\ell_n - 1} \neq 0}} (X_0 = 0, X_1 = i_1, \dots, X_{\ell_1 - 1} = i_{\ell_1 - 1}, X_{\ell_1} = 0, \dots, X_{\ell_{n-1}+1} = i_{\ell_{n-1}+1}, \dots, X_{\ell_n - 1} = i_{\ell_n - 1}, X_{\ell_n} = 0).$$

The probability of this event is

$$P(T_1 - T_0 = \ell_1, \dots, T_n - T_{n-1} = \ell_n) = \sum_{\substack{i_1, \dots, i_{\ell_1 - 1} \neq 0 \\ \cdots \\ i_{\ell_{n-1}+1}, \dots, i_{\ell_n - 1} \neq 0}} P(X_0 = 0, X_1 = i_1, \dots, X_{\ell_1 - 1} = i_{\ell_1 - 1}, X_{\ell_1} = 0, \dots, X_{\ell_{n-1}} = 0, X_{\ell_{n-1}+1} = i_{\ell_{n-1}+1}, \dots, X_{\ell_n - 1} = i_{\ell_n - 1}, X_{\ell_n} = 0). \quad (1)$$

To compute the right hand side, we show that

$$P(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} \quad \forall n \in \mathbb{N}, \forall i_1, i_2, \dots, i_n \in \mathbb{Z}^d. \quad (2)$$

where p_{ab} denotes $p(a, \{b\})$. It suffices to show

$$P(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = P(X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) p_{i_{n-1} i_n}. \quad (3)$$

Once (3) is proved, (2) follows by applying (3) sucessively. We have

$$P(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = E I_{X_0 = i_0, \dots, X_n = i_n}$$

$$= E I_{X_0 = i_0, \dots, X_{n-1} = i_{n-1}} I_{X_n = i_n}$$

$$= E[\underbrace{E(I_{X_n = i_n} \mid X_0, X_1, \dots, X_{n-1})}_{p(X_{n-1}, \{i_n\})} I_{X_0 = i_0, \dots, X_{n-1} = i_{n-1}}]$$

$$= E(p(i_{n-1}, \{i_n\}) I_{X_0 = i_0, \dots, X_{n-1} = i_{n-1}})$$

$$= p(i_{n-1}, \{i_n\}) E I_{X_0 = i_0, \dots, X_{n-1} = i_{n-1}}$$

$$= p_{i_{n-1} i_n} P(X_0 = i_0, \dots, X_{n-1} = i_{n-1}).$$

We have proved (3).

Applying (2) to compute the right hand side of (1), we get

$$\mathbb{P}(T_1-T_0=k_1,\dots,T_n-T_{n-1}=k_n)=\sum_{\substack{i_0=0,i_1,\dots,i_{k_1-1}\neq 0,\\ \dots\\ i_{k_{n-1}}=0,i_{k_{n-1}+1},\dots,i_{k_n-1}\neq 0,\\ i_{k_n}=0}} P_{i_0i_1}P_{i_1i_2}\cdots P_{i_{k_n-1}i_{k_n}}$$

$$=\Big(\sum_{\substack{i_0=0,i_1,\dots,i_{k_1-1}\neq 0,\\ i_{k_1}=0}} P_{i_0i_1}P_{i_1i_2}\cdots P_{i_{k_1-1}i_{k_1}}\Big)\cdots\Big(\sum_{\substack{i_{k_{n-1}}=0,i_{k_{n-1}+1},\dots,i_{k_n-1}\neq 0,\\ i_{k_n}=0}} P_{i_{k_{n-1}}i_{k_{n-1}+1}}\cdots P_{i_{k_n-1}i_{k_n}}\Big)$$

$$\overset{(2)}{=}\sum_{\substack{i_0=0,i_1,\dots,i_{k_1-1}\neq 0,\\ i_{k_1}=0}}\mathbb{P}(X_0=i_0,X_1=i_1,\dots,X_{k_1}=i_{k_1})\cdots\sum_{\substack{i_{k_{n-1}}=0,i_{k_{n-1}+1},\dots,i_{k_n-1}\neq 0\\ i_{k_n}=0}}\mathbb{P}(X_0=i_{k_{n-1}},\dots,X_{k_n}=i_{k_n})$$

$$=\mathbb{P}(X_k\neq 0\text{ for }1\le k<k_1,X_{k_1}=0)\cdots\mathbb{P}(X_k\neq 0\text{ for }1\le k<k_n,X_{k_n}=0)$$

$$=\mathbb{P}(T_1=k_1)\,\mathbb{P}(T_1=k_2)\cdots\mathbb{P}(T_n=k_n).$$

Therefore, $(Y_n)_{n\geq 1}$ is a renewal sequence.

Next, we show that $(X_n)_{n\geq 0}$ is either transient or null recurrent. By definition, $(X_n)_{n\geq 0}$ is transient if $\mathbb{P}(T_1<\infty)<1$. Suppose $\mathbb{P}(T_1<\infty)=1$. We show that $(X_n)_{n\geq 0}$ is null recurrent, i.e. $ET_1=\infty$. Define

$$\Psi(s)=\sum_{n=0}^{\infty}\mathbb{P}(X_n=0)\,s^n,$$
$$\Phi(s)=\sum_{n=0}^{\infty}\mathbb{P}(T_1=n)\,s^n.\qquad \forall s\in[0,1)$$

By Theorem 4, Fristedt-Gray page 493, $\Psi(s)=\dfrac{1}{1-\Phi(s)}$ for all $s\in[0,1)$.

Because $P(T_1=\infty)=0$, by Theorem 13, Fristedt-Gray page 71 we have

$$ET_1 = \lim_{s\to 1^-} \frac{1-\phi(s)}{1-s}.$$

Then
$$\frac{1}{ET_1} = \lim_{s\to 1^-} \frac{1-s}{1-\phi(s)} = \lim_{s\to 1^-} (1-s)H(s), \qquad (4)$$

with the convention $\frac{1}{\infty}=0$.

For $n\geq 0$ and $k\in\mathbb{Z}^d$, we put $u_n(k)=P(X_n=k)$. Then $u_0=\delta_0$, the Dirac measure at $0\in\mathbb{Z}^d$. For $n\geq 1$,

$$u_n(k) = EI_{X_n=k} = \sum_{l\in\mathbb{Z}^d} EI_{X_n=k} I_{X_{n-1}=l}$$

$$= \sum_{l\in\mathbb{Z}^d} E\big[\underbrace{E(I_{X_n=k} \mid X_0,\dots,X_{n-1})}_{=p(X_{n-1},\{k\})} I_{X_{n-1}=l}\big]$$

$$= \sum_{l\in\mathbb{Z}^d} E\left(p(l,\{k\}) I_{X_{n-1}=l}\right)$$

$$= \sum_{l\in\mathbb{Z}^d} p(l,\{k\})\, P(X_{n-1}=l)$$

$$= \sum_{l\in\mathbb{Z}^d} p(k-l)\, u_{n-1}(l) = (p * u_{n-1})(k).$$

Then $u_n = p*u_{n-1}$. Applying this identity sucessively, we get

$$u_n = \underbrace{p*p*\cdots*p}_{n \text{ times}} = p^{*n}.$$

Fix $s\in(0,1)$ and put

$$\Gamma(k) = \sum_{n=0}^{\infty} u_n(k)s^n = \sum_{n=0}^{\infty} p^{*n}(k)\, s^n. \qquad (5)$$

Denote $L^1(\mathbb{Z}^d) = \{u : \mathbb{Z}^d \to \mathbb{R} \mid \sum_{k\in\mathbb{Z}^d} |u(k)| < \infty\}$.

The Fourier transform of a function $u \in L^1(\mathbb{Z}^d)$ is defined as a function $\hat{u} : \mathbb{R}^d \to \mathbb{C}$,

$$\hat{u}(x) = \sum_{k\in\mathbb{Z}^d} u(k) e^{-2\pi i (x,k)} \qquad \forall x \in \mathbb{R}^d,$$

where (x,k) denotes the usual inner product in $\mathbb{R}^d$. We recall two following properties of Fourier transform:

(a) For $u, v \in L^1(\mathbb{Z}^d)$, we have $u * v \in L^1(\mathbb{Z}^d)$ and $\widehat{u*v} = \hat{u}\hat{v}$.

(b) For $u \in L^1(\mathbb{Z}^d)$,
$$u(k) = \int_U \hat{u}(x) e^{2\pi i (x,k)} dx \qquad \forall k \in \mathbb{Z}^d,$$
where $U = [0,1)^d$ is the unit cube in $\mathbb{R}^d$.

Proof of (a)

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} |(u*v)(k)| &= \sum_{k\in\mathbb{Z}^d} \Big|\sum_{\ell\in\mathbb{Z}^d} u(k-\ell) v(\ell)\Big| \le \sum_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d} |u(k-\ell)||v(\ell)| \\ &= \sum_{\ell\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} |u(k-\ell)|\,|v(\ell)| \\ &= \sum_{\ell\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d}|u(k)|\Big)|v(\ell)| \\ &= \Big(\sum_{k\in\mathbb{Z}^d}|u(k)|\Big)\Big(\sum_{\ell\in\mathbb{Z}^d}|v(\ell)|\Big) \\ &< \infty.\end{aligned}$$

Thus, $u*v \in L^1(\mathbb{Z}^d)$. We have

$$\widehat{u*v}(x) = \sum_{k\in\mathbb{Z}^d} (u*v)(k) e^{-2\pi i (x,k)} = \sum_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d} u(k-\ell)v(\ell) e^{-2\pi i (x,k)}.$$

The order of summing can be reversed because $\sum_{k\in\mathbb{Z}^d}\sum_{\ell\in\mathbb{Z}^d}|u(k-\ell)||v(\ell)|<\infty$.

Then
$$\widehat{u*v}(x) = \sum_{\ell\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d} u(k-\ell)\,v(\ell)\,e^{-2\pi i(x,k)}$$
$$= \sum_{\ell\in\mathbb{Z}^d} v(\ell)\,e^{-2\pi i(x,\ell)} \sum_{k\in\mathbb{Z}^d} u(k-\ell)\,e^{-2\pi i(x,k-\ell)}$$
$$= \sum_{\ell\in\mathbb{Z}^d} v(\ell)\,e^{-2\pi i(x,\ell)} \sum_{k\in\mathbb{Z}^d} u(k)\,e^{-2\pi i(x,k)}$$
$$= \hat{v}(x)\,\hat{u}(x).$$

Proof of (b)
$$\hat{u}(x)\,e^{2\pi i(x,k)} = \sum_{\ell\in\mathbb{Z}^d} u(\ell)\,e^{2\pi i(x,k-\ell)}$$

Because $\sum_{\ell\in\mathbb{Z}^d}|u(\ell)\,e^{2\pi i(x,k-\ell)}| = \sum_{\ell\in\mathbb{Z}^d}|u(\ell)|<\infty$, by the Dominated Convergence Theorem we have
$$\int_U \sum_{\ell\in\mathbb{Z}^d} u(\ell)\,e^{2\pi i(x,k-\ell)}\,dx = \sum_{\ell\in\mathbb{Z}^d} u(\ell)\int_U e^{2\pi i(x,k-\ell)}\,dx. \quad (6)$$

We compute $\int_U e^{2\pi i(x,r)}\,dx$. Write $x=(x_1,x_2,\dots,x_d)\in U$ and $r=(r_1,\dots,r_d)\in\mathbb{Z}^d$.
$$\int_U e^{2\pi i(x,r)}\,dx = \int_{[0,1)^d} e^{2\pi i(x_1r_1+\dots+x_dr_d)}\,dx = \int_{[0,1)^d} e^{2\pi i x_1 r_1}\cdots e^{2\pi i x_d r_d}\,dx$$
$$= \left(\int_0^1 e^{2\pi i x_1 r_1}\,dx_1\right)\cdots\left(\int_0^1 e^{2\pi i x_d r_d}\,dx_d\right)$$

Because
$$\int_0^1 e^{2\pi i x_j r_j}\,dx_j = \begin{cases} 1 & \text{if } r_j=0, \\ \frac{1}{2\pi i r_j}\,e^{2\pi i x_j r_j}\Big|_{x_j=0}^{x_j=1} = 0 & \text{if } r_j\neq 0, \end{cases}$$

we get $$\int_U e^{2\pi i(x,r)}\,dx = \begin{cases} 1 & \text{if } r=0, \\ 0 & \text{if } r\in\mathbb{Z}^d\setminus\{0\}. \end{cases}$$

Then (6) becomes $$\int_U \sum_{\ell\in\mathbb{Z}^d} u(\ell)\, e^{2\pi i(x,k-\ell)}\,dx = u(k).$$

Thus, $$u(k) = \int_U \hat{u}(k)\, e^{2\pi i(x,k)}\,dx.$$

$\blacksquare$

Return to the problem. Because p is a probability distribution on $\mathbb{Z}^d$, $p\in L^1(\mathbb{Z}^d)$. By Property (a), $p^{*n}\in L^1(\mathbb{Z}^d)$ and $\widehat{p^{*n}} = \hat{p}^n$. By (5),

$$\sum_{k\in\mathbb{Z}^d} |\Gamma(k)| \leq \sum_{k\in\mathbb{Z}^d}\sum_{n=0}^{\infty} |p^{*n}(k)|\, s^n = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}^d} |p^{*n}(k)|\, s^n$$
$$= \underbrace{\left(\sum_{n=0}^{\infty} s^n\right)}_{=\frac{1}{1-s}} \sum_{k\in\mathbb{Z}^d} |p^{*n}(k)| < \infty.$$

Thus, $\Gamma\in L^1(\mathbb{Z}^d)$. We have

$$\hat{\Gamma}(x) = \sum_{k\in\mathbb{Z}^d} \Gamma(k)\, e^{-2\pi i(x,k)} \overset{(5)}{=\!=} \sum_{k\in\mathbb{Z}^d}\sum_{n=0}^{\infty} p^{*n}(k)\, s^n\, e^{-2\pi i(x,k)}$$
$$\overset{\text{Fubini}}{=\!=} \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}^d} p^{*n}(k)\, e^{-2\pi i(x,k)}\, s^n$$
$$= \sum_{n=0}^{\infty} \widehat{p^{*n}}(x)\, s^n = \sum_{n=0}^{\infty} \hat{p}^n(x)\, s^n.$$

We have
$$|\hat{p}(x)| = \Big|\sum_{k\in\mathbb{Z}^d} p(k)\, e^{-2\pi i(x,k)}\Big| \leq \sum_{k\in\mathbb{Z}^d} |p(k)| = 1 \quad \forall x\in\mathbb{R}^d.$$

Thus, $|\hat{p}(x)s| \leq s < 1$. Then $\hat{\Gamma}(x) = \dfrac{1}{1-\hat{p}(x)s}$ for all $x\in\mathbb{R}^d$.

By Property (b),

$$\Gamma(0) = \int_{\mathcal{U}} \hat{p}(x) e^{2\pi i (x,0)} dx = \int_{\mathcal{U}} \hat{\Gamma}(x)\, dx = \int_{\mathcal{U}} \frac{1}{1-\hat{p}(x)s} dx.$$

Thus,

$$(1-s)\Psi(s) = (1-s)\Gamma(0) = \int_{\mathcal{U}} \frac{1-s}{1-\hat{p}(x)s} dx. \qquad \forall x \in (0,1)$$

Because of (4), to show that $ET_1 = \infty$ we need to show

$$\lim_{s\to 1^-} \int_{\mathcal{U}} \underbrace{\frac{1-s}{1-\hat{p}(x)s}}_{f(x,s)} dx = 0. \qquad (7)$$

First, we show that $\hat{p}(x) \neq 1$ for almost every $x \in \mathcal{U}$.

$$\operatorname{Re} \hat{p}(x) = \operatorname{Re} \sum_{k\in\mathbb{Z}^d} p(k) e^{-2\pi i (x,k)} = \sum_{k\in\mathbb{Z}^d} p(k) \cos 2\pi (x,k) \leq \sum_{k\in\mathbb{Z}^d} p(k) = 1.$$

The equality occurs if and only if

$$p(k)\cos 2\pi (x,k) = p(k) \quad \forall k \in \mathbb{Z}^d$$

$$\Leftrightarrow \quad p(k) = 0 \quad \text{or} \quad \cos 2\pi (x,k) = 1 \quad \forall k \in \mathbb{Z}^d$$

$$\Leftrightarrow \quad p(k) = 0 \quad \text{or} \quad (x,k) \in \mathbb{Z} \quad \forall k \in \mathbb{Z}^d.$$

Because $p(0) < 1$, there exists $k_0 \in \mathbb{Z}^d \setminus \{0\}$ such that $p(k_0) > 0$. Then $(x, k_0) \in \mathbb{Z}$. For each $j \in \mathbb{Z}$, the set $\{x \in \mathbb{R}^d : (x, k_0) = j\}$ is a hyperplane in $\mathbb{R}^d$, thus has measure zero. Then the set

$$\{x \in \mathcal{U} : \hat{p}(x) = 1\} \subset \{x \in \mathcal{U} : (x, k_0) \in \mathbb{Z}\} = \bigcup_{j\in\mathbb{Z}} \{x \in \mathcal{U} : (x, k_0) = j\}$$

$$= \mathcal{U} \cap \Big[\underbrace{\bigcup_{j\in\mathbb{Z}} \{x\in\mathbb{R}^d : (x, k_0) = j\}}_{\text{measure zero}} \Big]$$

is of measure zero. Thus, $\hat{p}(x) \neq 1$ for almost every $x \in \mathcal{U}$. Then

$$\lim_{s\to 1^-} f(x,s) = \frac{1-1}{1-\hat{p}(x)} = 0 \quad \text{a.e. } x\in\mathcal{U}.$$

Also, $|1-\hat{p}(x)s| \geq \mathrm{Re}(1-\hat{p}(x)s) = 1-(\mathrm{Re}\,\hat{p}(x))s \geq 1-|\hat{p}(x)|s \geq 1-s$.

Thus,
$$|\hat{f}(x,s)| = \frac{1-s}{|1-\hat{p}(x)s|} \leq 1 \quad \forall s\in(0,1), x\in\mathcal{U}.$$

By the Dominated Convergence Theorem,

$$\lim_{s\to 1^-} \int_{\mathcal{U}} f(x,s)\,dx = \int_{\mathcal{U}} \lim_{s\to 1^-} f(x,s)\,dx = \int_{\mathcal{U}} 0\,dx = 0.$$

Thus, (7) is proved.

wrong; redo on page 34

4) Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $S = \{1,2,3,\dots\}$ and transition kernel p. For any $k,l\in S$, denote $\pi_{kl} = \mathbb{P}_k(\exists n\geq 1 : X_n = l)$, where $\mathbb{P}_k$ is the probability measure corresponding to the initial state $X_0 \equiv k$. Let $i_0, j \in S$ be such that $\pi_{i_0 i_0} = 1$ and $\pi_{i_0 j} > 0$. We show that $\pi_{jj} =$

$= \pi_{j i_0} = 1$.

If $j = i_0$, the statement is true. Consider the case $j\neq i_0$. Let $(Y_n)_{n\geq 0}$ be a Markov chain with state space S, transition kernel p, and initial state $Y_0 \equiv i_0$. Let $\mathbb{P}$ be the corresponding probability measure. Then $\mathbb{P} = \mathbb{P}_{i_0}$

because it is determined by the transition kernel and the initial state (Theorem 21.2 in Professor Krylov's lecture notes; also Problem 8, Fristedt-Gray page 514). Then

$$\pi_{i_0 i_0} = P(\exists n \geq 1 : Y_n = i_0) = 1,$$
$$\pi_{i_0 j} = P(\exists n \geq 1 : Y_n = j) > 0.$$

This is not true

We first show that $Y_n' = I_{Y_n = j}$ is a renewal sequence. The proof is almost identical to a part of the solution to the previous problem (pages 10-12): introduce

$$T_0' = 0,$$
$$T_1' = \inf\{n > 0 : Y_n' = 1\},$$
$$T_m' = \inf\{n > T_{m-1} : Y_n' = 1\} \qquad \forall m \geq 2.$$

For any $n \geq 2$, $k_1, k_2, \dots, k_n \in \mathbb{N}$, we denote $l_0 = 0$, $l_{j'} = k_1 + k_2 + \dots + k_{j'}$ for $1 \leq j' \leq n$.

$$(T_1' - T_0' = k_1, \dots, T_n' - T_{n-1}' = k_n) = (T_1' = l_1, T_2' = l_2, \dots, T_n' = l_n)$$
$$= (Y_0 = i_0, Y_1, \dots, Y_{l_1 - 1} \neq j, Y_{l_1} = j, Y_{l_1+1}, \dots, Y_{l_2 - 1} \neq j, Y_{l_2} = j, \dots, Y_{l_{n-1}} = j,$$
$$Y_{l_{n-1}+1}, \dots, Y_{l_n - 1} \neq j, Y_{l_n} = j)$$

$$= \bigcup_{\substack{i_1, \dots, i_{l_1 - 1} \neq j, \\ \dots \\ i_{l_{n-1}+1}, \dots, i_{l_n - 1} \neq j}} (Y_0 = i_0, Y_1 = i_1, \dots, Y_{l_1 - 1} = i_{l_1 - 1}, Y_{l_1} = j, \dots, Y_{l_{n-1}+1} = i_{l_{n-1}+1},$$
$$\dots, Y_{l_n - 1} = i_{l_n - 1}, Y_{l_n} = j).$$

The probability of this event is

$$\mathbb{P}(T_1'-T_0'=k_1,\dots,T_n'-T_{n-1}'=k_n)$$

$$=\sum_{\substack{i_1,\dots,i_{l_1-1}\neq j,\\ \cdots\\ i_{l_{n-1}+1},\dots,i_{l_n-1}\neq j}} \mathbb{P}(Y_0=i_0, Y_1=i_1,\dots,Y_{l_1-1}=i_{l_1-1}, Y_{l_1}=j,\dots,Y_{l_{n-1}+1}=i_{l_{n-1}+1},\dots, Y_{l_n-1}=i_{l_n-1}, Y_{l_n}=j)$$

$$=\sum_{\substack{i_1,\dots,i_{l_1-1}\neq j,\, i_{l_1}=j,\\ \cdots\\ i_{l_{n-1}+1},\dots,i_{l_n-1}\neq j,\, i_{l_n}=j}} p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{l_n-1} i_{l_n}}$$

$$=\Bigg(\sum_{i_1,\dots,i_{l_1-1}\neq j,\, i_{l_1}=j} p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{l_1-1} i_{l_1}}\Bigg)\Bigg(\sum_{\substack{i_{l_1}=j,\, i_{l_1+1},\dots,i_{l_2-1}\neq j,\\ i_{l_2}=j}} p_{i_{l_1} i_{l_1+1}}\cdots p_{i_{l_2-1} i_{l_2}}\Bigg)\cdots$$

$$\cdots\Bigg(\sum_{\substack{i_{l_{n-1}}=j,\, i_{l_{n-1}+1},\dots,i_{l_n-1}\neq j\\ i_{l_n}=j}} p_{i_{l_{n-1}} i_{l_{n-1}+1}}\cdots p_{i_{l_n-1} i_{l_n}}\Bigg)$$

$$=\Bigg(\sum_{\substack{i_1,\dots,i_{l_1}\neq j,\\ i_{l_1}=j}} \mathbb{P}(Y_0=i_0, Y_1=i_1,\dots,Y_{k_1}=i_{l_1})\Bigg)\Bigg(\sum_{\substack{i_{l_1}=j,\, i_{l_1+1},\dots,i_{l_2-1}\neq j,\\ i_{l_2}=j}} \mathbb{P}(Y_0=i_{l_1},\dots,Y_{k_2}=i_{l_2})\Bigg)\cdots$$

$$\cdots\Bigg(\sum_{\substack{i_{l_{n-1}}=j,\, i_{l_{n-1}+1},\dots,i_{l_n-1}\neq j,\\ i_{l_n}=j}} \mathbb{P}(Y_0=i_{l_{n-1}},\dots,Y_{k_n}=i_{l_n})\Bigg)$$

$$=\mathbb{P}(Y_k\neq j \text{ for } 1\le k<k_1, Y_{k_1}=j)\cdots\mathbb{P}(Y_k\neq j \text{ for } 1\le k<k_n, Y_{k_n}=j)$$

wrong

$$= \mathbb{P}(T_1'=k_1)\,\mathbb{P}(T_1'=k_2)\cdots\mathbb{P}(T_1'=k_n).$$

Therefore, $(Y_n')_{n\geq 0}$ is a renewal sequence. Define

$$\Psi(s) = \sum_{n=0}^{\infty} P(Y_n'=1)s^n = \sum_{n=0}^{\infty} P(Y_n=j)s^n,$$

$$\phi(s) = \sum_{n=0}^{\infty} P(T_1'=n)s^n. \qquad \forall s\in[0,1).$$

By Theorem 4, Fristedt-Gray page 493, $\Psi(s) = \dfrac{1}{1-\phi(s)}$ for all $s\in[0,1)$.

Denote $A_n = \{\omega : Y_n(\omega)=j\}$. Then

$$P(\overline{\lim_{n\to\infty}} A_n) = P(\exists n\geq 1 : Y_n=j) = \pi_{i_0 j} > 0.$$

By Borel's lemma (Lemma 3, Fristedt-Gray page 78),

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

Since $Y_0 = i_0 \neq j$, $P(A_0)=0$. Then

$$\lim_{s\to 1^-} \Psi(s) = \sum_{n=0}^{\infty} P(Y_n=j) \quad \text{(by the Monotone Convergence Theorem)}$$

$$= \sum_{n=0}^{\infty} P(A_n) = \infty.$$

Then $$\phi(s) = 1 - \frac{1}{\Psi(s)} \to 1 \quad \text{as } s\to 1^-.$$

Then $$P(T_1'<\infty) = \sum_{n=0}^{\infty} P(T_1'=n)$$

$$= \lim_{s\to 1^-} \phi(s) \quad \text{(by the Monotone Convergence Theorem)}$$

$$= 1.$$

In other words, $\pi_{i_0 j} = P(\exists n\geq 1 : Y_n=j) = 1$.

Because T_1' is an almost surely finite stopping time, the random sequence $Z_n = Y_{T_1'+n}$ is a Markov chain with the same transition kernel as that of Y_n (the Strong Markov property, Theorem 7 in Fristedt-Gray page 521). By the definition of T_1', $Z_0 = Y_{T_1'} = j$ almost surely. Because the probability measure associate with a Markov chain depends only on the transition kernel and the initial state,

$$\pi_{jj} = P(\exists n \geq 1 : Z_n = j), \qquad (1)$$

$$\pi_{ji_0} = P(\exists n \geq 1 : Z_n = i_0). \qquad (2)$$

We have

$$P(T_2' < \infty) = \sum_{k_1=1}^{\infty} P(T_1' = k_1,\ T_2' \in (k_1, \infty))$$

$$= \sum_{k_1, k_2 = 1}^{\infty} P(T_1' = k_1,\ T_2' = k_1 + k_2)$$

$$= \sum_{k_1, k_2 = 1}^{\infty} P(T_1' - T_0' = k_1,\ T_2' - T_1' = k_2)$$

$$= \sum_{k_1, k_2 = 1}^{\infty} P(T_1' = k_1) P(T_1' = k_2) = \left(\sum_{k_1 = 1}^{\infty} P(T_1' = k_1) \right)^2$$

$$= \left(P(T_1' < \infty) \right)^2 = 1.$$

For every ω such that $T_2'(\omega) < \infty$,

$$Z_{T_2'(\omega) - T_1'(\omega)}(\omega) = Y_{T_2'(\omega)}(\omega) = j.$$

Thus, for almost every ω, there exists $n \geq 1$ (depending on ω) such that

$Z_n = j$. Therefore, $\pi_{jj} = 1$ according to (1).

We proved in the previous problem (pages 10-12) that $I_{Y_n = i_0}$ is a renewal sequence. Introduce

$$T_0 \equiv 0,$$
$$T_1 = \inf\{n \geq 1 : Y_n = i_0\},$$
$$T_m = \inf\{n > T_{m-1} : Y_n = i_0\} \quad \forall m \geq 2.$$

Then $P(T_1 < \infty) = P(\exists n \geq 1 : Y_n = i_0) = \pi_{i_0 i_0} = 1$. For every $m \geq 2$,

$$P(T_m < \infty) = \sum_{k_1, k_2, \dots, k_m = 1}^{\infty} P(T_1 = k_1, T_2 = k_1 + k_2, \dots, T_m = k_1 + k_2 + \dots + k_m)$$

$$= \sum_{k_1, k_2, \dots, k_m = 1}^{\infty} P(T_1 - T_0 = k_1, T_2 - T_1 = k_2, \dots, T_m - T_{m-1} = k_m)$$

$$= \sum_{k_1, k_2, \dots, k_m = 1}^{\infty} P(T_1 = k_1) P(T_1 = k_2) \dots P(T_1 = k_m)$$

$$= \left(\sum_{k_1 = 1}^{\infty} P(T_1 = k_1)\right)^m$$

$$= \left(P(T_1 < \infty)\right)^m = 1.$$

Then

$$P(\exists m \geq 1 : T_m = \infty) = P\left(\bigcup_{m=1}^{\infty} (T_m = \infty)\right) \leq \sum_{m=1}^{\infty} P(T_m = \infty) = 0.$$

Put $A = \{\omega : T_m(\omega) < \infty \ \forall m \geq 1\}$. Then $P(A) = 1$. For each $\omega \in A$, $\lim_{m \to \infty} T_m(\omega) = \infty$. Thus, there exists $r \geq 1$ (depending on ω) such that $T_r(\omega) > T_1'(\omega)$. Then $Z_{T_r(\omega) - T_1'(\omega)}(\omega) = Y_{T_r(\omega)}(\omega) = i_0$.

This means for almost every ω, there exists $n \geq 1$ (depending on ω) such that $Z_n = i_0$. Therefore, $\pi_{j i_0} = 1$ according to (2).

⑤ Let $(X_n)_{n\geq 0}$ be a renewal sequence with $X_0 \equiv 0$. For each $n \geq 1$, put $u_n = \mathbb{P}(X_n = 1)$. Suppose not all $u_n = 0$. Denote

$$S = \{n \geq 1 : u_n > 0\} \neq \phi,$$
$$\gamma = \gcd(S),$$
$$T_0 \equiv 0,$$
$$T_1 = \inf\{n \geq 1 : X_n = 1\},$$
$$T_m = \inf\{n > T_{m-1} : X_n = 1\} \quad \forall m \geq 2.$$

First, we show that $\mathbb{P}\left(\frac{T_1}{\gamma} \in \{\infty, 1, 2, 3, \dots\}\right) = 1$.

$$\left\{\omega : \frac{T_1(\omega)}{\gamma} \notin \{\infty, 1, 2, 3, \dots\}\right\} = \left\{\omega : T_1(\omega) = m\gamma + r \text{ for } m \in \{0, 1, 2, \dots\},\ r \in \{1, 2, \dots, \gamma - 1\}\right\}$$

$$= \bigcup_{\substack{m \geq 0 \\ 1 \leq r \leq \gamma - 1}} \underbrace{\{\omega : T_1(\omega) = m\gamma + r\}}_{A_{m,r}}.$$

We need to show that $\mathbb{P}(A_{m,r}) = 0$. For each $\omega \in A_{m,r}$,

$$X_{m\gamma + r}(\omega) = X_{T_1(\omega)}(\omega) = 1.$$

Thus, $A_{m,r} \subset (X_{m\gamma+r} = 1)$. Because γ does not devide $m\gamma + r$, $m\gamma + r \notin S$. Then $u_{m\gamma+r} = 0$. We get

$$\mathbb{P}(A_{m,r}) \leq \mathbb{P}(X_{m\gamma+r} = 1) = u_{m\gamma+r} = 0.$$

We have showed that $\mathbb{P}\left(\frac{T_1}{\gamma} \notin \{\infty, 1, 2, 3, \dots\}\right) = 0$.

Next, suppose $\beta \geq 1$ is an integer such that

$$\mathbb{P}\left(\frac{T_1}{\beta} \in \{\infty, 1, 2, 3, \dots\} \right) = 1.$$

To show $\beta \leq \gamma$, it suffices to show that β is a common divisor of all elements of S. We only need to consider the case $\beta \geq 2$. Our first step is to show

$$\mathbb{P}\left(\frac{T_m}{\beta} \in \{\infty, 1, 2, 3, \dots\} \right) = 1 \qquad \forall m \geq 2. \qquad (1)$$

Let $m \geq 2$. We have

$$\left\{ \omega : \frac{T_m(\omega)}{\beta} \notin \{\infty, 1, 2, 3, \dots\} \right\} = \bigcup_{\substack{\ell = 0, 1, 2, \dots \\ r = 1, 2, \dots, \beta - 1}} \underbrace{\{ \omega : T_m(\omega) = \ell\beta + r \}}_{B_{\ell, r}}.$$

We need to show $\mathbb{P}(B_{\ell,r}) = 0$ for each $\ell = 0, 1, 2, \dots$ and $r = 1, 2, \dots, \beta - 1$.

$$B_{\ell,r} = \bigcup_{\substack{k_1, \dots, k_{m-1} \geq 1 \\ k_1 + \dots + k_{m-1} < \ell\beta + r}} (T_1 = k_1, T_2 = k_1 + k_2, \dots, T_{m-1} = k_1 + \dots + k_{m-1}, T_m = \ell\beta + r)$$

$$= \bigcup_{\substack{k_1, \dots, k_{m-1} \geq 1 \\ k_1 + \dots + k_{m-1} < \ell\beta + r}} (T_1 - T_0 = k_1, T_2 - T_1 = k_2, \dots, T_{m-1} - T_{m-2} = k_{m-1}, T_m - T_{m-1} = \ell\beta + r - k_1 - \dots - k_{m-1})$$

$$= \bigcup_{\substack{k_1, \dots, k_m \geq 1 \\ k_1 + \dots + k_m = \ell\beta + r}} (T_1 - T_0 = k_1, \dots, T_m - T_{m-1} = k_m).$$

This is a union of pairwise disjoint sets. The probability of this event is

$$\mathbb{P}(B_{\ell,r}) = \sum_{\substack{k_1, \dots, k_m \geq 1 \\ k_1 + \dots + k_m = \ell\beta + r}} \mathbb{P}(T_1 - T_0 = k_1, \dots, T_m - T_{m-1} = k_m)$$

$$= \sum_{\substack{k_1, \dots, k_m \geq 1 \\ k_1 + \dots + k_m = \ell\beta + r}} \mathbb{P}(T_1 = k_1)\, \mathbb{P}(T_1 = k_2) \cdots \mathbb{P}(T_1 = k_m). \qquad (2)$$

For each ~~ty~~ tuple $(k_1, k_2, \dots, k_m)$ such that $k_1+\dots+k_m = l\beta + r$, there exists $i_0 \in \{1,2,\dots,m\}$ such that β does not divide k_{i_0}. Because $\frac{T_1}{\beta} \in \{\infty, 1,2,3,\dots\}$ almost surely, $P(T_1 = k_{i_0}) = 0$. Thus, each summand on the right hand side of (2) is equal to zero. This implies $P(B_{l,r}) = 0$. We have proved (1).

Our next step is to show that β divides n for every $n \in S$. Let $n \in S$.

$$\{\omega : X_n(\omega) = 1\} = \{\omega : T_m(\omega) = n \text{ for some } m \geq 1\}$$
$$= \bigcup_{m=1}^{\infty} \{\omega : T_m(\omega) = n\}.$$

Then
$$u_n = P(X_n = 1) = P\left(\bigcup_{m=1}^{\infty} (T_m = n)\right) \leq \sum_{m=1}^{\infty} P(T_m = n).$$

Because $u_n > 0$, there exists $m \geq 1$ such that $P(T_m = n) > 0$. Put

$$A = \left\{\omega : \frac{T_m(\omega)}{\beta} \in \{\infty, 1,2,3,\dots\}\right\},$$
$$B = \{\omega : T_m(\omega) = n\}.$$

Then $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 1 + P(B) - 1 = P(B) > 0$. Thus, there exists $\omega_0 \in A \cap B$.

$$\frac{n}{\beta} = \frac{T_m(\omega_0)}{\beta} \in \{\infty, 1,2,3,\dots\}$$

This implies β divides n.

6 Let $f : [0,1] \to \mathbb{R}$ be a monotone function. For each $x \in [0,1)$, put $b_n(x) = (k+1)2^{-n}$ and $a_n(x) = k2^{-n}$, where k is the unique integer satisfying $k2^{-n} \leq x < (k+1)2^{-n}$. We show that

$$\lim_{n\to\infty} \frac{f(b_n(x)) - f(a_n(x))}{b_n(x) - a_n(x)}$$

exists for almost every $x \in [0,1)$.

By replacing f with $-f$ if necessary, we can assume f is an increasing function. Denote $\Omega = [0,1)$, $\mathcal{F} = \mathcal{B}([0,1))$ (the Borel σ-field on $[0,1)$), $P =$ Lebesgue measure on $[0,1)$. Then $(\Omega, \mathcal{F}, P)$ is a probability space. We also introduce the following notations to transform the given problem to a probability problem.

$$x \rightsquigarrow \omega$$

$$a_n(x) \rightsquigarrow X_n(\omega)$$

$$b_n(x) \rightsquigarrow X_n(\omega) + 2^{-n}$$

$$\frac{f(b_n(x)) - f(a_n(x))}{b_n(x) - a_n(x)} \rightsquigarrow \frac{f(X_n(\omega)+2^{-n}) - f(X_n(\omega))}{2^{-n}} = Y_n(\omega).$$

We show that $\lim_{n\to\infty} Y_n$ exists almost surely. For each $n \geq 0$, define a σ-field

$$\mathcal{G}_n = \sigma\left(\{[k2^{-n}, (k+1)2^{-n}) : 0 \leq k < 2^n\}\right) \subset \mathcal{F}.$$

Because $[k2^{-n}, (k+1)2^{-n}) = [2k2^{-n-1}, (2k+1)2^{-n-1}) \cup [(2k+1)2^{-n-1}, (2k+2)2^{-n-1})$, $\mathcal{G}_n \subset \mathcal{G}_{n+1}$. For each $a \in \mathbb{R}$,

$$\{\omega : Y_n(\omega) < a\} = \left\{\omega : \frac{f(X_n(\omega)+2^{-n}) - f(X_n(\omega))}{2^{-n}} < a\right\}$$

$$= \bigcup_{k=0}^{2^n-1} \underbrace{\left\{\omega \in [k2^{-n}, (k+1)2^{-n}) : \frac{f(k2^{-n}+2^{-n}) - f(k2^{-n})}{2^{-n}} < a\right\}}_{= \phi \text{ or } [k2^{-n}, (k+1)2^{-n})}.$$

Then $\{\omega : Y_n(\omega) < a\} \in \mathcal{G}_n$. This implies $\sigma(Y_n) \subset \mathcal{G}_n$. Then

$$\sigma(Y_k) \subset \mathcal{G}_k \subset \mathcal{G}_n \quad \forall\, 0 \le k \le n.$$

Denote $\mathcal{F}_n = \sigma(Y_0, Y_1, \dots, Y_n)$. Then $\mathcal{F}_n = \sigma(\sigma(Y_0), \sigma(Y_1), \dots, \sigma(Y_n)) \subset \mathcal{G}_n$.

Also, $(\mathcal{F}_n)_{n\ge 0}$ is an increasing filtration.

We show that $(Y_n, \mathcal{F}_n)_{n\ge0}$ is a martingale. Since f is increasing, $Y_n \ge 0$.

$$EY_n = \int_0^1 Y_n(\omega)\,d\omega = \sum_{k=0}^{2^n-1} \int_{k2^{-n}}^{(k+1)2^{-n}} \frac{f(k2^{-n}+2^{-n}) - f(k2^{-n})}{2^{-n}}\,d\omega$$

$$= \sum_{k=0}^{2^n-1} [f(k2^{-n}+2^{-n}) - f(k2^{-n})]$$

$$= f(1) - f(0) \qquad \forall\, n \ge 0.$$

Thus, $\sup_{n\ge0} E(Y_n) = f(1) - f(0) < \infty$.

Take any $A \in \mathcal{F}_n$. Because $A \in \mathcal{G}_n$, A is of the form

$$A = [k_1 2^{-n}, (k_1+1)2^{-n}) \cup \dots \cup [k_m 2^{-n}, (k_m+1)2^{-n})$$

for $1 \le k_1 < k_2 < \dots < k_m \le 2^n - 1$. Then

$$E\, Y_{n+1} I_A = \int_0^1 Y_{n+1}(\omega) I_A(\omega)\,d\omega = \sum_{j=1}^m \underbrace{\int_{k_j 2^{-n}}^{(k_j+1)2^{-n}} Y_{n+1}(\omega)\,d\omega}_{\{1\}}. \qquad (1)$$

We have

$$\{1\} = \int_{2k_j 2^{-n-1}}^{(2k_j+1)2^{-n-1}} Y_{n+1}(\omega)\,d\omega + \int_{(2k_j+1)2^{-n-1}}^{(2k_j+2)2^{-n-1}} Y_{n+1}(\omega)\,d\omega =$$

$$= \int_{2k_j 2^{-n-1}}^{(2k_j+1)2^{-n-1}} \frac{f((2k_j+1)2^{-n-1}) - f(2k_j 2^{-n-1})}{2^{-n-1}}\, dw + \int_{(2k_j+1)2^{-n-1}}^{(2k_j+2)2^{-n-1}} \frac{f((2k_j+2)2^{-n-1}) - f((2k_j+1)2^{-n-1})}{2^{-n-1}}\, dw$$

$$= f((2k_j+1)2^{-n-1}) - f(2k_j 2^{-n-1}) + f((2k_j+2)2^{-n-1}) - f((2k_j+1)2^{-n-1})$$

$$= f((k_j+1)2^{-n}) - f(k_j 2^{-n}).$$

Then (1) can be written as

$$E Y_{n+1} I_A = \sum_{j=1}^{m} [f((k_j+1)2^{-n}) - f(k_j 2^{-n})]. \qquad (2)$$

On the other hand,

$$E Y_n I_A = \int_0^1 Y_n(w) I_A(w)\, dw = \sum_{j=0}^{m} \int_{k_j 2^{-n}}^{(k_j+1)2^{-n}} Y_n(w)\, dw$$

$$= \sum_{j=0}^{m} \int_{k_j 2^{-n}}^{(k_j+1)2^{-n}} \frac{f((k_j+1)2^{-n}) - f(k_j 2^{-n})}{2^{-n}}\, dw$$

$$= \sum_{j=0}^{m} [f((k_j+1)2^{-n}) - f(k_j 2^{-n})]. \qquad (3)$$

By (2) and (3), $E Y_{n+1} I_A = E Y_n I_A$. Thus, $E(Y_{n+1} | \mathcal{F}_n) = Y_n$. This implies $(Y_n, \mathcal{F}_n)_{n \geq 0}$ is a martingale. By the Martingale Convergence Theorem, $\lim_{n\to\infty} Y_n$ exists almost surely.

7 Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space (over $\mathbb{R}$). We show that

(a) the map $(x,y) \in X \times X \mapsto \|x+y\| \in \mathbb{R}$ is continuous,

(b) the map $(x,y) \in X \times X \mapsto \langle x,y \rangle \in \mathbb{R}$ is continuous.

Proof of (a)

Let $(x_0, y_0) \in X \times X$ and $\{(x_n, y_n)\}_{n\geq 1}$ be a sequence in $X \times X$ that converges to (x_0, y_0). Then $x_n \to x_0$ and $y_n \to y_0$ in X. Then

$$\|(x_n + y_n) - (x_0 + y_0)\| = \|(x_n - x_0) + (y_n - y_0)\| \leq \underbrace{\|x_n - x_0\|}_{\to 0} + \underbrace{\|y_n - y_0\|}_{\to 0} \to 0 \text{ as } n \to \infty.$$

Thus, $x_n + y_n \to x_0 + y_0$ as $n \to \infty$.

Proof of (b)

Let $(x_0, y_0) \in X \times X$ and $\{(x_n, y_n)\}_{n\geq 1}$ be a sequence in $X \times X$ that converges to (x_0, y_0). Then $x_n \to x_0$ and $y_n \to y_0$ in X. Because (y_n) is convergent, it is bounded. There exists a number $M > 0$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$.

$$\begin{aligned}
|\langle x_n, y_n\rangle - \langle x_0, y_0\rangle| &= |\langle x_n - x_0, y_n\rangle + \langle x_0, y_n - y_0\rangle| \\
&\leq |\langle x_n - x_0, y_n\rangle| + |\langle x_0, y_n - y_0\rangle| \\
&\leq \|x_n - x_0\|\,\|y_n\| + \|x_0\|\,\|y_n - y_0\| \\
&\leq M \underbrace{\|x_n - x_0\|}_{\to 0} + \|x_0\| \underbrace{\|y_n - y_0\|}_{\to 0} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, $\langle x_n, y_n\rangle \to \langle x_0, y_0\rangle$ as $n \to \infty$.

8 We determine all potential sequences beginning with two 1's : $(1, 1, \ldots)$. That is to determine all sequences $(u_n)_{n\geq 0}$ such that

$$\begin{cases} u_0 = u_1 = 1, \\ u_n = P(X_n = 1) \quad \forall n \geq 0, \end{cases}$$

for some renewal sequence $(X_n)_{n\geq 0}$.

Suppose (u_n) is such a sequence. The first waiting time is defined as

$$T_1 = \inf\{n \geq 1 : X_n = 1\}.$$

Because $X_1 = 1$ almost surely, $T_1 = 1$ almost surely. Then

$$P(T_1 = n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $s \in [0,1)$, we define

$$\Psi(s) = \sum_{n=0}^{\infty} u_n s^n = 1 + s + \sum_{n=2}^{\infty} u_n s^n,$$

$$\phi(s) = \sum_{n=0}^{\infty} P(T_1 = n) s^n = s.$$

By Theorem 4, Fristedt–Gray page 493, $\Psi(s) = \frac{1}{1-\phi(s)}$ for all $s \in [0,1)$.

Then
$$1 + s + \sum_{n=2}^{\infty} u_n s^n = \frac{1}{1-s} = 1 + s + \sum_{n=2}^{\infty} s^n \quad \forall s \in [0,1).$$

This implies $u_n = 1$ for all $n \geq 2$.

Conversely, we show that the constant sequence $(u_0, u_1, u_2, \dots) = (1, 1, 1, \dots)$ is a potential sequence. Let $(X_n)_{n\geq 0}$ be a sequence of constant random variables $X_n = 1$. Then $u_n = P(X_n = 1)$ for all $n \geq 0$. We verify by definition that $(X_n)_{n \geq 0}$ is a renewal sequence. Introduce

$$\begin{aligned} T_0 &\equiv 0, \\ T_1 &= \inf\{n \geq 1 : X_n = 1\} \equiv 1, \\ T_m &= \inf\{n > T_{m-1} : X_n = 1\} \qquad \forall m \geq 2. \end{aligned}$$

Suppose $T_{m-1} \equiv m-1$ for some $m \geq 2$. Then

$$T_m = \inf\{n > m-1 : X_n = 1\} \equiv m.$$

We get $T_m \equiv m$ for all $m = 0, 1, 2, \ldots$

For $n \geq 2$, and $k_1, k_2, \ldots, k_n \geq 1$,

$$P(T_1 - T_0 = k_1, \ldots, T_n - T_{n-1} = k_n) = P(T_1 = k_1, T_2 = k_1 + k_2, \ldots, T_n = k_1 + k_2 + \cdots + k_n). \quad (1)$$

If $k_j > 1$ for some j then $k_1 + k_2 + \cdots + k_j > j = T_j$. Then (1) gives

$$P(T_1 - T_0 = k_1, \ldots, T_n - T_{n-1} = k_n) = \begin{cases} P(T_1 = 1, T_2 = 2, \ldots, T_n = n) & \text{if } k_1 = \cdots = k_n = 1, \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } k_1 = \cdots = k_n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Because $T_1 \equiv 1$,

$$P(T_1 = k_1)\, P(T_1 = k_2) \cdots P(T_1 = k_n) = \begin{cases} 1 & \text{if } k_1 = \cdots = k_n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $P(T_1 - T_0 = k_1, \ldots, T_n - T_{n-1} = k_n) = P(T_1 = k_1)\, P(T_1 = k_2) \cdots P(T_1 = k_n)$.

We conclude that $(X_n)_{n \geq 0}$ is a renewal sequence.

9 Let $q \in (0,1)$, and μ be a probability distribution on $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ given by

$$\mu(\{x\}) = (1-q)\, q^x \quad \forall x \in \mathbb{Z}^+.$$

Let $(X_n)_{n \geq 0}$ be a branching process with branching distribution μ. Denote

$$\pi_{\{0\}}(x) = P_x(X_k = 0 \text{ for some } k \geq 0) \quad \forall x \in \mathbb{Z}^+,$$

where P_x is the probability measure associate with the Markov chain $(X_n)_{n \geq 0}$

with the initial state $X_0 = x$ almost surely. We compute $\pi_{\{0\}}(x)$.

First, we verify that μ is indeed a probability distribution on $\mathbb{Z}^+$. By definition,

$$\mu(A) = \sum_{x\in A} \mu(\{x\}) = \sum_{x\in A} (1-q)q^x \qquad \forall A \in \mathbb{Z}^+.$$

It is clear that $\mu(A) \geq 0$ and μ is σ-additive.

$$\mu(\mathbb{Z}^+) = \sum_{x\in\mathbb{Z}^+} (1-q)q^x = (1-q)\sum_{x=0}^{\infty} q^x = (1-q)\frac{1}{1-q} = 1.$$

Thus, μ is a probability distribution on $\mathbb{Z}^+$.

The probability generating function of μ is $\rho: [0,1] \to [0,1]$,

$$\rho(s) = \sum_{k=0}^{\infty} s^k \mu(\{k\}) \qquad \forall s \in [0,1].$$

We simplify $\rho(s)$ as follows.

$$\rho(s) = \sum_{k=0}^{\infty} s^k (1-q) q^k = (1-q)\sum_{k=0}^{\infty} (sq)^k = \frac{1-q}{1-sq} \qquad \forall s \in [0,1].$$

By Theorem 9, Fristedt-Gray page 524, $\pi_{\{0\}}(x) = c^x$ where c is the smallest fixed point of ρ. This formula was also established while solving Problem 3 of Homework 8. We compute all fixed point of ρ:

$$\rho(s) = s \iff \begin{cases} \dfrac{1-q}{1-sq} = s, \\ s \in [0,1] \end{cases}$$

$$\iff \begin{cases} s^2 q - s + 1 - q = 0, \\ s \in [0,1] \end{cases}$$

$$\Leftrightarrow \begin{cases} (s-1)(sq+q-1)=0, \\ s\in[0,1] \end{cases}$$

$$\Leftrightarrow \quad s=1 \quad \text{or} \quad \begin{cases} s=\dfrac{1-q}{q}, \\ s\in[0,1]. \end{cases}$$

If $0<q\le\frac{1}{2}$ then 1 is the only fixed point of ρ.

If $\frac{1}{2}<q<1$ then 1 and $\frac{1-q}{q}$ are the fixed points of ρ. The smaller one is $\frac{1-q}{q}$. Therefore,

$$\pi_{\{0\}}(x)=\begin{cases} 1 & \text{if } 0<q\le\frac{1}{2}, \\ \left(\dfrac{1-q}{q}\right)^x & \text{if } \frac{1}{2}<q<1 \end{cases} \qquad \forall x\in\mathbb{Z}^+.$$

④ Let $(X_n)_{n\ge 0}$ be a Markov chain with state space $S=\{1,2,3,\dots\}$ and transition kernel p. For any $k,l\in S$, denote $\pi_{kl}=\mathbb{P}_k(\exists n\ge 1: X_n=l)$ where $\mathbb{P}_k$ is the probability measure corresponding to the initial state $X_0\equiv k$. Let $i_0, j\in S$ be such that $\pi_{i_0i_0}=1$ and $\pi_{i_0j}>0$. We show that $\pi_{i_0j}=\pi_{jj}=\pi_{ji_0}=1$.

For $j=i_0$, the statement is true. Consider the case $j\ne i_0$. Let Σ be the family of all subsets of S, and $(\Omega,\mathcal{F})=\bigotimes_{n=0}^{\infty}(S,\Sigma)$. Let $(Y_n)_{n\ge 0}$ be the Markov chain with state space (S,Σ), transition kernel p, and initial state $Y_0\equiv i_0$, as constructed in the proof of Ionescu-Tulcea theorem (Problem 4, Homework 7). Then $\mathcal{F}=\sigma(Y_0,Y_1,Y_2,\dots)$ and the filtration for $(Y_n)_{n\geq 0}$ is

$$\mathcal{F}_n = \sigma(Y_0, Y_1, \dots, Y_n) \quad \forall n \geq 0.$$

Then $\mathbb{P} = \mathbb{P}_{i_0}$ because it is determined by the transition kernel and the initial state (Theorem 21.2 in Professor Krylov's lecture notes; also Problem 8, Fristedt-Gray page 514). Then

$$\pi_{i_0 i_0} = \mathbb{P}(\exists n \geq 1 : Y_n = i_0) = 1,$$

$$\pi_{i_0 j} = \mathbb{P}(\exists n \geq 1 : Y_n = j) > 0.$$

We proved in the previous problem (pages 10-12) that $I_{Y_n = i_0}$ is a renewal sequence. Introduce the waiting times:

$$T_0 \equiv 0,$$

$$T_1 = \inf\{n \geq 1 : Y_n = i_0\},$$

$$T_m = \inf\{n > T_{m-1} : Y_n = i_0\} \quad \forall m \geq 2.$$

Then $\mathbb{P}(T_1 < \infty) = \mathbb{P}(\exists n \geq 1 : Y_n = i_0) = \pi_{i_0 i_0} = 1$. For every $m \geq 2$,

$$\begin{aligned}
\mathbb{P}(T_m < \infty) &= \sum_{k_1, k_2, \dots, k_m = 1}^{\infty} \mathbb{P}(T_1 = k_1, T_2 = k_1 + k_2, \dots, T_m = k_1 + k_2 + \dots + k_m) \\
&= \sum_{k_1, k_2, \dots, k_m = 1}^{\infty} \mathbb{P}(T_1 - T_0 = k_1, T_2 - T_1 = k_2, \dots, T_m - T_{m-1} = k_m) \\
&= \sum_{k_1, k_2, \dots, k_m = 1}^{\infty} \mathbb{P}(T_1 = k_1)\, \mathbb{P}(T_1 = k_2) \cdots \mathbb{P}(T_1 = k_m) \\
&= \left(\sum_{k_1 = 1}^{\infty} \mathbb{P}(T_1 = k_1) \right)^m \\
&= \cancel{\sum} \left(\mathbb{P}(T_1 < \infty) \right)^m \\
&= 1.
\end{aligned}$$

We have

$$0 < \pi_{i_0 j} = P(\exists n \geq 1 : Y_n = j) = P\Big(\bigcup_{n=1}^{\infty}(Y_n = j)\Big) \leq \sum_{n=1}^{\infty} P(Y_n = j).$$

Thus, there exists $n_0 \geq 1$ such that $P(Y_{n_0} = j) > 0$. Put

$$A_k = \{\omega : Y_{T_{n_0 k}(\omega) + n_0}(\omega) = j\} = (Y_{T_{n_0 k} + n_0} = j) \qquad \forall k \geq 0.$$

Then $P(A_0) = P(Y_{n_0} = j) > 0$. Our goal is to show that A_k's have the same probability and are pairwise uncorrelated.

Fix $k \geq 1$ for a moment. We know that $T_{n_0 k}$ is an almost surely finite stopping time. By the Strong Markov property (Theorem 7, Fristedt-Gray page 521), the random sequence $Y_n' = Y_{T_{n_0 k} + n}$ is a Markov chain with the same kernel as that of $(Y_n)_{n \geq 0}$. By the definition of the waiting times, $Y_0'(\omega) = Y_{T_{n_0 k}(\omega) + 0} = i_0$ almost surely. Because the probability measure associate with a Markov chain depends only on the transition kernel and the initial state,

$$P(Y_{n_0}' = j) = P(Y_{n_0} = j) = P(A_0).$$

In other words, $P(A_k) = P(A_0) > 0$ for all $k \geq 1$.

Next, let $k, l \in \{0, 1, 2, \dots\}$, $k > l$. We show that A_k and A_l are uncorrelated. It suffices to show that $Y_{T_{k n_0} + n_0}$ and $Y_{T_{l n_0} + n_0}$ are independent.

Put $\mathcal{G} = \mathcal{F}_{T_{n_0 k}} = \{A \in \mathcal{F} : A \cap T_{n_0 k}^{-1}([0, n]) \in \mathcal{F}_n \text{ for all } n = 0, 1, 2, \dots\}$.

We show that $Y_{T_{n_0 \ell}+n_0}$ is $\mathcal{G}$-measurable. For each set $B \subset S$,

$$
\begin{aligned}
(Y_{T_{n_0\ell}+n_0} \in B) &= \{\omega : Y_{T_{n_0\ell}(\omega)+n_0}(\omega) \in B\} \\
&= \bigcup_{r=0}^{\infty} \{\omega \in T_{n_0\ell}^{-1}(r) : Y_{r+n_0}(\omega) \in B\} \\
&= \bigcup_{r=0}^{\infty} \Big[\underbrace{T_{n_0\ell}^{-1}(r) \cap Y_{r+n_0}^{-1}(B)}_{B_r}\Big]. \qquad (1)
\end{aligned}
$$

Let $n \geq 0$. Suppose there exists $\omega \in B_r \cap T_{n_0k}^{-1}([0,n])$. Then $T_{n_0\ell}(\omega) = r$ and $T_{n_0k}(\omega) \leq n$. By the definition of the waiting times,

$$T_{n_0k}(\omega) \geq T_{n_0\ell}(\omega) + n_0k - n_0\ell = r + n_0(k-\ell) \geq r + n_0.$$

This implies $r + n_0 \leq n$. Thus, $B_r \cap T_{n_0k}^{-1}([0,n]) = \phi$ if $r + n_0 > n$. Consider the case $r + n_0 \leq n$. Because $T_{n_0\ell}$ is a stopping time,

$$T_{n_0\ell}^{-1}(r) \in \mathcal{F}_{r+1} \subset \mathcal{F}_{r+n_0} \subset \mathcal{F}_n.$$

Because T_{n_0k} is a stopping time, $T_{n_0k}^{-1}([0,n]) \in \mathcal{F}_n$. Also, $Y_{r+n_0}^{-1}(B) \in \mathcal{F}_{r+n_0} \subset \mathcal{F}_n$.
~~Thus~~ Also, $Y_{r+n_0}^{-1}(B) \in \mathcal{F}_{r+n_0} \subset \mathcal{F}_n$. Thus,

$$B_r \cap T_{n_0k}^{-1}([0,n]) = T_{n_0\ell}^{-1}(r) \cap Y_{r+n_0}^{-1}(B) \cap T_{n_0k}^{-1}([0,n]) \in \mathcal{F}_n.$$

Then $B_r \in \mathcal{G}$. By (1) we conclude that $(Y_{T_{n_0\ell}+n_0} \in B) \in \mathcal{G}$.

Now that $Y_{T_{n_0\ell}+n_0}$ is $\mathcal{G}$-measurable, to show that $Y_{T_{n_0k}+n_0}$ and $Y_{T_{n_0\ell}+n_0}$ are independent, it suffices to show $Y_{T_{n_0k}+n_0}$ and $\mathcal{G}$ are independent.

That is to show

$$\mathbb{P}(Y_{T_{n_0 k}+n_0} = i \mid \mathcal{G}) = \mathbb{P}(Y_{T_{n_0 k}+n_0} = i) \quad \forall i \in S. \qquad (2)$$

The Strong Markov property says that $Y'_n = Y_{T_{n_0 k}+n}$ is a Markov chain with transition kernel p, initial ~~sp~~ state $Y'_0 = Y_{T_{n_0 k}} \equiv i_0$, and with respect to the filtration $\mathcal{F}'_n = \mathcal{F}_{T_{n_0 k}+n}$. Because the probability measure associate with a Markov chain is determined by the transition kernel and the initial state, the probability measure associate with $(Y'_n)_{n\geq 0}$ is also $\mathbb{P}$.

$$RHS(2) = \mathbb{P}(Y'_{n_0} = i) = \mathbb{P}(Y_{n_0} = i).$$

$$LHS(2) = \mathbb{P}(Y_{T_{n_0 k}+n_0} = i \mid \mathcal{F}_{T_{n_0 k}}) = \mathbb{P}(Y'_{n_0} = i \mid \mathcal{F}'_0) = \mathbb{P}(Y_n = i \mid \mathcal{F}_0).$$

Because $\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathbb{P}(Y_{n_0} = i \mid \mathcal{F}_0) = \mathbb{P}(Y_{n_0} = i)$. Thus, (2) is proved.

We have showed that A_k and A_l are uncorrelated for all $0 \leq l < k$.

Also, $\displaystyle\sum_{k=0}^{\infty} \mathbb{P}(A_k) = \sum_{k=0}^{\infty} \mathbb{P}(A_0) = \infty$.

By Borel-Cantelli's lemma, (Lemma 5, Fristedt-Gray page 79)

$$\mathbb{P}(\overline{\lim_{k\to\infty}} A_k) = 1.$$

We have

$$\begin{aligned}\overline{\lim_{k\to\infty}} A_k &= \{\omega : \exists k \geq 0 \text{ such that } \omega \in A_k\} \\ &= \{\omega : \exists k \geq 0 \text{ such that } Y_{T_{n_0 k}(\omega)+n_0}(\omega) = j\} \\ &\subset \{\omega : \exists n \geq 1 \text{ such that } Y_n(\omega) = j\}.\end{aligned}$$

Then $\mathbb{P}(\exists n \geq 1 : Y_n = j) = 1$. We get $\pi_{i_0 j} = 1$.

Next, we show that $\pi_{jj} = 1$. Introduce a stopping time

$$T_1' = \inf\{n \geq 1 : Y_n = j\}.$$

Then $P(T_1' < \infty) = P(\exists n \geq 1 : Y_n = j) = \pi_{i_0 j} = 1$. Then T_1' is an almost surely finite stopping time. By the Strong Markov property (Theorem 7, Fristedt-Gray page 521), $Z_n = Y_{T_1'+n}$ is a Markov chain with transition kernel p and initial state $Z_0 = Y_{T_1'} \equiv j$. Because the probability measure associate with a Markov chain is determined by the transition kernel and the initial state, the probability measure associate with $(Z_n)_{n\geq 0}$ is P_j.

$$\pi_{jj} = P_j(\exists n \geq 1 : Z_n = j) = P(\exists n \geq 1 : Y_{T_1'+n} = j).$$

For each $m \geq 0$, we know that

$$\sum_{k=m}^{\infty} P(A_k) = \sum_{\ell=m}^{\infty} P(A_0) = \infty,$$

A_k and A_ℓ are uncorrelated if $m \leq \ell < k$.

By Borel-Cantelli lemma, $P(\overline{\lim_{\substack{k\to\infty \\ k \geq m}}} A_k) = 1$.

$$\overline{\lim_{\substack{k\to\infty \\ k\geq m}}} A_k = \{\omega : \exists k \geq m \text{ such that } Y_{T_{n_0 k}(\omega)+n_0}(\omega) = j\}$$

$$\subset \{\omega : \exists n \geq m \text{ such that } Y_n(\omega) = j\}.$$

$$(\text{because } T_{n_0 k}(\omega) + n_0 \geq n_0 k + n_0 > m)$$

Thus, $P(\exists n \geq m : Y_n = j) = 1$.

Put $U_0 = (T_1' < \infty)$,

$U_m = (\exists n \geq m : Y_n = j) \quad \forall m \geq 1$,

$A = \bigcap_{m=0}^{\infty} U_m$.

Then $\quad \mathbb{P}(A^c) = \mathbb{P}(\bigcup_{m=0}^{\infty} U_m^c) \leq \sum_{m=0}^{\infty} \mathbb{P}(U_m^c) = 0.$

Then $\mathbb{P}(A) = 1$. For each $\omega \in A$, $T_1'(\omega) = m < \infty$. Since $\omega \in U_m$, there exists $n_1 > m$ such that $Y_{n_1} = j$. Then $\omega \in (\exists n \geq 1 : Y_{T_1'+n} = j)$. We get

$$\pi_{jj} = \mathbb{P}(\exists n \geq 1 : Y_{T_1'+n} = j) = 1.$$

Now we show that $\pi_{ji_0} = 1$.

$$\pi_{ji_0} = \mathbb{P}_j(\exists n \geq 1 : Z_n = i_0) = \mathbb{P}(\exists n \geq 1 : Y_{T_1'+n} = i_0).$$

Put $V_0 = (T_1' < \infty)$,

$V_m = (T_m < \infty) \quad \forall m \geq 1$,

$B = \bigcap_{m=0}^{\infty} V_m$.

Then $\mathbb{P}(B^c) = \mathbb{P}(\bigcup_{m=0}^{\infty} V_m^c) \leq \sum_{m=0}^{\infty} \mathbb{P}(V_m^c) = 0$. Thus, $\mathbb{P}(B) = 1$.

For each $\omega \in B$, $T_1'(\omega) = m < T_{m+1}(\omega) < \infty$. Put $n_2 = T_{m+1}(\omega) - T_1'(\omega) \geq 1$.

Then $Y_{T_1'(\omega)+n_2} = Y_{T_{m+1}(\omega)} = i_0$. Thus, $\omega \in (\exists n \geq 1 : Y_{T_1'+n} = i_0)$.

We conclude that

$$\pi_{ji_0} = \mathbb{P}(\exists n \geq 1 : Y_{T_1'+n} = i_0) = 1.$$

(10) Let $(X_n)_{n\geq 0}$ be a Markov chain on a countable state space $S=\{1,2,3,\dots\}$. Let i_0 and j be two states that are accessible from each other. We show that these states

(a) have the same period,

(b) are of the same type: transient, positive recurrent or null recurrent.

Introduce the following notations:

$$\pi_{k\ell} = P_k(\exists n\geq 1 : X_n=\ell) \quad \forall k,\ell\in S,\ k\neq \ell,$$

$$p_{k\ell}^{n} = P_k(X_n=\ell) \quad \forall k,\ell\in S,\ \forall n\geq 0,$$

$$\gamma_k = \gcd\{n\geq 0 : P_k(X_n=k)>0\} \quad \forall k\in S,$$

$$m_k = E_k T^{(k)},$$ where $T^{(k)}$ is the first waiting time of the Markov chain with initial state $X_0\equiv k$,

$$q_{ij} = p(i,\{j\}) \quad \forall i,j\in S,$$ where p is the transition kernel.

First, we prove the following lemma, which is known as the Chapman–Kolmogorov equations.

[Lemma $$p_{k\ell}^{n+m} = \sum_{i=1}^{\infty} p_{ki}^{n}\, p_{i\ell}^{m} \quad \forall k,\ell\in S,\ \forall m,n\geq 0.$$]

Proof of the lemma

In Problem 3, page 11, we verified that

$$P_k(X_0=k, X_1=j_1,\dots,X_n=j_n) = q_{kj_1}q_{j_1j_2}\cdots q_{j_{n-1}j_n} \quad \forall k,j_1,\dots,j_n\in S.$$

This identity is applied in the following computation.

$$p_{k\ell}^{n+m} = P_k(X_{n+m}=\ell) = \sum_{i\in S} P_k(X_{n+m}=\ell,\ X_n=i)$$

$$= \sum_{i\in S} \sum_{\substack{j_1,\dots,j_{n-1},j_{n+1},\dots,j_{n+m-1}\in S\\ j_0=k,\ j_n=i,\ j_{n+m}=\ell}} P_k(X_0=j_0, X_1=j_1,\dots, X_{n+m}=j_{n+m})$$

$$= \sum_{i\in S} \sum_{\substack{j_1,\dots,j_{n-1},j_{n+1},\dots,j_{n+m-1}\in S\\ j_0=k,\ j_n=i,\ j_{n+m}=\ell}} q_{j_0 j_1} q_{j_1 j_2} \cdots q_{j_{n+m-1} j_{n+m}}$$

$$= \sum_{i\in S} \left(\sum_{\substack{j_1,\dots,j_{n-1}\in S\\ j_0=k,\ j_n=i}} q_{j_0 j_1} \cdots q_{j_{n-1} j_n} \right) \left(\sum_{\substack{j_{n+1},\dots,j_{n+m-1}\in S\\ j_n=i,\ j_{n+m}=\ell}} q_{j_n j_{n+1}} \cdots q_{j_{n+m-1} j_{n+m}} \right)$$

$$= \sum_{i\in S} \left(\sum_{\substack{j_1,\dots,j_{n-1}\in S\\ j_0=k,\ j_n=i}} P_k(X_0=j_0, X_1=j_1,\dots,X_n=j_n) \right) \left(\sum_{\substack{j_{n+1},\dots,j_{n+m-1}\in S\\ j_n=i,\ j_{n+m}=\ell}} P_i(X_0=j_n,\dots,X_m=j_{n+m}) \right)$$

$$= \sum_{i\in S} P_k(X_n=i)\, P_i(X_m=\ell)$$

$$= \sum_{i\in S} p_{ki}^{n}\, p_{i\ell}^{m}.$$

The lemma is proved. $\blacksquare$

Return to the problem. Because i_0 and j are accessible from each other, $\pi_{i_0 j}>0$ and $\pi_{j i_0}>0$. If $i_0=j$, the claims (a) and (b) are true. Consider the case $i_0\neq j$. We have

$$0 < \pi_{i_0 j} = P_{i_0}(\exists n\geq 1: X_n=j) = P_{i_0}\Big(\bigcup_{n=1}^{\infty}(X_n=j)\Big) \leq \sum_{n=1}^{\infty} P_{i_0}(X_n=j) = \sum_{n=1}^{\infty} p_{i_0 j}^{n}.$$

Thus, there exists $n_0\geq 1$ such that $p_{i_0 j}^{n_0}>0$. Similarly,

$$0 < \pi_{j i_0} = P_j(\exists n \geq 1 : X_n = i_0) = P_j\left(\bigcup_{n=1}^{\infty}(X_n = i_0)\right) \leq \sum_{n=1}^{\infty} P_j(X_n = i_0) = \sum_{n=1}^{\infty} p_{j i_0}^n .$$

There exists $n_1 \geq 1$ such that $p_{j i_0}^{n_1} > 0$.

Proof of (a)

Put $\quad U = \{n \geq 0 : P_{i_0}(X_n = i_0) > 0\}$,

$\quad V = \{n \geq 0 : P_j(X_n = j) > 0\}$.

Then $\gamma_{i_0} = \gcd U$ and $\gamma_j = \gcd V$. By the lemma,

$$p_{i_0 i_0}^{n_0+n_1} \geq p_{i_0 j}^{n_0} p_{j i_0}^{n_1} > 0,$$

$$p_{jj}^{n_0+n_1} \geq p_{j i_0}^{n_1} p_{i_0 j}^{n_0} > 0.$$

Then $n_0+n_1 \in U \cap V$. Then $U, V \neq \{0\}$. Thus, $\gamma_{i_0}, \gamma_j < \infty$. Since $n_0+n_1 \in U$, it is divisible by γ_{i_0}. For each $m \in V$,

$$p_{i_0 i_0}^{n_0+n_1+m} \geq p_{i_0 j}^{n_0} p_{j i_0}^{n_1+m} \geq p_{i_0 j}^{n_0}\left(p_{jj}^{m} p_{j i_0}^{n_1}\right) > 0.$$

Thus, $n_0+n_1+m \in U$. Then it is divisible by γ_{i_0}. Then $m = (n_0+n_1+m) - (n_0+n_1)$ is divisible by γ_{i_0}. Then γ_{i_0} is a common divisor of all elements of V. Thus, $\gamma_{i_0} \leq \gamma_j$. Because states i_0 and j play an equal role, we also have $\gamma_j \leq \gamma_{i_0}$. Therefore, $\gamma_j = \gamma_{i_0}$.

Proof of (b)

Suppose state i_0 is recurrent. Then $\pi_{i_0 i_0} = 1$. Because $\pi_{i_0 i_0} = 1$ and $\pi_{i_0 j} > 0$, we get $\pi_{jj} = 1$ by Problem 4. We conclude that (i_0 is recurrent $\Rightarrow$ j is recurrent). Because the two states play an equal role, we also have

(j is recurrent $\Rightarrow$ i_0 is recurrent). Then (i_0 is recurrent $\Leftrightarrow$ j is recurrent). Equivalently, (i_0 is transient $\Leftrightarrow$ j is transient).

Suppose that state i_0 is positive recurrent, i.e. $m_{i_0} < \infty$. By Part (a), we can denote $\gamma = \gamma_{i_0} = \gamma_j \geq 1$. By the Renewal Theorem for Markov sequences (Theorem 10, Fristedt-Gray page 536),

$$\lim_{k \to \infty} \frac{1}{\gamma} \sum_{r=0}^{\gamma-1} p_{i_0 i_0}^{r+k} = \frac{1}{m_{i_0}} > 0 \qquad (1)$$

$$\lim_{k \to \infty} \frac{1}{\gamma} \sum_{r=0}^{\gamma-1} p_{i_0 j}^{r+k} = \frac{\pi_{i_0 j}}{m_j}. \qquad (2)$$

We have $\quad LHS(1) = \frac{1}{\gamma} \lim_{k \to \infty} \sum_{r=0}^{\gamma-1} p_{i_0 i_0}^{r+k} \leq \frac{1}{\gamma} \sum_{r=0}^{\gamma-1} \overline{\lim_{k\to\infty}} \, p_{i_0 i_0}^{r+k}.$

Thus, there exists $r_0 \in \{0, 1, \dots, \gamma-1\}$ such that $\overline{\lim_{k\to\infty}} \, p_{i_0 i_0}^{r_0+k} = \varepsilon > 0$. There exists a strictly increasing sequence $k_1, k_2, k_3, \dots$ such that $p_{i_0 i_0}^{r_0+k_\ell} > \frac{\varepsilon}{2}$ for all $\ell = 1, 2, 3, \dots$ We have

$$p_{i_0 j}^{r_0+k_\ell+n_0} \geq p_{i_0 i_0}^{r_0+k_\ell} p_{i_0 j}^{n_0} > \frac{\varepsilon}{2} p_{i_0 j}^{n_0} > 0 \quad \forall \ell \geq 1.$$

Then (2) gives us

$$\frac{\pi_{i_0 j}}{m_j} = \lim_{k\to\infty} \frac{1}{\gamma} \sum_{r=0}^{\gamma-1} p_{i_0 j}^{r+k} = \lim_{\ell\to\infty} \frac{1}{\gamma} \sum_{r=0}^{\gamma-1} p_{i_0 j}^{r_0+k_\ell+n_0} \geq \frac{\varepsilon}{2} p_{i_0 j}^{n_0} > 0.$$

We get $m_j < \infty$. This shows (i_0 is positive recurrent $\Rightarrow$ j is positive recurrent). Because states i_0 and j play an equal role, we also have (i_0 is positive recurrent $\Leftrightarrow$ j is positive recurrent). Since ($m_{i_0} < \infty \Leftrightarrow m_j < \infty$), we have ($m_{i_0} = \infty \Leftrightarrow m_j = \infty$). This shows ($i_0$ is null recurrent $\Leftrightarrow$ j is null recurrent).