Theory of Probability and Measure Theory – Math 8652

Take-home final exam

1) Let w_t, $t \in [0, \infty)$, be a Wiener process. Find the average time it spends in [-1,1] before exiting (-2,2).

2) Let w_t, $t \in [0, \infty)$, be a Wiener process. Find the distribution of

$$\sup_{t \geq 0}(w_t - t).$$

3) Let X be a Markov chain on $\mathbb{Z}^d$ starting at the origin and having a transition kernel

$$p(x, B) = \sum_{y \in B} p(y - x),$$

where $p(y)$ is a probability distribution on $\mathbb{Z}^d$ (not necessarily on the nearest neighbors of the origin) such that $p(0) < 1$. Such X are called random walks. Prove that the origin is either transient or null recurrent.

4) Assume that we are givin a Markov chain with state space $S = \{1, 2, ...\}$ and, for an $i_0 \in S$, we have $P_{i_0}(\exists n \geq 1 : X_n = i_0) = 1$. Prove that, if, for a j, we have $\pi_{i_0 j} > 0$, then $\pi_{i_0 j} = \pi_{jj} = \pi_{j i_0} = 1$.

5) Let $U = (u_0, u_1, u_2, ...)$ be a potential sequence of a renewal sequence such that not all $u_n = 0$. Prove that $\gamma(U)$, introduced in Definition 10/1 of the Lecture Notes, is the greatest integer in $\{1, 2, ...\}$ such that $P(T_1/\gamma \in \{\infty, 1, 2, ...\}) = 1$.

6) (Close to Problem 24.44 in the textbook) Prove the following version of Lebesgue's differentiation theorem. Let $f(t)$ be a finite monotone function on [0,1]. For $x \in [0, 1)$ and integer $n \geq 0$, write $x = k2^{-n} + \epsilon$, where k is an integer and $0 \leq \epsilon < 2^{-n}$, and define $a_n(x) = k2^{-n}$ and $b_n(x) = (k+1)2^{-n}$. Prove that

$$\lim_{n \to \infty} \frac{f(b_n(x)) - f(a_n(x))}{b_n(x) - a_n(x)}$$

exists for almost every $x \in [0, 1]$.

7) Let X be an inner product space with scalar product $\langle ., . \rangle$ and norm $||.||$. Prove that $||x + y||$ is a continuous of x and y: $||x_n + y_n|| \to 0$ if $||x_n - x||, ||y_n - y|| \to 0$. Also prove that $\langle x, y \rangle$ is a continuous function of x and y: $\langle x_n, y_n \rangle \to 0$ if

$||x_n - x||, ||y_n - y|| \to 0$.

8) (Problem 25.10 in the textbook) Describe all potential sequences beginning with two 1's: (1,1,...).

9) (Problem 26.33 in the textbook) Let $q \in (0,1)$ and $\mu\{x\} = (1-q)q^x$, $x \in \mathbb{Z}^+$. Calculate $x \mapsto \pi_{\{0\}}(x)$ for a branching process with branching distribution μ.

10) (Problem 26.39 in the textbook) Show that if two states are accessible from each other then they both have the same period and are of the same type: transient, positive recurrent or null recurrent.