Name : Tuan Pham
ID : 4652218

Math 8652: Theory of Probability
Homework #2

(1) Let X be a $[0,\infty]$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\phi : (0,\infty) \to \mathbb{R}$

$$\phi(u) = \mathbb{E}\, e^{-uX} \quad \forall\, u \in (0,\infty)$$

be its moment generating function.

First, we show that ϕ is continuous on $(0,\infty)$. Take $u_0 > 0$ and let (u_n) be a sequence in $(0,\infty)$ converging to u_0. Put $Y_n = e^{-u_n X}$ and $Y = e^{-u_0 X}$. For $\omega \in \Omega$,

$$Y_n(\omega) = \begin{cases} e^{-u_n X(\omega)} & \text{if } X(\omega) \in [0,\infty), \\ 0 & \text{if } X(\omega) = \infty. \end{cases}$$

$$Y(\omega) = \begin{cases} e^{-u_0 X(\omega)} & \text{if } X(\omega) \in [0,\infty), \\ 0 & \text{if } X(\omega) = \infty. \end{cases}$$

Thus, $\lim_{n\to\infty} Y_n(\omega) = Y(\omega)$ for all $\omega \in \Omega$. Moreover, $0 \le Y_n(\omega) \le 1$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$. By the Dominated Convergence Theorem, $\lim_{n\to\infty} \mathbb{E}\, Y_n = \mathbb{E}\, Y$. This means $\lim_{n\to\infty} \phi(u_n) = \phi(u_0)$. We have showed that ϕ is continuous on $(0,\infty)$.

Next, define $\phi(0) := 1$. Assume $\mathbb{P}(X<\infty) = 1$. We show that ϕ is continuous at 0. According to the assumption, the set

$$A = \{\omega \in \Omega : X(\omega) = \infty\}$$

is of measure zero. Let (u_n) be a sequence in $(0,\infty)$ converging to 0. We show $\lim_{n\to\infty} \phi(u_n) = 1$. For each $\omega \in \Omega \setminus A$,

$$\lim_{n\to\infty} Y_n(\omega) = \lim_{n\to\infty} e^{-u_n X(\omega)} = e^{-0.X(\omega)} = 1.$$

Hence, $Y_n \to 1$ almost everywhere in Ω. We also have $0 \le Y_n(\omega) \le 1$ for all $n \in \mathbb{N}$ and $\omega \in \Omega \setminus A$. By the Dominated Convergence Theorem, $\lim_{n\to\infty} EY_n = E(1) = 1$. In other words, $\lim_{n\to\infty} \phi(u_n) = 1$.

(2) Let $X_1, X_2, X_3, \dots$ be an independent and identically distributed ~~of~~ sequence of random variables. Suppose that they are $[0,\infty]$-valued and that $P(X_1 > 0) > 0$. We show $\sum_{k=1}^{\infty} X_k = \infty$ a.s.

Method 1 (using Borel-Cantelli's lemma)

$$\{\omega \in \Omega : X_1(\omega) > 0\} = \bigcup_{n=1}^{\infty} \underbrace{\{\omega \in \Omega : X_1(\omega) > \tfrac{1}{n}\}}_{\text{increasing sequence of events}}$$

Thus, $P(X_1 > 0) = \lim_{n\to\infty} P(X_1 > \frac{1}{n})$. Because $P(X_1 > 0) > 0$, there exists $n_0 \in \mathbb{N}$ such that $P(X_1 > \frac{1}{n_0}) > 0$. Put $\varepsilon = \frac{1}{n_0}$. For each $n \in \mathbb{N}$, we put

$$A_n = \{\omega \in \Omega : X_n(\omega) > \varepsilon\}.$$

Because $X_1, X_2, X_3, \dots$ is an independent sequence, the events $A_1, A_2, A_3, \dots$ are independent. Because $X_1, X_2, X_3, \dots$ are identically distributed, $P(A_n) = P(A_1) > 0$ for all $n \in \mathbb{N}$. Thus,

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} P(A_1) = \infty.$$

By Borel-Cantelli's lemma, $P(\overline{\lim} A_n) = 1$. For each $\omega \in \overline{\lim} A_n$, $\omega \in A_k$ for infinitely many k's, which implies $X_k(\omega) > \varepsilon$ for infinitely many k's. Thus

$$\sum_{n=1}^{\infty} X_n(\omega) = \infty.$$

Therefore, $\sum_{n=1}^{\infty} X_n = \infty$ a.s.

Method 2 (using moment generating functions)

Let $\phi : [0,\infty) \to \mathbb{R}$ be the common moment generating function of $X_1, X_2, X_3, \dots$

$$\phi(u) = E e^{-uX_1} \quad \forall u \geq 0$$

(with the convention that $e^{-0.\infty} = 1$). Because $0 \leq e^{-uX_1} \leq 1$, $0 \leq \phi(u) \leq 1$ for all $u \geq 0$. We show that $\phi(u) < 1$ for all $u > 0$. Suppose by contradiction that $\phi(u_0) = 1$ for some $u_0 > 0$. Then

$$E(\underbrace{1 - e^{-u_0 X_1}}_{\geq 0}) = 1 - E e^{-u_0 X_1} = 1 - \phi(u_0) = 0.$$

Thus, $1 - e^{-u_0 X_1} = 0$ a.s. Taking logarithm both sides of the equation $e^{-u_0 X_1} = 1$, we get $u_0 X_1 = 0$ a.s. Since $u_0 \neq 0$, $X_1 = 0$ a.s. This contradicts the fact that $P(X_1 > 0) > 0$.

Put $S_n = \sum_{k=1}^{n} X_k$ and $S = \sum_{k=1}^{\infty} X_k$. Then (S_n) converges pointwise to S. This implies (S_n) converges to S in distribution. Because $X_1, X_2, \dots, X_n$ are independent, the moment generating function of S_n is

$$\phi_n(u) = \underbrace{\phi(u)\,\phi(u) \cdots \phi(u)}_{n \text{ times}} = \phi(u)^n.$$

For $u > 0$, $\lim \phi_n(u) = \lim \phi(u)^n = 0$ because $\phi(u) \in [0,1)$. Together with the fact that $\phi_n(0) = 1$ for all $n \in \mathbb{N}$, we get

$$\lim_{n\to\infty} \phi_n(u) = \begin{cases} 0 & \text{if } u \in (0,\infty), \\ 1 & \text{if } u = 0. \end{cases}$$

Denote this function by $\Psi(u)$. By Theorem 14.21, Fristedt-Gray page 262, $\Psi(u)$ is the moment generating function of S. ~~Put $A=\{\omega\in\Omega : S(\omega)<\infty\}$.~~ Then $Ee^{-S} = \Psi(1) = 0$. Thus, $e^{-S}=0$ a.s. This implies $S=\infty$ a.s.

3 Let $X_1, X_2, X_3, \dots$ be an independent and identically distributed sequence of $[0,\infty]$-valued random variables. Let N be a $\{\infty, 1, 2, 3, \dots\}$-valued random variable which is independent of the sequence $X_1, X_2, X_3, \dots$ Introduce a map $S : \Omega \to [0,\infty]$,

$$S(\omega) = \sum_{k=1}^{N(\omega)} X_k(\omega).$$

First we show that S is a random variable. Write

$$S(\omega) = \sum_{k=1}^{\infty} X_k(\omega) \underbrace{I_{k\le N(\omega)}}_{Y_k(\omega)}.$$

$Y_k = I_{N^{-1}(\{\infty, k, k+1, \dots\})}$ is a random variable. Thus, $S = \sum_{k=1}^{\infty} X_k Y_k$ is also a random variable.

Let $\phi : [0,\infty) \to [0,1]$ be the common moment generating function of $X_1, X_2, X_3, \dots$ and $\rho : [0,\infty) \to [0,1]$ be the probability generating function of N, and $\Psi : [0,\infty) \to [0,1]$ be the moment generating function of S. We show that $\Psi = \rho\circ\phi$. For $u\in(0,\infty)$,

$$\Psi(u) = Ee^{-uS} = \underbrace{\int_\Omega e^{-uS} I_{N<\infty}\, P(d\omega)}_{\{1\}} + \underbrace{\int_\Omega e^{-uS} I_{N=\infty}\, P(d\omega)}_{\{2\}}. \quad (1)$$

If $P(X_1>0)>0$ then $\sum_{k=1}^{\infty} X_k = \infty$ a.s. by Problem 2. Then

$$\{2\} = \int_\Omega e^{-u\sum_{k=1}^{\infty} X_k} I_{N=\infty} P(d\omega) = \int_\Omega 0 \cdot I_{N=\infty} P(d\omega) = 0.$$

If $P(X_1>0)=0$ then $P(X_n>0)=0$ for all $n \in \mathbb{N}$. Then

$$\{\omega : \sum_{k=1}^{\infty} X_k(\omega) > 0\} = \bigcup_{k=1}^{\infty} \underbrace{\{\omega : X_k(\omega)>0\}}_{\text{null-event}}$$

is a null event. Then $\sum_{k=1}^{\infty} X_k = 0$ a.s. Then

$$\{2\} = \int_\Omega e^{-u\sum_{k=1}^{\infty} X_k} I_{N=\infty} P(d\omega) = \int_\Omega 1 \cdot I_{N=\infty} P(d\omega) = P(N=\infty).$$

Combining the two cases, we get

$$\{2\} = \begin{cases} 0 & \text{if } P(X_1>0)>0, \\ P(N=\infty) & \text{if } P(X_1>0)=0. \end{cases}$$

By Problem ② (Method 2), if $P(X_1>0)>0$ then $\phi(u) \in [0,1)$ for all $u \in (0,\infty)$.

In case $P(X_1>0)=0$, we have $X_1 = 0$ a.s. and

$$\phi(u) = E e^{-uX_1} = E e^{-u \cdot 0} = 1 \quad \forall u \in (0,\infty).$$

Thus, $P(X_1>0)>0$ if and only if $\phi(u) \in [0,1)$. We get

$$\{2\} = \begin{cases} 0 & \text{if } \phi(u) \in [0,1), \\ P(N=\infty) & \text{if } \phi(u) = 1. \end{cases}$$

We have

$$\begin{aligned}
\{1\} &= \int_\Omega e^{-uS} \sum_{k=1}^{\infty} I_{N=k} P(d\omega) \\
&= \sum_{k=1}^{\infty} \int_\Omega e^{-uX_1 - uX_2 - \cdots - uX_k} I_{N=k} P(d\omega) \quad \text{(by Monotone Convergence Theorem)} \\
&= \sum_{k=1}^{\infty} E(e^{-uX_1} e^{-uX_2} \cdots e^{-uX_k} I_{N=k}) \\
&= \sum_{k=1}^{\infty} (Ee^{-uX_1})(Ee^{-uX_2}) \cdots (Ee^{-uX_k})(EI_{N=k})
\end{aligned}$$

(since $N, X_1, X_2, \dots, X_k$ are independent)

$$= \sum_{k=1}^{\infty} \phi(u)^k P(N=k).$$

We can now rewrite (1) as follows.

$$\Psi(u) = \sum_{k=1}^{\infty} \phi(u)^k P(N=k) + \begin{cases} 0 & \text{if } \phi(u) \in [0,1), \\ P(N=\infty) & \text{if } \phi(u) = 1. \end{cases}$$

$$= \rho(\phi(u)).$$

Therefore, $\Psi = \rho \circ \Phi$.

Next, we compute ES and $\operatorname{Var} S$ in terms of the means and variances of X_1 and N.

$$ES = -\Psi'(0) = -\rho'(\phi(0))\,\phi'(0) = \rho'(1)\, EX_1.$$

By Theorem 13, Fristedt-Gray page 71,

$$\rho^{(n)}(1) = E\left(\prod_{m=0}^{n-1} (N-m)\right) \quad \forall n \in \mathbb{N}.$$

Thus, $\rho'(1) = EN$. We obtain $ES = (EN)(EX_1)$.

$$ES^2 = \Psi''(0) = \rho''(\phi(0))\,\phi'(0)^2 + \rho'(\phi(0))\,\phi''(0)$$
$$= \rho''(1)\,(EX_1)^2 + \rho'(1)\, EX_1^2$$
$$= E(N(N-1))\,(EX_1)^2 + (EN)\, EX_1^2.$$

$$\operatorname{Var} S = ES^2 - (ES)^2 = E(N^2-N)(EX_1)^2 + (EN)EX_1^2 - (EN)^2(EX_1)^2$$
$$= (EN^2 - (EN)^2)(EX_1)^2 + (EN)(EX_1^2 - (EX_1)^2)$$
$$= (\operatorname{Var} N)(EX_1)^2 + (EN)\operatorname{Var} X_1.$$

④ Let (Q_n) be a sequence of probability distributions on $[0,\infty)$ and $\phi_n : [0,\infty) \to [0,1]$

$$\phi_n(u) = \int_{[0,\infty)} e^{-ux}\, Q_n(dx)$$

be the moment generating function of Q_n. First, suppose (Q_n) converges to a distribution Q on $[0,\infty)$ whose moment generating function is $\phi : [0,\infty) \to [0,1]$. By the definition of convergence of distributions,

$$\lim_{n\to\infty} \int_{[0,\infty)} f(x)\, Q_n(dx) = \int_{[0,\infty)} f(x)\, Q(dx)$$

for every continuous and bounded function $f : [0,\infty) \to \mathbb{R}$. For $u \in [0,\infty)$, we take $f(x) = e^{-ux}$ which is continuous and bounded in $[0,\infty)$. Then

$$\lim_{n\to\infty} \int_{[0,\infty)} e^{-ux}\, Q_n(dx) = \int_{[0,\infty)} e^{-ux}\, Q(dx).$$

Thus, $\lim_{n\to\infty} \phi_n(u) = \phi(u)$ for every $u \in [0,\infty)$. Let X be a $[0,\infty)$-valued random variable whose distribution is Q. Then $\phi(u)$ is the moment generating function of X. By Problem 1, ϕ is continuous on $[0,\infty)$.

Now we show the converse. Suppose (ϕ_n) converges pointwise to a function $\phi : [0,\infty) \to [0,1]$ which is continuous at 0. We show that (Q_n) converges to a distribution Q on $[0,\infty)$ whose moment generating function is ϕ.

Assume we could show a weaker statement, namely every subsequence of (Q_n) has a convergent subsequence. Let (Q_{n_k}) be a subsequence of (Q_n) that converges to a distribution Q_0 on $[0,\infty)$. By the first part, the moment generating function of Q_0 is the pointwise limit of (ϕ_n), which is ϕ. Thus, the limit of every convergent subsequence of (Q_n) is Q_0. Suppose by contradiction that (Q_n)

does not converge to Q_0. Then there exists a continuous and bounded function $f: [0,\infty) \to \mathbb{R}$ such that

$$\int_{[0,\infty)} f(x)\, Q_n(dx) \not\to \int_{[0,\infty)} f(x)\, Q_0(dx) \quad \text{as } n\to\infty.$$

There exists $\varepsilon > 0$ and a subsequence (Q_{m_k}) of (Q_n) such that

$$\left| \int_{[0,\infty)} f(x) Q_{m_k}(dx) - \int_{[0,\infty)} f(x) Q_0(dx) \right| \geq \varepsilon \quad \forall k \in \mathbb{N}.$$

We know that (Q_{m_k}) has a subsequence converging to Q_0. However, the above inequality forbits that from happening. We got a contradiction. Thus, (Q_n) converges to Q_0.

The remaining task is to show that every subsequence of (Q_n) has a convergent subsequence. In fact, we only need to show that (Q_n) has a convergent subsequence (i.e. ~~weakly~~ relatively weakly compact). Once it is done, we can apply this result to each subsequence of (Q_n). It suffices to show that (Q_n) is tight. Take $\varepsilon > 0$. We are to find a number $R > 0$ such that $Q_n((R,\infty)) < \varepsilon$ for all $n \in \mathbb{N}$.

For $u \in (0,\infty)$, we have

$$\phi_n(u) = \int_{[0,\infty)} e^{-ux} Q_n(dx) = \int_{[0,\infty)} \int_{[0,\infty)} u e^{-ut} I_{t\geq x}\, dt\, Q_n(dx).$$

Now Fubini's theorem is applicable because the integrand is nonnegative.

$$\phi_n(u) = \int_{[0,\infty)} \int_{[0,\infty)} u e^{-ut} I_{t\geq x} Q_n(dx)\, dt = \int_{[0,\infty)} u e^{-ut} Q_n([0,t])\, dt$$

$$= \underbrace{\int_{[0,\infty)} u e^{-ut} dt}_{=1} - \int_{[0,\infty)} u e^{-ut} Q_n((t,\infty))\, dt.$$

For each $R \in (0,\infty)$,

$$1-\phi_n(u) = \int_{[0,\infty)} u e^{-ut} Q_n((t,\infty))\,dt \geq \int_{[0,\infty)} u e^{-ut} Q_n((t,\infty))\, I_{[0,R]}(t)\,dt$$

$$\geq Q_n((R,\infty)) \int_{[0,\infty)} u e^{-ut} I_{[0,R]}(t)\,dt = (1-e^{-uR})Q_n((R,\infty)).$$

Thus, $\quad Q_n((R,\infty)) \leq \dfrac{1-\phi_n(u)}{1-e^{-uR}} \qquad \forall u, R \in (0,\infty),\ \forall n \in \mathbb{N}.$

Set $u = \frac{1}{R}$. We get

$$Q_n((R,\infty)) \leq \frac{1-\phi_n\left(\frac{1}{R}\right)}{1-e^{-1}} \leq 2\left(1-\phi_n\left(\tfrac{1}{R}\right)\right) \quad \forall n \in \mathbb{N}\ \forall R \in (0,\infty).$$

Since ϕ is continuous at 0, there exists $R_1 > 0$ such that $1-\phi\left(\frac{1}{R_1}\right) < \frac{\varepsilon}{4}$.

Because $\lim_{n\to\infty} \phi_n\left(\frac{1}{R_1}\right) = \phi\left(\frac{1}{R_1}\right)$, there exists $N \in \mathbb{N}$ such that

$$\left|\phi_n\left(\tfrac{1}{R_1}\right) - \phi\left(\tfrac{1}{R_1}\right)\right| < \frac{\varepsilon}{4} \qquad \forall n > N.$$

Because $\lim_{x\to 0^+} \phi_k(x) = 1$ for every $1 \leq k \leq N$, there exists $R_2 > R_1$ such that

$$1-\phi_k\left(\tfrac{1}{R_2}\right) < \frac{\varepsilon}{2} \qquad \forall 1 \leq k \leq N.$$

For $1 \leq n \leq N$, $\quad Q_n((R_2,\infty)) \leq 2\left(1-\phi_n\left(\frac{1}{R_2}\right)\right) < 2\cdot\frac{\varepsilon}{2} = \varepsilon.$

For $n > N$,

$$Q_n((R_2,\infty)) \leq Q_n((R_1,\infty)) \leq 2\left(1-\phi_n\left(\tfrac{1}{R_1}\right)\right)$$

$$= 2\left(1-\phi\left(\tfrac{1}{R_1}\right)\right) + 2\left(\phi\left(\tfrac{1}{R_1}\right) - \phi_n\left(\tfrac{1}{R_1}\right)\right)$$

$$< 2\cdot\frac{\varepsilon}{4} + 2\cdot\frac{\varepsilon}{4} = \varepsilon.$$

Therefore, $Q_n((R_2,\infty)) < \varepsilon$ for all $n \in \mathbb{N}$.