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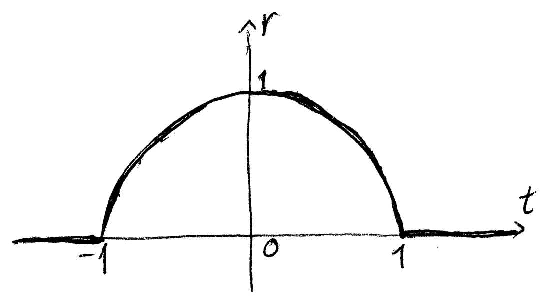
Math 8652: Theory of Probability

Homework #3

① We show that the function $r(t) = \sqrt{(1-t^2)_+}$ is not positive definite. Suppose by contradiction that $r(t)$ is positive definite. For each $x \in \mathbb{R}$,

$$\int_{-1}^{1} r(t) e^{ixt}\,dt = \int_{-\infty}^{\infty} r(t) e^{ixt}\,dt = \int_{-\infty}^{\infty} r(t-s) e^{ix(t-s)}\,dt \quad \forall s \in \mathbb{R}.$$

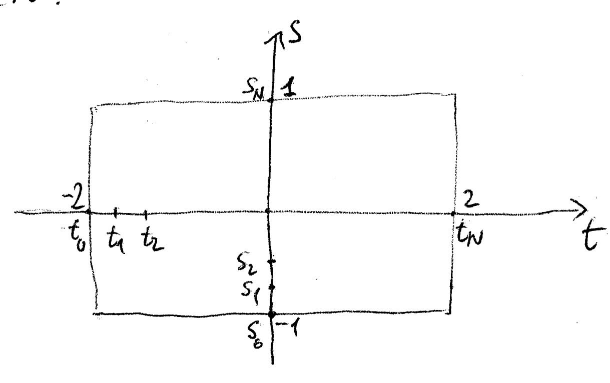


Thus,
$$\int_{-1}^{1} r(t) e^{ixt}\,dt = \frac{1}{2}\int_{-1}^{1}\int_{-\infty}^{\infty} r(t-s)\, e^{ix(t-s)}\,dt\,ds$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-2}^{2} r(t-s)\, e^{ixt} e^{-ixs}\,dt\,ds \qquad (1)$$

(because r is supported in $[-1,1]$).

Let $N \in \mathbb{N}$. Divide $[-1,1]$ into N subintervals of length $\frac{2}{N}$ by the nodes s_j, $0 \le j \le N$. Divide $[-2,2]$ into N subintervals of length $\frac{4}{N}$ by the nodes t_k, $0 \le k \le N$.



Then the double integral (1) is equal to the limit of the Riemann sum as $N\to\infty$. Thus,

$$\int_{-1}^{1} r(t)e^{ixt}\,dt = \frac{1}{2}\lim_{N\to\infty}\frac{8}{N^2}\sum_{j,k=0}^{N} r(t_k-s_j)\,e^{ixt_k}\,e^{-ixs_j}.$$

For each $N\in\mathbb{N}$, the sum inside the limit is nonnegative because r is positive definite. Thus, the limit as $N\to\infty$ is also nonnegative. We get

$$\int_{-1}^{1} r(t)\,e^{ixt}\,dt \geqslant 0 \qquad \forall\, x\in\mathbb{R}.$$

The real part of the left hand side must be nonnegative. Using the fact that $r(t)\cos(xt)$ is an even function in t, we get

$$\int_{0}^{1} r(t)\cos(xt)\,dt \geqslant 0 \qquad \forall x\in\mathbb{R}.$$

Taking $x=\frac{3\pi}{2}$, we get

$$\int_0^1 \sqrt{1-t^2}\,\cos\left(\tfrac{3\pi}{2}t\right)dt \geqslant 0. \qquad (2)$$

We show that this inequality is false.

For $t\in(0,\frac{1}{3})$, $\sqrt{1-t^2}\leqslant 1-\frac{t^2}{2}$ and $\cos\left(\frac{3\pi}{2}t\right)>0$.

For $t\in(\cancel{0}\frac{1}{3},1)$, $\sqrt{1-t^2}\geqslant 1-t^2$ and $\cos\left(\frac{3\pi}{2}t\right)<0$.

Then

$$\begin{aligned}\mathrm{LHS}(2) &= \int_0^{1/3}\sqrt{1-t^2}\cos\left(\tfrac{3\pi}{2}t\right)dt + \int_{1/3}^{1}\sqrt{1-t^2}\cos\left(\tfrac{3\pi}{2}t\right)dt \\ &\leqslant \underbrace{\int_0^{1/3}\left(1-\tfrac{t^2}{2}\right)\cos\left(\tfrac{3\pi}{2}t\right)dt}_{A} + \underbrace{\int_{1/3}^{1}(1-t^2)\cos\left(\tfrac{3\pi}{2}t\right)dt}_{B}. \qquad (3)\end{aligned}$$

Each integrand is the product of a quadratic polynomial and a cosine function. We can compute their indefinite integrals by integrating by parts (twice for each integral). The results are

$$A = \frac{1}{27} \frac{17\pi^2+8}{\pi^3} \quad \text{and} \quad B = -\frac{16}{27} \frac{2+\pi^2}{\pi^3}.$$

Then (3) becomes

$$LHS(2) \leq A+B = \frac{\pi^2-24}{27\pi^3} < 0.$$

This contradicts (2).

(2) Let $f, g : (0, \infty) \to \mathbb{R}$ be completely monotone functions, i.e. f and g are infinitely differentiable and

$$(-1)^k f^{(k)}(t) \geq 0 \quad \forall k \in \mathbb{N},$$
$$(-1)^k g^{(k)}(t) \geq 0 \quad \forall k \in \mathbb{N}.$$

We show that $h = fg$ is also completely monotone. Because f and g are nonnegative, h is also nonnegative. Because f and g are infinitely differentable, h is also infinitely differentiable. We show by induction on $n \in \mathbb{N}$ the identity

$$h^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t)\, g^{(n-k)}(t) \qquad \forall t \in (0, \infty). \qquad (1)$$

We have

$$h'(t) = (fg)'(t) = f'(t)g(t) + f(t)g'(t).$$

Thus, (1) is true for $n = 1$. Suppose (1) is true for some $n \in \mathbb{N}$. Differentiating both sides of (1), we get

$$h^{(n+1)}(t) = \sum_{k=0}^{n} \binom{n}{k} \left[f^{(k+1)}(t) g^{(n-k)}(t) + f^{(k)}(t) g^{(n-k+1)}(t) \right]$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(t) g^{(n-k)}(t) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t) g^{(n-k+1)}(t)$$

$$= \sum_{j=1}^{n+1} \binom{n}{j-1} f^{(j)}(t) g^{(n+1-j)}(t) + \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(t) g^{(n+1-j)}(t)$$

$$= \underbrace{\binom{n}{n}}_{=\binom{n+1}{n+1}} f^{(n+1)}(t) g(t) + \sum_{j=1}^{n} \underbrace{\left[\binom{n}{j-1} + \binom{n}{j} \right]}_{=\binom{n+1}{j}} f^{(j)}(t) g^{(n+1-j)}(t) + \underbrace{\binom{n}{0}}_{=\binom{n+1}{0}} f(t) g^{(n+1)}(t)$$

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} f^{(j)}(t) g^{(n+1-j)}(t).$$

This means (1) is true for $n+1$. Thus, it is true for all $n \in \mathbb{N}$.

Multiplying both sides of (1) by $(-1)^n$, we get

$$(-1)^n h^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} \underbrace{(-1)^k f^{(k)}(t)}_{\geq 0} \underbrace{(-1)^{n-k} g^{(n-k)}(t)}_{\geq 0} \geq 0.$$

Therefore, h is completely monotone.

③ We give two methods to compute $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(k+1) n^k}{(1+n)^{k+2}}$.

Put $$S_n = \sum_{k=0}^{n} \frac{(k+1) n^k}{(1+n)^{k+2}}.$$

<u>Method 1</u> (direct computation)

For fixed $n \in \mathbb{N}$, we define function $F: (0, n+1) \to \mathbb{R}$,

$$F(x) = \frac{1}{1+n} \sum_{k=0}^{n} \left(\frac{x}{1+n}\right)^{k+1}.$$

Then $S_n = F'(n)$. Put $q = q(x) = \frac{x}{1+n} \in (0,1)$.

Then $$F(x) = \frac{1}{1+n} \sum_{k=0}^{n} q^{k+1} = \frac{1}{1+n} \frac{q - q^{n+2}}{1-q}.$$

$$F'(x) = \frac{1}{1+n} \frac{(q' - (n+2) q' q^{n+1})(1-q) + q'(q - q^{n+2})}{(1-q)^2}$$

$$= \underbrace{\frac{q'}{(1+n)(1-q)^2}}_{A} \left[1 - q^{n+1} - \underbrace{(n+1)(1-q)}_{B}\, q^{n+1}\right]. \qquad (1)$$

At $x = n$, $q' = \frac{1}{1+n} = 1-q$. Then $A = B = 1$. Then (1) gives us

$$F'(n) = 1 - 2q^{n+1} = 1 - 2\left(1 - \frac{1}{1+n}\right)^{n+1}.$$

Using the identity $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$, we get

$$\lim_{n\to\infty} S_n = 1 - 2 \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^{n+1} = 1 - 2e^{-1}.$$

<u>Method 2</u> (using moment generating functions)

Let Q be a probability distribution on $\mathbb{R}$ whose density function is

$$f(x) = e^{-x} I_{(0,\infty)}(x) = \begin{cases} e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

This is the exponential distribution with parameter $a = 1$. Its moment generating function is $\phi:(0,\infty)\to\mathbb{R}$,

$$\phi(u)=\int_{\mathbb{R}} e^{-ux}Q(dx)=\int_{-\infty}^{\infty} e^{-ux}f(x)\,dx=\int_0^{\infty} e^{-(u+1)x}\,dx=\frac{1}{1+u}.$$

Let (X_1,X_2) be an independent pair of random variables, each having the same distribution Q. Put $X=X_1+X_2$. Then the density function of X is

$$f_X(x)=(f*f)(x)=\int_{-\infty}^{\infty} f(y)f(x-y)\,dy=\int_{-\infty}^{\infty} e^{-y}e^{y-x}I_{y>0}\,I_{y<x}\,dy$$
$$=e^{-x}\lambda\big((0,\infty)\cap(-\infty,x)\big),$$

where λ is the Lebesgue measure on $\mathbb{R}$. Thus,

$$f_X(x)=\begin{cases} xe^{-x} & \text{if } x>0,\\ 0 & \text{if } x\le 0.\end{cases}$$

Let $F_X:\mathbb{R}\to[0,1]$ be the distribution function of X. Because f_X is integrable over $\mathbb{R}$, F_X is continuous. The moment generating function of X is

$$\phi_X(u)=\phi(u)\,\phi(u)=(1+u)^{-2}\quad\forall u\in(0,\infty).$$

The inversion formula says that

$$F_X(x)=\lim_{n\to\infty}\sum_{0\le k\le nx}\frac{(-n)^k}{k!}\phi_X^{(k)}(n)\qquad\forall x\in[0,\infty).$$

At $x=1$,
$$F_X(1)=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-n)^k}{k!}\phi_X^{(k)}(n).\qquad(2)$$

We have
$$\phi_X^{(k)}(u)=\frac{d^k}{du^k}(1+u)^{-2}=\frac{d^{k-1}}{du^{k-1}}(-2)(1+u)^{-3}$$
$$=\frac{d^{k-2}}{du^{k-2}}(-2)(-3)(1+u)^{-4}$$

$$= \dots = (-2)(-3)\cdots(-k-1)\,(1+u)^{-k-2}$$

$$= \frac{(-1)^k (k+1)!}{(1+u)^{k+1}}.$$

Then (2) becomes

$$F_X(1) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n^k}{k!} \frac{(k+1)!}{(1+n)^{k+1}} = \lim_{n\to\infty} S_n .$$

Therefore,

$$\lim_{n\to\infty} S_n = F_X(1) = \int_{-\infty}^{1} f_X(x)dx = \int_0^1 x e^{-x} dx = -xe^{-x}\Big|_0^1 - \int_0^1 -e^{-x} dx$$

$$= -e^{-1} + (1-e^{-1})$$

$$= 1 - 2e^{-1}.$$

(4) Let $(X_1, X_2, \dots)$ be an independent and identically distributed sequence of real-valued random variables. Put $S_n = X_1 + X_2 + \dots + X_n$ and $S = \overline{\lim_{n\to\infty}} \frac{S_n}{n}$.

First, we show that S is equal to a constant in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ almost surely. Define a map $g: \prod_{i=1}^{\infty} \mathbb{R} \to \overline{\mathbb{R}}$, $g(x_1, x_2, \dots) = \overline{\lim_{n\to\infty}} \frac{s_n}{n}$ where $s_n = x_1 + x_2 + \dots + x_n$. We show that g is measurable. Each s_n can be viewed as a function $s_n : \mathbb{R}^n \to \mathbb{R}$, $s_n(x_1, \dots, x_n) = x_1 + \dots + x_n$. It is continuous and, thus, is measurable. For each $a \in \mathbb{R}$,

$$\{x = (x_1, x_2, \dots) ;\, g(x) < a\} = \bigcup_{n=1}^{\infty} \{x : \sup_{k\geq n} \frac{s_k}{k} < a\}$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \{x : \sup_{k\geq n} \frac{s_k}{k} \leq a - \frac{1}{m}\}$$

$$= \bigcup_{m,n=1}^{\infty} \bigcup_{k=n}^{\infty} \{x : \frac{s_k}{k} \leq a - \frac{1}{m}\}$$

$$= \bigcup_{m,n=1}^{\infty} \bigcup_{k=n}^{\infty} \underbrace{S_k^{-1}\left(\left(-\infty, k\left(a-\tfrac{1}{m}\right)\right]\right) \times \mathbb{R} \times \mathbb{R} \times \cdots}_{\text{measurable in } \prod_{i=1}^{\infty} \mathbb{R}}$$

Hence, $\{x : g(x) < a\}$ is measurable in $\prod_{i=1}^{\infty} \mathbb{R}$. We have showed that g is measurable.

Next, we show that $g(X_1, X_2, \ldots)$ is equal to a constant in $\overline{\mathbb{R}}$ almost surely. Let π be a permutation of $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. We have

$$g(x_{\pi(1)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots) = \overline{\lim_{m\to\infty}} \frac{S_m(x_{\pi(1)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots)}{m}$$

$$= \overline{\lim_{m\to\infty}} \frac{S_m(x_1, \ldots, x_n, x_{n+1}, \ldots)}{m}$$

$$= g(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots).$$

Thus, g is exchangeable. Define two maps $f_1, f_2 : \prod_{i=1}^{\infty} \mathbb{R} \to \mathbb{R}$,

$$f_1(x) = \begin{cases} g(x) & \text{if } g(x) \in \mathbb{R}, \\ 1 & \text{if } g(x) = \infty, \\ -1 & \text{if } g(x) = -\infty, \end{cases}$$

$$f_2(x) = \begin{cases} g(x) & \text{if } g(x) \in \mathbb{R}, \\ 2 & \text{if } g(x) = \infty, \\ -2 & \text{if } g(x) = -\infty. \end{cases}$$

Then f_1 and f_2 are measurable and exchangeable maps. By Hewitt-Savage 0-1 Law, $f_1(X_1, X_2, \ldots)$ and $f_2(X_1, X_2, \ldots)$ are equal to constants almost surely. Write

$$Y_1 := f_1(X_1, X_2, \dots) = c_1 \in \mathbb{R} \text{ a.s.}$$

$$Y_2 := f_2(X_1, X_2, \dots) = c_2 \in \mathbb{R} \text{ a.s.}$$

If $c_1 \notin \{\pm 1\}$ then $g(X_1, X_2, \dots) = f_1(X_1, X_2, \dots) = c_1 \in \mathbb{R}$ a.s.

If $c_2 \notin \{\pm 2\}$ then $g(X_1, X_2, \dots) = f_2(X_1, X_2, \dots) = c_2 \in \mathbb{R}$ a.s.

Consider the case $c_1 \in \{\pm 1\}$ and $c_2 \in \{\pm 2\}$. Suppose $c_1 = 1$. Since $Y_1 = 1$ a.s., $\{\omega : Y_1 = -1\}$ is of measure zero. Thus, $\{\omega : g(X_1, X_2, \dots) = -\infty\}$ is of measure zero. Then $\{\omega : Y_2 = -2\}$ is of measure zero. This implies $c_2 = 2$.

$$\{\omega : g(X_1, X_2, \dots) \in \mathbb{R}\} \subset \{\omega : Y_1 = Y_2\}$$

which is of measure zero because $Y_1 = 1$ a.s. and $Y_2 = 2$ a.s. Therefore, $g(X_1, X_2, \dots) = \infty$ a.s. We do similarly to the case $c_1 = -1$. In that situation, $g(X_1, X_2, \dots) = -\infty$ a.s. We have showed that $g(X_1, X_2, \dots)$ is equal to a constant in $\overline{\mathbb{R}}$ almost surely.

Write $\overline{\lim_{n\to\infty}} \frac{S_n}{n} = c \in \overline{\mathbb{R}}$ a.s. Next, we show that the probability that there exist infinitely many $n \in \mathbb{N}$ such that $S_n > cn$ is equal to 0 or 1. Put

$$A = \{x = (x_1, x_2, \dots) \in \prod_{i=1}^{\infty} \mathbb{R} : \exists \text{ infinitely many } n \in \mathbb{N} \text{ such that } \tfrac{S_n}{n} > c\}$$

We show that A is measurable.

$$A^c = \{x \in \prod_{i=1}^{\infty} \mathbb{R} : \exists \text{ finitely many } n \in \mathbb{N} \text{ such that } \tfrac{S_n}{n} > c\}$$

$$= \left\{ x \in \prod_{i=1}^{\infty} \mathbb{R} : \exists m \in \mathbb{N},\ \frac{S_n}{n} \leq c \ \ \forall n \geq m \right\}$$

$$= \bigcup_{m=1}^{\infty} \left\{ x : \frac{S_n}{n} \leq c \ \ \forall n \geq m \right\}$$

$$= \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ x : \frac{S_n}{n} \leq c \right\}$$

$$= \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \underbrace{S_n^{-1}((-\infty, cn]) \times \mathbb{R} \times \mathbb{R} \times \cdots}_{\text{measurable in } \prod_{i=1}^{\infty} \mathbb{R}}$$

Thus, A is measurable. Define a map $h : \prod_{i=1}^{\infty} \mathbb{R} \to \mathbb{R}$, $h = I_A$. This is a measurable map. Let π be a permutation of $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Then

$$(x_1, x_2, \ldots) \in A \iff \exists \text{ infinitely many } m \in \mathbb{N} \text{ such that } \frac{x_1 + \cdots + x_m}{m} > c$$

$$\iff \exists \text{ infinitely many } m \in \mathbb{N},\ m > n \text{ such that } \frac{x_{\pi(1)} + \cdots + x_{\pi(n)} + x_{n+1} + \cdots + x_m}{m} > c$$

$$\iff (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots) \in A.$$

Thus, $h(x_1, x_2, \ldots) = h(x_{\pi(1)}, \ldots, x_{\pi(n)}, x_{n+1}, \ldots)$. In other words, h is exchangeable. By Hewitt-Savage 0-1 Law, $h(X_1, X_2, \ldots)$ is equal to a constant almost surely. This means $I_A = 1$ a.s. or $I_A = 0$ a.s. Then

$$P((X_1, X_2, \ldots) \in A) = \int_{\Omega} I_A(X_1, X_2, \ldots)\, P(d\omega) = 1 \text{ or } 0.$$