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Math 8652: Theory of Probability
Homework #5

① Let $X_1, \dots, X_n \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathrm{Span}(X_1, \dots, X_n)$ the linear span of $\{X_1, X_2, \dots, X_n\}$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. We show by induction in $n \in \mathbb{N}$ that $\mathrm{Span}(X_1, \dots, X_n)$ is closed in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

For $n = 1$, $\mathrm{Span}(X_1) = \{cX_1 : c \in \mathbb{R}\}$. If $X_1 = 0$ a.s. then $\mathrm{Span}(X_1) = \{0\}$, which is closed in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Consider the case X_1 is not almost-surely equal to 0. For each $m \in \mathbb{N}$, put

$$A_m = \{\omega \in \Omega : X_1(\omega) > \tfrac{1}{m}\},$$

$$B_m = \{\omega \in \Omega : X_1(\omega) < -\tfrac{1}{m}\}.$$

Then $\left(\bigcup_{m=1}^{\infty} A_m\right) \cup \left(\bigcup_{m=1}^{\infty} B_m\right) = \{\omega \in \Omega : X_1(\omega) \neq 0\}$, which has positive measure.

Then

$$0 < \mathbb{P}\left(\left(\bigcup_{m=1}^{\infty} A_m\right) \cup \left(\bigcup_{m=1}^{\infty} B_m\right)\right) \leq \sum_{m=1}^{\infty} \mathbb{P}(A_m) + \sum_{m=1}^{\infty} \mathbb{P}(B_m).$$

This implies there exists $m_0 \in \mathbb{N}$ such that $\mathbb{P}(A_{m_0}) > 0$ or $\mathbb{P}(B_{m_0}) > 0$. By replacing X_1 with $-X_1$ if necessary, we can assume $\mathbb{P}(A_{m_0}) > 0$. Let (Z_m) be a sequence in $\mathrm{Span}(X_1)$ that converges to $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Write $Z_m = c_m X_1$. Then

$$\int_\Omega Z_m I_{A_{m_0}} \,\mathbb{P}(d\omega) = c_m \underbrace{\int_\Omega X_1 I_{A_{m_0}} \,\mathbb{P}(d\omega)}_{a}. \qquad (1)$$

$$a = \int_\Omega X_1 I_{A_{m_0}} P(d\omega) \geq \frac{1}{m_0} P(A_{m_0}) > 0.$$

$$\left| \int_\Omega (Z_m - Z) I_{A_{m_0}} P(d\omega) \right| \leq \int_\Omega |Z_m - Z| P(d\omega) = \|Z_m - Z\|_{L^1} \to 0 \quad \text{as } m \to \infty.$$

Thus, $$\lim_{m\to\infty} \int_\Omega Z_m I_{A_{m_0}} P(d\omega) = \int_\Omega Z I_{A_{m_0}} P(d\omega).$$

Then (1) implies

$$c_m = \frac{1}{a} \int_\Omega Z_m I_{A_{m_0}} P(d\omega) \longrightarrow \frac{1}{a} \int_\Omega Z I_{A_{m_0}} P(d\omega) := c \quad \text{as } m \to \infty.$$

Then $Z_m = c_m X_1 \to c X_1$ pointwise as $m \to \infty$. For all m sufficiently large, $|c_m - c| \leq 1$. Then $|Z_m - cX_1| = |c_m - c||X_1| \leq |X_1|$, which is an integrable function. By the Dominated Convergence Theorem, $Z_m \to cX_1$ in $L^1(\Omega, F, P)$. Thus, $Z = cX_1 \in \mathrm{Span}(X_1)$.

Let $n \geq 2$. Suppose $\mathrm{Span}(Y_1, \dots, Y_r)$ is closed in $L^1(\Omega, F, P)$ for any $1 \leq r < n$ and $Y_1, Y_2, \dots, Y_r \in L^1(\Omega, F, P)$. Take $X_1, X_2, \dots, X_n \in L^1(\Omega, F, P)$. We show that $\mathrm{Span}(X_1, \dots, X_n)$ is closed in $L^1(\Omega, F, P)$. Let $(\tilde{X}_m)$ be a sequence in $\mathrm{Span}(X_1, \dots, X_n)$ that converges to some $X \in L^1(\Omega, F, P)$.

Because $\mathrm{Span}(X_1, \dots, X_n)$ is a finite dimensional vector space, it has a basis $\{Y_1, Y_2, \dots, Y_s\}$ for some $0 \leq s \leq n$. If $s = 0$ then $\mathrm{Span}(X_1, \dots, X_n) = \{0\}$, which is closed in $L^1(\Omega, F, P)$. If $s = 1$ then $\mathrm{Span}(X_1, \dots, X_n) = \mathrm{Span}(Y_1)$, which has been proved to be closed in $L^1(\Omega, F, P)$. Consider the case $s \geq 2$. Write

$$\tilde{X}_m = \alpha_{1m} Y_1 + \alpha_{2m} Y_2 + \dots + \alpha_{sm} Y_s$$

for $\alpha_{1m}, \alpha_{2m}, \dots, \alpha_{sm} \in \mathbb{R}$. For each $j \in \{1, 2, \dots, s\}$, we show that the sequence $(\alpha_{jm})_{m \in \mathbb{N}}$ is bounded. Suppose otherwise. Without loss of generality, we can assume the sequence $(\alpha_{sm})_{m \in \mathbb{N}}$ is unbounded. By replacing $(\tilde{X}_m)$ with a suitable subsequence, we can assume $|\alpha_{sm}| \to \infty$ as $m \to \infty$. Then

$$\begin{aligned} Y_s &= \frac{1}{\alpha_{sm}} \left(\tilde{X}_m - \alpha_{1m} Y_1 - \dots - \alpha_{s-1,m} Y_{s-1} \right) \\ &= \frac{1}{\alpha_{sm}} \tilde{X}_m + \sum_{j=1}^{s-1} \left(- \frac{\alpha_{jm}}{\alpha_{sm}} \right) Y_j \qquad \forall m \in \mathbb{N}. \qquad (2) \end{aligned}$$

Because $\tilde{X}_m \to X$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $|\alpha_{sm}| \to \infty$ as $m \to \infty$, $\frac{1}{\alpha_{sm}} \tilde{X}_m \to 0$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Letting $m \to \infty$ in (2), we get

$$Y_s = \lim_{m \to \infty} \sum_{j=1}^{s-1} \left(- \frac{\alpha_{jm}}{\alpha_{sm}} \right) Y_j .$$

This implies Y_s belongs to the closure in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ of $\operatorname{Span}(Y_1, \dots, Y_{s-1})$. Since $1 \le s-1 < n$, by the induction hypothesis, $\operatorname{Span}(Y_1, \dots, Y_{s-1})$ is closed in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then $Y_s \in \operatorname{Span}(Y_1, \dots, Y_{s-1})$. This is a contradiction because $Y_1, Y_2, \dots, Y_s$ are linearly independent.

We have showed that the sequence $(\alpha_{jm})_{m \in \mathbb{N}}$ is bounded for every $j \in \{1, 2, \dots, s\}$. (α_{1m}) has a convergent subsequence (α_{1m_k}). Denote by $\tilde{X}^{(1)}$ the subsequence $(\tilde{X}_{m_k})$ of $(\tilde{X}_m)$. If we regard $\tilde{X}^{(1)}$ instead of $(\tilde{X}_m)$, we can assume (α_{1m}) converges. (α_{2m}) has a convergent subsequence $(\alpha_{2m'_k})$. Denote by

$\tilde{X}^{(2)}$ the subsequence $(\tilde{X}^{(1)}_{m'_k})$ of $\tilde{X}^{(1)}$. If we regard $\tilde{X}^{(2)}$ instead of $(\tilde{X}_m)$, we can assume (α_{1m}) and (α_{2m}) converge. Continue this process s times. We then conclude that $(\tilde{X}_m)$ has a subsequence such that $(\alpha_{1m}), (\alpha_{2m}), \dots, (\alpha_{sm})$ converge as $m \to \infty$. Replace $(\tilde{X}_m)$ by this subsequence. Put $\alpha_j = \lim_{m\to\infty} \alpha_{jm}$ for each $j \in \{1, 2, \dots, s\}$. Put $Y = \alpha_1 Y_1 + \dots + \alpha_s Y_s$. Then

$$\tilde{X}_m = \alpha_{1m} Y_1 + \dots + \alpha_{sm} Y_s \longrightarrow \alpha_1 Y_1 + \dots + \alpha_s Y_s = Y \text{ pointwise as } m \to \infty.$$

$$|\tilde{X}_m - Y| \le \sum_{j=1}^{s} |\alpha_{jm} - \alpha_j|\,|Y_j| \le \sum_{j=1}^{s} |Y_j| \in L^1(\Omega, \mathcal{F}, P)$$

for all m sufficiently large. By the Dominated Convergence Theorem, $\tilde{X}_m \to Y$ in $L^1(\Omega, \mathcal{F}, P)$. Hence, $X = Y \in \mathrm{Span}(Y_1, \dots, Y_s) = \mathrm{Span}(X_1, \dots, X_n)$.

(2) Let $\Omega = \dot{\bigcup}_{n=1}^{\infty} A_n$ be a partition of Ω into disjoint sets $A_n \in \mathcal{F}$ such that $P(A_n) > 0$ for every $n \in \mathbb{N}$. Put $\mathcal{G} = \sigma(A_n : n \in \mathbb{N})$. Take $\xi \in L^1(\Omega, \mathcal{F}, P)$. We show that

$$E(\xi \mid \mathcal{G}) = \frac{1}{P(A_n)} E(\xi I_{A_n}).$$

on A_n for every $n \in \mathbb{N}$. Equivalently, we need to show

$$E(\xi \mid \mathcal{G}) = \sum_{n=1}^{\infty} \frac{1}{P(A_n)} E(\xi I_{A_n}) I_{A_n}. \qquad (1)$$

Denote

$$X = \sum_{n=1}^{\infty} \frac{1}{P(A_n)} E(\xi A_n) I_{A_n},$$

$$X_m = \sum_{n=1}^{m} \frac{1}{P(A_n)} E(\xi A_n) I_{A_n}.$$

The indicators $I_{A_1}, I_{A_2}, I_{A_3}, \dots$ are $\mathcal{G}$-measurable. Thus, X_m is $\mathcal{G}$-measurable.

Because $A_1, A_2, A_3, \ldots$ are pairwise disjoint, $X_m = X$ on $\bigcup_{n=1}^{m} A_n$. ~~Thus~~,

~~A_m~~ Because $\bigcup_{n=1}^{\infty} A_n = \Omega$,

$$X(\omega) = \lim_{n\to\infty} X_m(\omega) \quad \forall \omega \in \Omega.$$

Then X is also $\mathcal{G}$-measurable.

Next, we show that $X \in L^1(\Omega, \mathcal{F}, P)$ and $EX = E\xi$.

$$\begin{aligned}
\int_\Omega |X_m| \, P(d\omega) &\leq \int_\Omega \sum_{n=1}^{m} \frac{1}{P(A_n)} E(|\xi| I_{A_n}) I_{A_n} \, P(d\omega) \\
&= \sum_{n=1}^{m} \frac{1}{P(A_n)} E(|\xi| I_{A_n}) P(A_n) \\
&= \sum_{n=1}^{m} E(|\xi| I_{A_n}) \\
&= E(|\xi| I_{\bigcup_{n=1}^{m} A_n}). \qquad (2)
\end{aligned}$$

Then
$$\begin{aligned}
\int_\Omega |X| \, P(d\omega) &\overset{\text{Fatou}}{\leq} \liminf_{m\to\infty} \int_\Omega |X_m| \, P(d\omega) \leq \liminf_{m\to\infty} E(|\xi| I_{\bigcup_{n=1}^{m} A_n}) \\
&= E(|\xi| I_\Omega) \quad \text{(by Monotone Convergence Theorem)} \\
&= E|\xi| < \infty.
\end{aligned}$$

While we was deriving (2), we also derived $|X_m| \leq |\xi| I_{\bigcup_{n=1}^{m} A_n} \leq |\xi|$.

Thus, by the Dominated Convergence Theorem,

$$\begin{aligned}
EX = \int_\Omega X \, P(d\omega) &= \lim_{m\to\infty} \int_\Omega X_m \, P(d\omega) = \lim_{m\to\infty} \int_\Omega \sum_{n=1}^{m} \frac{1}{P(A_n)} E(\xi I_{A_n}) I_{A_n} \\
&= \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{P(A_n)} E(\xi I_{A_n}) P(A_n)
\end{aligned}$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} E(\xi I_{A_n}) = \lim_{m\to\infty} E(\xi I_{\bigcup_{n=1}^{m} A_n})$$

$$= E(\xi I_{\Omega}) \quad \text{(by the Dominated Convergence Theorem)}$$

$$= E\xi .$$

To show that $X = E(\xi | \mathcal{G})$, we need to show

$$E(X I_A) = E(\xi I_A) \quad \forall A \in \mathcal{G}.$$

Put $\mathcal{G}_0 = \{A \in \mathcal{G} : E(X I_A) = E(\xi I_A)\}$. We now show that $\mathcal{G}_0$ is a λ-system.

Since $E(X I_\phi) = E(\xi I_\phi) = 0$, $\phi \in \mathcal{G}_0$. Let $B_1, B_2, B_3, \dots$ be a sequence of pairwise disjoint members of $\mathcal{G}_0$.

$$E(X I_{B_n}) = E(\xi I_{B_n}) \quad \forall n \in \mathbb{N}.$$

Then
$$E(X I_{\bigcup_{n=1}^{m} B_n}) = \sum_{n=1}^{m} E(X I_{B_n}) = \sum_{n=1}^{m} E(\xi I_{B_n}) = E(\xi I_{\bigcup_{n=1}^{m} B_n}) \quad \forall m \in \mathbb{N}. \tag{3}$$

Put $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{G}$. Then $|X I_{\bigcup_{n=1}^{m} B_n}| \le |X|$. By the Dominated Convergence Theorem,

$$\lim_{m\to\infty} E(X I_{\bigcup_{n=1}^{m} B_n}) = E(X I_B).$$

Similarly, $\lim_{m\to\infty} E(\xi I_{\bigcup_{n=1}^{m} B_n}) = E(\xi I_B)$.

Letting $m\to\infty$ in (3), we get $E(X I_B) = E(\xi I_B)$. Thus $B \in \mathcal{G}_0$.

Take any $A \in \mathcal{G}_0$. Then

$$E(X I_{A^c}) = E(X - X I_A) = EX - E(X I_A),$$

$$E(\xi I_{A^c}) = E(\xi - \xi I_A) = E\xi - E(\xi I_A).$$

Because $EX = E\xi$ and $E(XI_A) = E(\xi I_A)$, we get $E(XI_{A^c}) = E(\xi I_{A^c})$. Then $A^c \in \mathcal{G}_0$. We have showed that $\mathcal{G}_0$ is a λ-system.

For each $k \in \mathbb{N}$, $XI_{A_k} = \frac{1}{P(A_k)} E(\xi I_{A_k}) I_{A_k}$ because

$$I_{A_k} I_{A_n} = I_{A_k \cap A_n} = \begin{cases} I_{A_k} & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

Thus,
$$E(XI_{A_k}) = \int_\Omega \frac{1}{P(A_k)} E(\xi I_{A_k}) I_{A_k} P(d\omega) = \frac{1}{P(A_k)} E(\xi I_{A_k}) P(A_k)$$
$$= E(\xi I_{A_k}).$$

This means $A_k \in \mathcal{G}_0$. The family $\mathcal{a} = \{\phi, A_1, A_2, A_3, \dots\}$ is a π-system because $A_1, A_2, A_3, \dots$ are pairwise disjoint. Then $\mathcal{a}$ is a π-system contained in the λ-system $\mathcal{G}_0$. By the Sierpinski Class theorem, $\mathcal{G}_0$ contains $\sigma(\mathcal{a}) = \sigma(A_1, A_2, A_3, \dots) = \mathcal{G}$. Thus, $\mathcal{G}_0 = \mathcal{G}$.

(3) Let $d \in \mathbb{N}$ and $a_{ij}, b_i, c \in \mathbb{R}$ for $1 \leq i,j \leq d$. Suppose the function $f: \mathbb{R}^d \to \mathbb{R}$

$$f(x) = \sum_{i,j=1}^{d} a_{ij} x_i x_j + 2 \sum_{i=1}^{d} b_i x_i + c$$

is nonnegative for every $x \in \mathbb{R}^d$. We show that f has a minimum value.

Denote ~~by~~ $a = (a_{ij})_{1 \leq i,j \leq d} \in M_d(\mathbb{R})$, $b = (b_i)_{1 \leq i \leq d} \in \mathbb{R}^d$. Denote by $(\cdot,\cdot)$ the usual dot product in $\mathbb{R}^d$. Then

$$f(x) = (ax, x) + 2(b, x) + c. \qquad (1)$$

We observe that $\sum_{i,j=1}^{d} a_{ij} x_i x_j = \sum_{i,j=1}^{d} \frac{a_{ij} + a_{ji}}{2} x_i x_j$.

This means if we replace matrix a with $\frac{a + a^T}{2}$, the value of $f(x)$ does not change. Thus, we can assume a is symmetric, i.e. $a_{ji} = a_{ij}$ for all $1 \le i, j \le d$.

$$f(tx) = t^2(ax, x) + 2t(b, x) + c \geq 0 \quad \forall t \in \mathbb{R}. \qquad (2)$$

If $(ax, x) < 0$ for some $x \in \mathbb{R}^d$ then (2) is not satisfied as $t \to \infty$. Thus, $(ax, x) \geq 0$ for all $x \in \mathbb{R}^d$. We view LHS(2) as a quadratic polynomial in t. The discriminant must be nonpositive, i.e.

$$(b, x)^2 \le c(ax, x) \quad \forall x \in \mathbb{R}^d.$$

Because a is symmetric, the transpose matrix of a is $a^* = a$. For each $x \in \ker a^*$,

$$(b, x)^2 \leq c(ax, x) = c(a^* x, x) = c(0, x) = 0.$$

Thus, $(b, x) = 0$. Then $b \in (\ker a^*)^\perp$. In linear algebra, we know the identity

$$\text{range}(a) = (\ker a^*)^\perp.$$

Thus, $b \in \text{range}(a)$. There exists $x_0 \in \mathbb{R}^d$ such that $b = ax_0$. Then (1) gives

$$\begin{aligned}
f(x) &= (ax, x) + 2(ax_0, x) + c \\
&= (ax, x) + (ax_0, x) + (ax_0, x) + c \\
&= (ax, x) + (ax_0, x) + \underbrace{(x_0, ax)}_{=(ax, x_0)} + c \\
&= (ax + ax_0, x + x_0) - (ax_0, x_0) + c \\
&= \underbrace{(a(x + x_0), x + x_0)}_{\geq 0} - (ax_0, x_0) + c \\
&\geq -(ax_0, x_0) + c.
\end{aligned}$$

The equality holds when $x = -x_0$. Therefore, $\min_{x \in \mathbb{R}^d} f(x) = f(-x_0)$.

4. Consider a random walk with steps $X_1, X_2, X_3, \dots$ Suppose $P(X_1=1) = P(X_1=-1) = \frac{1}{2}$. Put $S_n = X_1 + X_2 + \dots + X_n$, and $\xi_n = \left(\frac{5}{4}\right)^{-n} 2^{S_n}$. First, we show that $\xi_n \to 0$ a.s.

We have $EX_1 = 1\,P(X_1=1) + (-1)\,P(X_1=-1) = \frac{1}{2} - \frac{1}{2} = 0$. By the Strong law of large numbers, $\frac{S_n}{n} \to EX_1 = 0$ a.s. Then

$$\log \xi_n = S_n \log 2 - n \log \frac{5}{4} = n \log 2 \underbrace{\left(\frac{S_n}{n} - \frac{\log \frac{5}{4}}{\log 2} \right)}_{\to -\frac{\log 5/4}{\log 2} < 0} \longrightarrow -\infty \quad \text{a.s.}$$

Thus, $\xi_n = e^{\log \xi_n} \to 0$ a.s.

Next, we show $E \sup_{n \in \mathbb{N}} \xi_n = \infty$. Put $\xi = \sup_{n \in \mathbb{N}} \xi_n \geq 0$. Suppose by contradiction that $E\xi < \infty$. Because $0 \leq \xi_n \leq \xi$ and $\xi_n \to 0$ a.s., by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} E\xi_n = E\left(\lim_{n\to\infty} \xi_n\right) = E(0) = 0. \qquad (1)$$

On the other hand,

$$E\xi_n = \left(\frac{5}{4}\right)^{-n} 2^{S_n} = \left(\frac{5}{4}\right)^{-n} E\left(2^{X_1} 2^{X_2} \dots 2^{X_n}\right)$$

$$= \left(\frac{5}{4}\right)^{-n} (E2^{X_1})(E2^{X_2}) \dots (E2^{X_n}) \quad \text{(because } X_1, X_2, \dots \text{ are independent)}$$

$$= \left(\frac{5}{4}\right)^{-n} (E2^{X_1})^n \quad \text{(because } X_1, X_2, \dots \text{ have the same distribution)}$$

$$= \left(\frac{5}{4}\right)^{-n} \Big(2^1 \underbrace{P(X_1=1)}_{=1/2} + 2^{-1} \underbrace{P(X_1=-1)}_{=1/2}\Big)^n$$

$$= \left(\frac{5}{4}\right)^{-n} \left(\frac{5}{4}\right)^n = 1.$$

This contradicts (1).